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On the Domains of Bessel Operators

Jan Dereziński and Vladimir Georgescu

Abstract. We consider the Schrödinger operator on the halfline with the potential $(m^2-\frac{1}{4})\frac{1}{x^2}$, often called the Bessel operator. We assume that m is complex. We study the domains of various closed homogeneous realizations of the Bessel operator. In particular, we prove that the domain of its minimal realization for $|\mathrm{Re}(m)|<1$ and of its unique closed realization for $\mathrm{Re}(m)>1$ coincide with the minimal second-order Sobolev space. On the other hand, if $\mathrm{Re}(m)=1$ the minimal second-order Sobolev space is a subspace of infinite codimension of the domain of the unique closed Bessel operator. The properties of Bessel operators are compared with the properties of the corresponding bilinear forms.

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1. Introduction

1.1. Overview of Closed Realizations of the Bessel Operator

The Schrödinger operator on the half-line given by the expression

$$L_\alpha := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \Big(\alpha - \frac{1}{4}\Big)\frac{1}{x^2} \tag{1.1}$$

is often called *the Bessel operator*. The name is justified by the fact that its eigenfunctions and many other related objects can be expressed in terms of Bessel-type functions.

There exists a large literature devoted to self-adjoint realizations of (1.1) for real α. The theory of closed realizations of (1.1) for complex α is also interesting. Let us recall the basic elements of this theory, following [6,9].

For any complex α, there exist two most obvious realizations of L_α: the *minimal* $L_\alpha^{\min}$, and the *maximal* $L_\alpha^{\max}$. The complex plane is divided into two regions by the parabola defined by

$$\alpha = (1 + \mathrm{i}\omega)^2, \quad \omega \in \mathbb{R}, \tag{1.2}$$

(or, if we write $\alpha = \alpha_{\mathrm{R}} + \mathrm{i}\alpha_{\mathrm{I}}$, by $\alpha_{\mathrm{R}} + \sqrt{\alpha_{\mathrm{R}}^2 + \alpha_{\mathrm{I}}^2} = 2$). To the right of this parabola, that is, for $|\mathrm{Re}\sqrt{\alpha}| \geq 1$, we have $L_\alpha^{\min} = L_\alpha^{\max}$. For $|\mathrm{Re}\sqrt{\alpha}| < 1$, that is to the left of (1.2), $\mathcal{D}(L_\alpha^{\min})$ has codimension 2 inside $\mathcal{D}(L_\alpha^{\max})$. The operators $\mathcal{D}(L_\alpha^{\min})$ and $\mathcal{D}(L_\alpha^{\max})$ are homogeneous of degree -2.

Let us note that in the region $|\mathrm{Re}\sqrt{\alpha}| < 1$ the operators $L_\alpha^{\min}$ and $L_\alpha^{\max}$ are not the most important realizations of L_α. Much more useful are closed realizations of L_α situated between $L_\alpha^{\min}$ and $L_\alpha^{\max}$, defined by boundary conditions near zero. (Among these realizations, the best known are self-adjoint ones corresponding to real α and real boundary conditions.) All of this is described in [9].

Among these realizations for $\alpha \neq 0$ only two, and for $\alpha = 0$ only one, are homogeneous of degree -2. All of them are covered by the holomorphic family of closed operators H_m, introduced in [6] and defined for $\mathrm{Re}(m) > -1$ as the restriction of $L_{m^2}^{\max}$ to functions that behave as $x^{\frac{1}{2}+m}$ near zero. Note that

$$L_{m^2}^{\min} = H_m = L_{m^2}^{\max}, \quad \mathrm{Re}(m) \geq 1; \tag{1.3}$$

$$L_{m^2}^{\min} \subsetneq H_m \subsetneq L_{m^2}^{\max}, \quad |\mathrm{Re}(m)| < 1. \tag{1.4}$$

1.2. Main Results

Our new results give descriptions of the domains of various realizations of L_α for $\alpha \in \mathbb{C}$. First of all, we prove that for $|\mathrm{Re}\sqrt{\alpha}| < 1$ the domain of $L_\alpha^{\min}$ does not depend on α and coincides with the *minimal 2nd order Sobolev space*

$$\mathcal{H}_0^2(\mathbb{R}_+) := \{f \in \mathcal{H}^2(\mathbb{R}_+) \mid f(0) = f'(0) = 0\}, \tag{1.5}$$

where

$$\mathcal{H}^2(\mathbb{R}_+) := \{f \in L^2(\mathbb{R}_+) \mid f'' \in L^2(\mathbb{R}_+)\} \tag{1.6}$$

is the *(full) second-order Sobolev space*. We also show that

$$\{\alpha \mid |\mathrm{Re}\sqrt{\alpha}| < 1\} \ni \alpha \mapsto L_\alpha^{\min} \tag{1.7}$$

is a holomorphic family of closed operators.

