ON THE DIVERGENCE OF TAYLOR SERIES
IN DE BRANGES–ROVNYAK SPACES

PIERRE-OLIVIER PARISÉ AND THOMAS RANSFORD

ABSTRACT. It is known that there exist functions in certain de Branges–Rovnyak spaces whose Taylor series diverge in norm, even though polynomials are dense in the space. This is often proved by showing that the sequence of Taylor partial sums is unbounded in norm. In this note we show that it can even happen that the Taylor partial sums tend to infinity in norm. We also establish similar results for lower-triangular summability methods such as the Cesàro means.

1. INTRODUCTION

The Hilbert function spaces now known as de Branges–Rovnyak spaces were first introduced by de Branges and Rovnyak in the appendix of [3], in relation to scattering models. Later on, thanks largely to Sarason's work culminating in [14], these spaces turned out to have deep connections with operator theory. For a recent account of the theory of de Branges–Rovnyak spaces, see the two-volume monograph [5, 6].

Let H^∞ denote the space of bounded holomorphic functions on the open unit disk $\mathbb{D}$. Given b in the unit ball of H^∞, the *de Branges–Rovnyak space* $\mathcal{H}(b)$ is the (unique) reproducing kernel Hilbert space on $\mathbb{D}$ with kernel $k^b(z,w) = (1 - \overline{b(w)}b(z))/(1 - \overline{w}z)$, for $w, z \in \mathbb{D}$. It is a Hilbert space of holomorphic functions on $\mathbb{D}$, contractively contained in the Hardy space H^2.

The properties enjoyed by $\mathcal{H}(b)$ depend strongly on whether b is or is not an extreme point of the unit ball of H^∞. In particular, $\mathcal{H}(b)$ contains the polynomials if and only if b is non-extreme, and in this case the polynomials are dense in $\mathcal{H}(b)$ (see [15, IV-3]). However, even if the polynomials are dense in $\mathcal{H}(b)$, the Taylor series of functions in $\mathcal{H}(b)$ may diverge in norm. In this note, we examine this phenomenon in detail.

It was shown in [2, Theorem 5.5] that there exists a non-extreme function b in the unit ball of H^∞ and a function $f \in \mathcal{H}(b)$ whose radial dilates satisfy

$$(1) \qquad \limsup_{r \to 1^-} \|f_r\|_{\mathcal{H}(b)} = \infty.$$

Somewhat later, in [4, Theorem 3.1], an example was given of a non-extreme b and an $f \in \mathcal{H}(b)$ with the stronger property that

$$(2) \qquad \liminf_{r \to 1^-} \|f_r\|_{\mathcal{H}(b)} = \infty.$$

Another construction of such an example can be found in [9, Theorem 1.1].

As was pointed out in [4, Corollary 3.4], the relation (1) implies that

$$(3) \qquad \limsup_{n \to \infty} \|s_n(f)\|_{\mathcal{H}(b)} = \infty \quad \text{and} \quad \limsup_{n \to \infty} \|\sigma_n(f)\|_{\mathcal{H}(b)} = \infty,$$

where $s_n(f)$ is the sequence of Taylor partial sums of f and $\sigma_n(f)$ is the sequence of their Cesàro means. However, the corresponding stronger form of (3), namely

$$(4) \qquad \liminf_{n \to \infty} \|s_n(f)\|_{\mathcal{H}(b)} = \infty \quad \text{and} \quad \liminf_{n \to \infty} \|\sigma_n(f)\|_{\mathcal{H}(b)} = \infty,$$

does not immediately follow from (2). It raises the question as to whether there exist examples where (4) holds. Our aim in this note is to present an affirmative answer to this question. This is not a trivial endeavour since, as we shall see, there exist de Branges–Rovnyak spaces $\mathcal{H}(b)$ in which $\limsup_{n \to \infty} \|s_n(f)\|_{\mathcal{H}(b)} = \infty$ for some f, yet $\liminf_{n \to \infty} \|s_n(f)\|_{\mathcal{H}(b)} < \infty$ for all f.