We find the constancy of the domain of the minimal operator quite surprising and interesting. It contrasts with the fact that $\mathcal{D}(L_\alpha^{\max})$ for $|\mathrm{Re}\sqrt{\alpha}| < 1$ depends on α. Similarly, $\mathcal{D}(H_m)$ for $|\mathrm{Re}(m)| < 1$ depends on m.

The holomorphic family $L_\alpha^{\min}$ for $|\mathrm{Re}\sqrt{\alpha}| < 1$ consists of operators whose spectrum covers the whole complex plane. Therefore, the usual approach to holomorphic families of closed operators based on the study of the resolvent is not available.

We also study H_m for $\mathrm{Re}(m) \geq 1$ (which by (1.3) coincides with $L_{m^2}^{\min}$ and $L_{m^2}^{\max}$). We prove that for $\mathrm{Re}(m) > 1$ its domain also coincides with $\mathcal{H}_0^2(\mathbb{R}_+)$. The most unusual situation occurs in the case $\mathrm{Re}(m) = 1$. In this case, we show that the domain of H_m is always larger than $\mathcal{H}_0^2(\mathbb{R}_+)$ and depends on m.

Specifying to real α, the main result of our paper can be summarized as follows: Let $L_\alpha^{\min}$ be the closure in $L^2(\mathbb{R}_+)$ of the operator $-\partial_x^2 + \frac{\alpha - \frac{1}{4}}{x^2}$ with domain $C_{\mathrm{c}}^\infty(\mathbb{R}_+)$.

(1) If $\alpha < 1$ then $L_\alpha^{\min}$ is Hermitian (symmetric) but not self-adjoint and its domain is $\mathcal{H}_0^2(\mathbb{R}_+)$.
(2) If $\alpha = 1$ then $L_\alpha^{\min}$ is self-adjoint and $\mathcal{H}_0^2(\mathbb{R}_+)$ is a dense subspace of infinite codimension of its domain.
(3) If $\alpha > 1$ then $L_\alpha^{\min}$ is self-adjoint with domain $\mathcal{H}_0^2(\mathbb{R}_+)$.

As a side remark, let us mention two open problems about Bessel operators.

Open Problem 1.1.

1. *Can the holomorphic family H_m be extended beyond* $\mathrm{Re}(m) > -1$*? (Probably not).*
2. *Can the holomorphic family $L_\alpha^{\min}$ (hence also $L_\alpha^{\max}$) be extended beyond* $|\mathrm{Re}\sqrt{\alpha}| < 1$*? (Probably not).*

Question 1 has already been mentioned in [6]. We hope that both questions can be answered by methods of [10].

1.3. Bilinear Bessel Forms

With every operator T on a Hilbert space $\mathcal{H}$, one can associate the sesquilinear form

$$(f|Tg), \qquad f, g \in \mathcal{D}(T). \tag{1.8}$$

One can try to extend (1.8) to a larger domain. If T is self-adjoint, there is a natural extension to the so-called *form domain of* T, $\mathcal{Q}(T) := \mathcal{D}(\sqrt{|T|})$. Interpreting T as a bounded map from $\mathcal{Q}(T)$ to its anti-dual, we obtain the sesquilinear form

$$(f|Tg), \qquad f, g \in \mathcal{Q}(T), \tag{1.9}$$

which extends (1.8).

We would like to have a similar construction for Bessel operators, including non-self-adjoint ones. Before we proceed, we should realize that identities involving non-self-adjoint operators do not like complex conjugation. Therefore, instead of sesquilinear forms it is more natural to use bilinear forms.

Our analysis of bilinear Bessel forms is based on the pair of formal factorizations of the Bessel operator

$$-\partial_x^2 + \Big(m^2 - \frac{1}{4}\Big)\frac{1}{x^2} = \Big(\partial_x + \frac{\frac{1}{2}+m}{x}\Big)\Big(-\partial_x + \frac{\frac{1}{2}+m}{x}\Big) \tag{1.10}$$

$$= \Big(\partial_x + \frac{\frac{1}{2}-m}{x}\Big)\Big(-\partial_x + \frac{\frac{1}{2}-m}{x}\Big). \tag{1.11}$$

In Theorems 8.2 and 8.3, for each $\mathrm{Re}(m) > -1$ we interpret (1.10) and (1.11) as factorizations of the Bessel operator H_m into two closed 1st order operators. They define natural bilinear forms, which we call *Bessel forms*. For each $\mathrm{Re}(m) > -1$, the corresponding Bessel form is unique, except for $\mathrm{Re}(m) = 0$, $m \neq 0$, when the two factorizations yield two distinct Bessel forms.

Instead of $\mathcal{H}_0^2(\mathbb{R}_+)$, the major role is now played by the *minimal 1st order Sobolev space*

$$\mathcal{H}_0^1(\mathbb{R}_+) := \{f \in \mathcal{H}^1(\mathbb{R}_+) \mid f(0) = 0\}, \tag{1.12}$$

subspace of the *(full) 1st order Sobolev space*

$$\mathcal{H}^1(\mathbb{R}_+) := \{f \in L^2(\mathbb{R}_+) \mid f' \in L^2(\mathbb{R}_+)\}. \tag{1.13}$$

Note that $\mathcal{H}_0^1(\mathbb{R}_+)$ is the domain of Bessel forms for $\mathrm{Re}(m) > 0$.

The analysis of Bessel forms and their factorizations shows a variety of behaviors depending on the parameter m. In particular, there is a kind of a phase transition at $\mathrm{Re}(m) = 0$. Curiously, in the analysis of the domain of Bessel operators the phase transition occurs elsewhere: at $\mathrm{Re}(m) = 1$.

1.4. Comparison with Literature

The fact that $\mathcal{D}(L_\alpha^{\min})$ does not depend on α for real $\alpha \in [0, 1[$ was first proven in [1], see also [2,3]. Actually, the arguments of [1] are enough to extend the result to complex α such that $|\alpha - \frac{1}{4}| < \frac{3}{4}$. The proof is based on the bound $\|Q\| = \frac{3}{4}$ of the operator Q on $L^2(\mathbb{R}_+)$ given by the integral kernel

$$Q(x, y) = \frac{1}{x^2}(x - y)\theta(x - y), \tag{1.14}$$

where θ is the Heaviside function. Our proof is quite similar. Instead of (1.14), we consider for $|\mathrm{Re}(m)| < 1$ the operator Q_{m^2} with the kernel

$$Q_{m^2}(x, y) = \frac{1}{2mx^2}(x^{\frac{1}{2}+m}y^{\frac{1}{2}-m} - x^{\frac{1}{2}-m}y^{\frac{1}{2}+m})\theta(x - y). \tag{1.15}$$

Note that $Q_{\frac{1}{4}}$ coincides with (1.14). We prove that the norm of Q_{m^2} is the inverse of the distance of m^2 to the parabola (1.2). A simple generalization of the Kato–Rellich Theorem to closed operators implies our result about $\mathcal{D}(L_\alpha^{\min})$.

In the paper [6] on page 567 it is written "If $m \neq 1/2$, then $\mathcal{D}(L_m^{\min}) \neq \mathcal{H}_0^2$." (In that paper $L_{m^2}^{\min}$ was denoted $L_m^{\min}$). This sentence was not formulated as a proposition, and no proof was provided. Anyway, in view of the results of [3] and of this paper, this sentence was wrong.

The analysis of Bessel forms in the self-adjoint case, that is for real $m > -1$, is well known—it is essentially equivalent to the famous *Hardy inequality*. This subject is discussed, e.g., in the monograph [4] and in a recent interesting paper [11] about a refinement of the one-dimensional Hardy's inequality. The latter paper contains in particular many references about factorizations of Bessel operators in the self-adjoint case.