2. Statement of main result

We shall treat the case of a general lower-triangular summability method. Let $\Gamma = (\gamma_{nk})_{n,k \geq 0}$ be an infinite lower-triangular matrix of complex numbers. Given a formal power series $f(z) = \sum_{k \geq 0} \widehat{f}(k) z^k$, for each $n \geq 0$ we define

$$S_n^\Gamma(f)(z) := \sum_{k=0}^n \gamma_{nk} \widehat{f}(k) z^k.$$

The following theorem is our main result.

Theorem 2.1. *Let b be a non-extreme point of the unit ball of H^∞, and let $\Gamma = (\gamma_{nk})_{n,k \geq 0}$ be an infinite lower-triangular matrix of complex numbers.*

(i) *If $\sup_{n \geq 0}(|\gamma_{nn}| \|z^n\|_{\mathcal{H}(b)}) = \infty$, then there exists $f \in \mathcal{H}(b)$ such that*

$$\limsup_{n \to \infty} \|S_n^\Gamma(f)\|_{\mathcal{H}(b)} = \infty.$$

(ii) *If $\sum_{n \geq 0}(|\gamma_{nn}|^{-2} \|z^n\|_{\mathcal{H}(b)}^{-2}) < \infty$, then there exists $f \in \mathcal{H}(b)$ such that*

$$\liminf_{n \to \infty} \|S_n^\Gamma(f)\|_{\mathcal{H}(b)} = \infty.$$

(iii) *If $\sum_{n \geq 0}(|\gamma_{nn}|^{-1} \|z^n\|_{\mathcal{H}(b)}^{-1}) < \infty$, then there exist $f, g \in \mathcal{H}(b)$ such that*

$$\liminf_{n \to \infty} |\langle S_n^\Gamma(f), g \rangle_{\mathcal{H}(b)}| = \infty.$$

An important special case is obtained by taking $\gamma_{nk} = \binom{n}{k}/\binom{n+\alpha}{k}$ for $0 \le k \le n$, where $\alpha \ge 0$. Then $S_n^{\Gamma}(f)$ is just the generalized Cesàro mean $\sigma_n^{\alpha}(f)$. In particular $S_n^{\Gamma}(f) = s_n(f)$ if $\alpha = 0$ and $S_n^{\Gamma}(f) = \sigma_n(f)$ if $\alpha = 1$. Since

$$|\gamma_{nn}| = \binom{n+\alpha}{n}^{-1} \asymp n^{-\alpha} \quad (n \to \infty),$$

we immediately obtain the following corollary.

Corollary 2.2. *Let b be a non-extreme point of the unit ball of H^{∞}, and let $\alpha \ge 0$.*

(i) *If $\sup_{n \ge 0}(n^{-\alpha}\|z^n\|_{\mathcal{H}(b)}) = \infty$, then there exists $f \in \mathcal{H}(b)$ such that*

$$\limsup_{n \to \infty} \|\sigma_n^{\alpha}(f)\|_{\mathcal{H}(b)} = \infty.$$

(ii) *If $\sum_{n \ge 0}(n^{2\alpha}/\|z^n\|_{\mathcal{H}(b)}^2) < \infty$, then there exists $f \in \mathcal{H}(b)$ such that*

$$\liminf_{n \to \infty} \|\sigma_n^{\alpha}(f)\|_{\mathcal{H}(b)} = \infty.$$

(iii) *If $\sum_{n \ge 0}(n^{\alpha}/\|z^n\|_{\mathcal{H}(b)}) < \infty$, then there exist $f, g \in \mathcal{H}(b)$ such that*

$$\liminf_{n \to \infty} |\langle \sigma_n^{\alpha}(f), g \rangle_{\mathcal{H}(b)}| = \infty.$$

Remark. Part (i) of Corollary 2.2 was already known. It was established in [7, Theorem 6.10]. Our proof here is quite different, and also leads to parts (ii) and (iii), which we believe to be new, even in the special cases of $s_n(f)$ and $\sigma_n(f)$.

3. Proof of main result

The proof is based on the following uniform boundedness principles from functional analysis.

Theorem 3.1. *Let (T_n) be a sequence of bounded operators on a complex Hilbert space H.*

(i) *If $\sup_n \|T_n\| = \infty$, then there exists $x \in H$ such that*

$$\limsup_{n \to \infty} \|T_n x\| = \infty.$$

(ii) *If $\sum_n 1/\|T_n\|^2 < \infty$, then there exists $x \in H$ such that*

$$\liminf_{n \to \infty} \|T_n x\| = \infty.$$

(iii) *If $\sum_n 1/\|T_n\| < \infty$, then there exist $x, y \in H$ such that*

$$\liminf_{n \to \infty} |\langle T_n x, y \rangle| = \infty.$$

Proof. Part (i) is just the standard uniform bounded principle. Parts (ii) and (iii) are proved in [12, Theorems 3 (ii) and 6 (ii)], where they are derived as a consequence of Ball's solution to the complex plank problem [1]. □

We are going to apply these results with $T_n = S_n^\Gamma$ and $H = \mathcal{H}(b)$. Each S_n^Γ is a bounded linear operator on $\mathcal{H}(b)$. The following lemma provides a lower bound for its operator norm. Once this lemma is established, Theorem 2.1 will follow immediately.

Notice that b is a non-extreme point of the unit ball of H^∞ if and only if there exists an outer function $a \in H^\infty$ with $a(0) > 0$ such that $|b|^2 + |a|^2 = 1$ a.e. on $\mathbb{T}$. The function a is then uniquely determined. For more details, see [14, Chapter IV]. We call (b, a) a *Pythagorean pair*.

Lemma 3.2. *Let (b, a) be a Pythagorean pair, and let $\Gamma = (\gamma_{nk})$ be a lower-triangular matrix of complex numbers. Then, for all $n \geq 0$,*

$$\|S_n^\Gamma : \mathcal{H}(b) \to \mathcal{H}(b)\| \geq a(0)|\gamma_{nn}|\|z^n\|_{\mathcal{H}(b)}.$$

Proof. It is known that, if $h \in H^2$, then $ah \in \mathcal{H}(b)$ and $\|ah\|_{\mathcal{H}(b)} \leq \|h\|_{H^2}$ (see [13, Lemma 4]). In particular, setting $g_n(z) := z^n a(z)$, we have that $g_n \in \mathcal{H}(b)$ and $\|g_n\|_{\mathcal{H}(b)} \leq 1$. Also $g_n(z) = a(0)z^n + (\text{terms of higher degree})$, so $S_n^\Gamma(g_n) = \gamma_{nn}a(0)z^n$. Combining these observations, we deduce that

$$\|S_n^\Gamma : \mathcal{H}(b) \to \mathcal{H}(b)\| \geq \frac{\|S_n^\Gamma(g_n)\|_{\mathcal{H}(b)}}{\|g_n\|_{\mathcal{H}(b)}} \geq \frac{|\gamma_{nn}a(0)|\|z^n\|_{\mathcal{H}(b)}}{1}.$$

This proves the lemma. $\square$

4. EXAMPLES

In order to exploit the results of the preceding section, we need to be able to calculate, or at least estimate, the norms of monomials $\|z^n\|_{\mathcal{H}(b)}$. There is a well-known formula that enables us to do this. The following result was established in [13, p.81].

Proposition 4.1. *Let (b, a) be a Pythagorean pair and let $\phi := b/a$, say $\phi(z) = \sum_{j=0}^\infty c_j z^j$. Then*

$$\|z^n\|_{\mathcal{H}(b)}^2 = 1 + \sum_{j=0}^n |c_j|^2 \quad (n \geq 0).$$

Clearly, if (b, a) is a Pythagorean pair, then $\phi := b/a \in N^+$, the Smirnov class. Conversely, given $\phi \in N^+$, then there exists a unique Pythagorean pair (b, a) such that $\phi = b/a$ (see [16, Proposition 3.1]). Thus we can identify $\mathcal{H}(b)$ by specifying the corresponding function $\phi \in N^+$, and this is what we shall do in the examples below.

Example 4.2. Let $\phi(z) := z/(\zeta - z)$, where $\zeta \in \mathbb{T}$.