Results about Bessel forms and their factorizations for complex parameters are borrowed to a large extent from [6]. We include them in this paper, because they provide an interesting complement to the analysis of domains of Bessel operators.

2. Basic Closed Realizations of the Bessel Operator

The main topic of this preliminary section are closed homogeneous realizations of L_α. We recall their definitions following [6,9].

We will denote by $\mathbb{R}_+$ the open positive half-line, that is $]0,\infty[$. We will use $L^2(\mathbb{R}_+)$ as our basic Hilbert space. We define $L_\alpha^{\max}$ to be the operator given by the expression L_α with the domain

$$\mathcal{D}(L_\alpha^{\max}) = \{f \in L^2(\mathbb{R}_+) \mid L_\alpha f \in L^2(\mathbb{R}_+)\}.$$

We also set $L_\alpha^{\min}$ to be the closure of the restriction of $L_\alpha^{\max}$ to $C_\mathrm{c}^\infty(\mathbb{R}_+)$.

We will often write m for one of the square roots of α, that is, $\alpha = m^2$. It is easy to see that the space of solutions of the differential equation

$$L_\alpha f = 0 \tag{2.1}$$

is spanned for $\alpha \neq 0$ by $x^{\frac12+m}$, $x^{\frac12-m}$, and for $\alpha = 0$ by $x^{\frac12}$, $x^{\frac12}\log x$. Note that both solutions are square integrable near 0 iff $|\mathrm{Re}(m)| < 1$. This is used in [6] to show that we have

$$\mathcal{D}(L_\alpha^{\max}) = \mathcal{D}(L_\alpha^{\min}) + \mathbb{C}x^{\frac12+m}\xi + \mathbb{C}x^{\frac12-m}\xi, \quad |\mathrm{Re}\sqrt{\alpha}| < 1,\ \alpha \neq 0; \tag{2.2}$$

$$\mathcal{D}(L_0^{\max}) = \mathcal{D}(L_0^{\min}) + \mathbb{C}x^{\frac12}\xi + \mathbb{C}x^{\frac12}\log(x)\xi, \quad \alpha = 0; \tag{2.3}$$

$$\mathcal{D}(L_\alpha^{\max}) = \mathcal{D}(L_\alpha^{\min}), \quad |\mathrm{Re}\sqrt{\alpha}| \geq 1. \tag{2.4}$$

Above (and throughout the paper) ξ is any $C_\mathrm{c}^\infty[0,\infty[$ function such that $\xi = 1$ near 0.

Following [6], for $\mathrm{Re}(m) > -1$ we also introduce another family of closed realizations of Bessel operators: the operators H_m are defined as the restrictions of $L_{m^2}^{\max}$ to

$$\mathcal{D}(H_m) := \mathcal{D}(L_{m^2}^{\min}) + \mathbb{C}x^{\frac12+m}\xi. \tag{2.5}$$

We will use various basic concepts and facts about one-dimensional Schrö -dinger operators with complex potentials. We will use [8] as the main reference, but clearly most of them belong to the well-known folklore. In particular, we will use two kinds of Green's operators. Let us recall this concept, following [8]. Let $L_\mathrm{c}^1(\mathbb{R}_+)$ be the set of integrable functions of compact support in $\mathbb{R}_+$. We will say that an operator $G : L_\mathrm{c}^1(\mathbb{R}_+) \to AC^1(\mathbb{R}_+)$ is a *Green's operator of* L_α if for any $g \in L_\mathrm{c}^1(\mathbb{R}_+)$

$$L_\alpha G g = g. \tag{2.6}$$

3. The Forward Green's Operator

Let us introduce the operator $G_\alpha^\rightarrow$ defined by the kernel

$$G_\alpha^\rightarrow(x,y) := \frac{1}{2m}\big(x^{\frac12+m}y^{\frac12-m} - x^{\frac12-m}y^{\frac12+m}\big)\theta(x-y), \quad \alpha \neq 0; \tag{3.1}$$

$$G_0^\rightarrow(x,y) := x^{\frac12}y^{\frac12}\log\Big(\frac{x}{y}\Big)\theta(x-y), \quad \alpha = 0. \tag{3.2}$$

Note that $G_\alpha^\rightarrow$ is a Green's operator in the sense of (2.6). Besides,

$$\mathrm{supp} G_\alpha^\rightarrow g \subset \mathrm{supp} g + \mathbb{R}_+, \tag{3.3}$$

which is why it is sometimes called the *forward Green's operator*.