The Taylor coefficients c_j of ϕ satisfy $c_0 = 0$ and $|c_j| = 1$ for $j \geq 1$, so by Proposition 4.1 we have $\|z^n\|_{\mathcal{H}(b)} = \sqrt{n+1}$. Corollary 2.2 implies that there exists $f \in \mathcal{H}(b)$ such that $\limsup_{n \to \infty} \|s_n(f)\|_{\mathcal{H}(b)} = \infty$. However, this result does not permit us to deduce the existence of an $f \in \mathcal{H}(b)$ with $\liminf_{n \to \infty} \|s_n(f)\|_{\mathcal{H}(b)} = \infty$ or $\limsup_{n \to \infty} \|\sigma_n(f)\|_{\mathcal{H}(b)} = \infty$. In fact, we

claim that no $f \in \mathcal{H}(b)$ satisfies either of these conclusions, indicating the Corollary 2.2 is rather sharp, at least for this function ϕ.

To justify the claim, we note that, for this particular ϕ, it is known that $b(z) = (1 - \tau)z/(\zeta - \tau z)$, where $\tau = (3 - \sqrt{5})/2$, and that $\mathcal{H}(b)$ is equal to the local Dirichlet space $\mathcal{D}_\zeta$, with equality of norms (see [15, Proposition 2]). It was proved in [11, Theorem 2.4] that

$$(5) \qquad \liminf_{n \to \infty} \|s_n(f)\|_{\mathcal{D}_\zeta} = \|f\|_{\mathcal{D}_\zeta} \quad (f \in \mathcal{D}_\zeta).$$

Also, it was shown in [10, Theorem 1.6] that $\|\sigma_n(f) - f\|_{\mathcal{D}_\zeta} \to 0$ for all $f \in \mathcal{D}_\zeta$, so in particular

$$(6) \qquad \limsup_{n \to \infty} \|\sigma_n(f)\|_{\mathcal{D}_\zeta} = \|f\|_{\mathcal{D}_\zeta} \quad (f \in \mathcal{D}_\zeta).$$

Since $\mathcal{D}_\zeta = \mathcal{H}(b)$, it follows that both (5) and (6) hold with $\mathcal{D}_\zeta$ replaced everywhere by $\mathcal{H}(b)$, confirming the claim made above.

Example 4.3. Let $\phi(z) = z^M/(1 - z)^N$, where M, N are integers with $M \geq 0$ and $N \geq 1$.

In this case $|c_j| \asymp j^{N-1}$, so by Proposition 4.1 we have $\|z^n\|_{\mathcal{H}(b)} \asymp n^{N-1/2}$ as $n \to \infty$. Applying Corollary 2.2, we deduce that, if $0 \leq \alpha < N - 1$, then there exists $f \in \mathcal{H}(b)$ such that $\|\sigma_n^\alpha(f)\|_{\mathcal{H}(b)} \to \infty$ as $n \to \infty$. In particular, if $N \geq 2$, then there exists $f \in \mathcal{H}(b)$ with $\|s_n(f)\|_{\mathcal{H}(b)} \to \infty$. Likewise, if $N \geq 3$, then there exists $f \in \mathcal{H}(b)$ with $\|\sigma_n(f)\|_{\mathcal{H}(b)} \to \infty$. This answers the question posed in the introduction.

Example 4.4. Let $\phi(z) = \exp(\beta/(1 - z)^\gamma)$, where $\beta > 0$ and $\gamma \in (0, 1)$.

In this case, it can be shown that

$$c_j \asymp \frac{\exp(Cj^{\gamma/(\gamma+1)})}{j^{(\gamma+2)/(2\gamma+2)}} \quad (j \to \infty),$$

where C is a positive constant depending on β, γ (see Appendix A below). By Proposition 4.1, it follows that

$$\|z^n\|_{\mathcal{H}(b)}^2 \gtrsim \frac{\exp(Cn^{\gamma/(\gamma+1)})}{n^{(\gamma+2)/(2\gamma+2)}} \quad (n \to \infty).$$

By Corollary 2.2, for each $\alpha \geq 0$, there exist functions $f, g \in \mathcal{H}(b)$ such that $|\langle \sigma_n^\alpha(f), g \rangle_{\mathcal{H}(b)}| \to \infty$. In particular, $\|\sigma_n^\alpha(f)\|_{\mathcal{H}(b)} \to \infty$.

This raises the following question: does there exist a function $f \in \mathcal{H}(b)$ such that $\|\sigma_n^\alpha(f)\|_{\mathcal{H}(b)} \to \infty$ for all $\alpha \geq 0$?

Appendix A. The Taylor coefficients of $\exp(\beta/(1 - z)^\gamma))$

Let $\phi(z) := \exp(\beta/(1 - z)^\gamma)$, where $\beta > 0$ and $\gamma \in (0, 1)$. This function belongs to the Smirnov class, since it is the exponential of a function in the Hardy space H^1. To estimate its Taylor coefficients, we use the following generalization of Stirling's formula due to Hayman [8, p.69].

Theorem A.1. *Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an admissible holomorphic function on $\mathbb{D}$ such that $f(r) > 0$ for all $r \in (0,1)$. For $r \in (0,1)$, define*

$$M(r) := \sup_{|z|=r} |f(z)|,$$

$$A(r) := r(\log M(r))',$$

$$B(r) := rA'(r).$$

Then, as $n \to \infty$,

$$(7) \qquad c_n \sim \frac{f(r_n)}{r_n^n \sqrt{2\pi B(r_n)}},$$

where $r_n \in (0,1)$ is the unique solution to the equation $A(r_n) = n$.

The definition of admissible function is complicated, and we do not give it here. However, the function $\phi(z)$ defined above is admissible (see [8, p.93]), so Theorem A.1 does indeed apply to it. Here is what we deduce.

Theorem A.2. *Let $\phi(z) = \exp(\beta/(1-z)^\gamma)$, where $\beta > 0$ and $\gamma \in (0,1)$. Then $\phi(z) = \sum_{n=0}^{\infty} c_n z^n$, where*

$$(8) \qquad c_n \sim \frac{\exp(Cn^{\gamma/(\gamma+1)})}{Dn^{(\gamma+2)/(2\gamma+2)}} \quad (n \to \infty),$$

and where

$$(9) \qquad C = (\beta\gamma)^{1/(\gamma+1)}(1 + 1/\gamma) \quad and \quad D = \sqrt{\frac{2\pi(\gamma+1)}{(\beta\gamma)^{1/(\gamma+1)}}}.$$

Proof. We apply Theorem A.1. Simple computations give

$$M(r) = \exp\left(\frac{\beta}{(1-r)^\gamma}\right), \quad A(r) = \frac{\beta\gamma r}{(1-r)^{\gamma+1}}, \quad B(r) = \frac{\beta\gamma r(1+\gamma r)}{(1-r)^{\gamma+2}}.$$

Also, the relation $A(r_n) = n$ is equivalent to $\beta\gamma r_n/n = (1-r_n)^{\gamma+1}$. Writing $s_n := 1 - r_n$, we obtain

$$s_n = \left(\frac{\beta\gamma}{n}\right)^{1/(\gamma+1)}(1 - s_n)^{1/(\gamma+1)}.$$

In particular $s_n = O(n^{-1/(\gamma+1)})$ as $n \to \infty$, and so

$$s_n = \left(\frac{\beta\gamma}{n}\right)^{1/(\gamma+1)}\left(1 + O(n^{-1/(\gamma+1)})\right).$$

A computation gives that

$$\log \phi(r_n) = \frac{\beta}{(1-r_n)^\gamma} = \frac{\beta}{s_n^\gamma} = \beta\left(\frac{n}{\beta\gamma}\right)^{\gamma/(\gamma+1)}\left(1 + O(n^{-1/(\gamma+1)})\right)$$

$$= \beta\left(\frac{n}{\beta\gamma}\right)^{\gamma/(\gamma+1)} + o(1),$$

so

$$(10) \qquad \phi(r_n) \sim \exp\left(\frac{\beta^{1/(\gamma+1)}}{\gamma^{\gamma/(\gamma+1)}}n^{\gamma/(\gamma+1)}\right).$$

Also,

$$\log(r_n^n) = n \log(1 - s_n) = -ns_n + O(ns_n^2)$$
$$= -n\Big(\frac{\beta\gamma}{n}\Big)^{1/(\gamma+1)}\Big(1 + O(n^{-1/(\gamma+1)})\Big) + O(n^{(\gamma-1)/(\gamma+1)})$$
$$= -(\beta\gamma)^{1/(\gamma+1)}n^{\gamma/(\gamma+1)} + o(1),$$

so

$$\text{(11)} \qquad r_n^n \sim \exp\Big(-(\beta\gamma)^{1/(\gamma+1)}n^{\gamma/(\gamma+1)}\Big).$$