Unfortunately, the operator $G_\alpha^\rightarrow$ is unbounded on $L^2(\mathbb{R}_+)$. To make it bounded, for any $a > 0$ we can compress it to the finite interval $[0, a]$, by introducing the operator $G_\alpha^{a\rightarrow}$ with the kernel

$$G_\alpha^{a\rightarrow}(x, y) := \mathbb{1}_{[0,a]}(x) G_\alpha^\rightarrow(x, y) \mathbb{1}_{[0,a]}(y). \tag{3.4}$$

It is also convenient to consider the operator L_α restricted to $[0, a]$. One of its closed realizations is defined by the zero boundary condition at 0 and no boundary conditions at a (see [8] Def. 4.14). It will be denoted $L_{\alpha,0}^a$. By Prop. 7.3 of [8], we have $G_\alpha^{a\rightarrow} = (L_{\alpha,0}^a)^{-1}$, and hence,

$$\mathcal{D}(L_{\alpha,0}^a) = G_\alpha^{a\rightarrow} L^2[0, a]. \tag{3.5}$$

Now we can describe the domain of $L_\alpha^{\min}$ with the help of the forward Green's operator.

Proposition 3.1. *Suppose that $f \in \mathcal{D}(L_\alpha^{\max})$. Then the following statements are equivalent:*

1. *$f \in \mathcal{D}(L_\alpha^{\min})$.*
2. *For some $a > 0$ and $g^a \in L^2[0, a]$ we have $f\Big|_{[0,a]} = G_\alpha^\rightarrow g^a\Big|_{[0,a]}$.*
3. *For all $a > 0$ there exists $g^a \in L^2[0, a]$ such that $f\Big|_{[0,a]} = G_\alpha^\rightarrow g^a\Big|_{[0,a]}$.*

Proof. The boundary space ([8] Def. 5.2) of L_α is trivial at ∞ (see [8] Prop. 5.15). Therefore, for any $a > 0$ we have

$$f \in \mathcal{D}(L_\alpha^{\min}) \;\Leftrightarrow\; f\Big|_{[0,a]} \in \mathcal{D}(L_{\alpha,0}^a). \tag{3.6}$$

Hence it is enough to apply (3.5). □

Define the operator $Q_\alpha := \frac{1}{x^2} G_\alpha^\rightarrow$. Its integral kernel is

$$Q_\alpha(x, y) = \frac{1}{2m}\left(x^{-\frac{3}{2}+m} y^{\frac{1}{2}-m} - x^{-\frac{3}{2}-m} y^{\frac{1}{2}+m}\right)\theta(x - y), \quad \alpha \neq 0; \tag{3.7}$$

$$Q_0(x, y) := x^{-\frac{3}{2}} y^{\frac{1}{2}} \log\left(\frac{x}{y}\right)\theta(x - y), \quad \alpha = 0. \tag{3.8}$$

Proposition 3.2. *Assume that $|\mathrm{Re}\sqrt{\alpha}| < 1$. Then the operator Q_α is bounded on $L^2(\mathbb{R}_+)$, and*

$$\|Q_\alpha\| = \frac{1}{\mathrm{dist}\big(\alpha, (1 + \mathrm{i}\mathbb{R})^2\big)} \tag{3.9}$$

Proof. Introduce the unitary operator $U : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ given by

$$(Uf)(t) := \mathrm{e}^{\frac{t}{2}} f(\mathrm{e}^t). \tag{3.10}$$

Note that if an operator K has the kernel $K(x,y)$, then UKU^{-1}, has the kernel $\mathrm{e}^{\frac{t}{2}}K(\mathrm{e}^t,\mathrm{e}^s)\mathrm{e}^{\frac{s}{2}}$. Therefore, for any α the operator $UQ_\alpha U^{-1}$ has the kernel

$$\frac{1}{2m}(\mathrm{e}^{-(t-s)(1-m)}-\mathrm{e}^{-(t-s)(1+m)})\theta(t-s),\quad \alpha\neq 0; \tag{3.11}$$