Lastly, we have

$$\text{(12)} \qquad B(r_n) = \frac{\beta\gamma r_n(1 + \gamma r_n)}{(1 - r_n)^{\gamma+2}} \sim \frac{\beta\gamma(1 + \gamma)}{s_n^{\gamma+2}} \sim \frac{1 + \gamma}{(\beta\gamma)^{1/(\gamma+1)}}n^{(\gamma+2)/(\gamma+1)}.$$

Feeding (10), (11) and (12) into (7), we obtain (8) and (9). $\qquad\square$

References

1. K. M. Ball, *The complex plank problem*, Bull. London Math. Soc. **33** (2001), no. 4, 433–442. MR 1832555

2. N. Chevrot, D. Guillot, and T. Ransford, *De Branges-Rovnyak spaces and Dirichlet spaces*, J. Funct. Anal. **259** (2010), no. 9, 2366–2383. MR 2674117

3. L. de Branges and J. Rovnyak, *Canonical models in quantum scattering theory*, Perturbation Theory and its Applications in Quantum Mechanics (Proc. Adv. Sem. Math. Res. Center, U.S. Army, Theoret. Chem. Inst., Univ. of Wisconsin, Madison, Wis., 1965), Wiley, New York, 1966, pp. 295–392. MR 0244795

4. O. El-Fallah, E. Fricain, K. Kellay, J. Mashreghi, and T. Ransford, *Constructive approximation in de Branges-Rovnyak spaces*, Constr. Approx. **44** (2016), no. 2, 269–281. MR 3544000

5. E. Fricain and J. Mashreghi, *The theory of $\mathcal{H}(b)$ spaces. Vol. 1*, New Mathematical Monographs, vol. 20, Cambridge University Press, Cambridge, 2016. MR 3497010

6. ———, *The theory of $\mathcal{H}(b)$ spaces. Vol. 2*, New Mathematical Monographs, vol. 21, Cambridge University Press, Cambridge, 2016. MR 3617311

7. S. Ghara, J. Mashreghi, and T. Ransford, *Summability and duality*, Publ. Mat. (to appear).

8. W. K. Hayman, *A generalisation of Stirling's formula*, J. Reine Angew. Math. **196** (1956), 67–95. MR 80749

9. J. Mashreghi and T. Ransford, *Outer functions and divergence in de Branges–Rovnyak spaces*, Complex Anal. Oper. Theory **12** (2018), no. 4, 987–995. MR 3780508

10. ———, *Hadamard multipliers on weighted Dirichlet spaces*, Integral Equations Operator Theory **91** (2019), no. 6, Paper No. 52, 13. MR 4032210

11. J. Mashreghi, M. Shirazi, and M. Withanachchi, *Partial Taylor sums on local Dirichlet spaces*, Anal. Math. Phys. **12** (2022), no. 6, Paper No. 147, 26. MR 4507242

12. V. Müller and J. Vršovský, *Orbits of linear operators tending to infinity*, Rocky Mountain J. Math. **39** (2009), no. 1, 219–230. MR 2476809

13. D. Sarason, *Doubly shift-invariant spaces in H^2*, J. Operator Theory **16** (1986), no. 1, 75–97. MR 847333

14. ———, *Sub-Hardy Hilbert spaces in the unit disk*, University of Arkansas Lecture Notes in the Mathematical Sciences, vol. 10, John Wiley & Sons, Inc., New York, 1994, A Wiley-Interscience Publication. MR 1289670

15. ———, *Local Dirichlet spaces as de Branges-Rovnyak spaces*, Proc. Amer. Math. Soc. **125** (1997), no. 7, 2133–2139. MR 1396993

16. _____, *Unbounded Toeplitz operators*, Integral Equations Operator Theory **61** (2008), no. 2, 281–298. MR 2418122

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAWAII AT MANOA, HONOLULU, HI 96822, USA.
  Email address: `parisepo@hawaii.edu`

DÉPARTEMENT DE MATHÉMATIQUES ET DE STATISTIQUE, UNIVERSITÉ LAVAL, QUÉBEC CITY (QUÉBEC), G1V 0A6, CANADA.
  Email address: `ransford@mat.ulaval.ca`