$$\mathrm{e}^{-(t-s)}(t-s)\theta(t-s),\quad \alpha=0. \tag{3.12}$$

Thus, it is the convolution by the function

$$t\to\frac{1}{2m}(\mathrm{e}^{-t(1-m)}-\mathrm{e}^{-t(1+m)})\theta(t),\quad \alpha\neq 0; \tag{3.13}$$

$$t\to\mathrm{e}^{-t}t\theta(t),\quad \alpha=0. \tag{3.14}$$

Assume now that $|\mathrm{Re}\sqrt{\alpha}|<1$. Then the function (3.13) is integrable, and we can apply the Fourier transformation defined by $(\mathcal{F}u)(\omega)=(2\pi)^{-1/2}\int\mathrm{e}^{-i\omega t}u(t)\,\mathrm{d}t$. After this transformation, the operator $UQ_\alpha U^{-1}$ becomes the multiplication wrt the Fourier transform of (3.13) or (3.14), that is

$$\omega\mapsto\frac{1}{(1+\mathrm{i}\omega)^2-m^2}. \tag{3.15}$$

Thus the norm of $UQ_\alpha U^{-1}$, and hence also of Q_α, is the supremum of the absolute value of (3.15). $\square$

Remark 3.3. The operator Q_α belongs to the class of operators analyzed in [17] on p. 271, which goes back to Hardy–Littlewood–Polya [13] p. 229.

Proposition 3.2 for $\alpha=\frac{1}{4}$ is especially important and simple. This case was noted in cf. [6, p. 566] and [1, Lemma 2.2]. It can be written as

$$g(x):=x^{-2}\int_0^x(x-y)f(y)\mathrm{d}y\ \Rightarrow\ \|g\|\leq\frac{4}{3}\|f\|. \tag{3.16}$$

One can remark that (3.16) is essentially equivalent to the one-dimensional version of the classical *Rellich's inequality*, see e.g., [4, (6.1.1)]:

$$\int_0^\infty\frac{|u|^2}{x^4}\mathrm{d}x\leq\frac{16}{9}\int_0^\infty|u''|^2\mathrm{d}x,\quad u\in C_\mathrm{c}^\infty(\mathbb{R}_+), \tag{3.17}$$

where we identify $f=u''$ and $g=\frac{u}{x^2}$.

The proof of the following proposition uses only the simple estimate (3.16).

Proposition 3.4. *$\mathcal{D}(L_\alpha^{\max})\cap\mathcal{D}(L_\beta^{\max})=\mathcal{H}_0^2(\mathbb{R}_+)$ if $\alpha\neq\beta$.*

Proof. We have $f\in\mathcal{D}(L_\alpha^{\max})$ if and only if $f\in L^2(\mathbb{R}_+)$ and $-f''+(\alpha-1/4)x^{-2}f\in L^2(\mathbb{R}_+)$; hence, if we also have $f\in\mathcal{D}(L_\beta^{\max})$ then $(\alpha-\beta)x^{-2}f\in L^2(\mathbb{R}_+)$ and since $\alpha\neq\beta$ we get $x^{-2}f\in L^2(\mathbb{R}_+)$ hence $f''\in L^2(\mathbb{R}_+)$. Recall that $f,f''\in L^2(\mathbb{R}_+)$ implies $f\in\mathcal{H}^1(\mathbb{R}_+)$ and $\|f'\|^2_{L^2(\mathbb{R}_+)}\leq\|f\|^2_{L^2(\mathbb{R}_+)}\|f''\|^2_{L^2(\mathbb{R}_+)}$. It follows that f is absolutely continuous and $f(x)=a+\int_0^x f'(y)\mathrm{d}y$ for some constant a and f' is absolutely continuous and $f'(x) = b + \int_0^x f''(y)\mathrm{d}y$ for some constant b, then

$$f(x) = a + bx + \int_0^x \int_0^y f''(z)\mathrm{d}z\mathrm{d}y = a + bx + x^2 g(x),$$
$$g(x) := x^{-2}\int_0^x (x-y)f''(y)\mathrm{d}y.$$

Then, by (3.16)

$$\|g\|_{L^2(\mathbb{R}_+)} \leq \frac{4}{3}\|f''\|_{L^2(\mathbb{R}_+)}. \tag{3.18}$$

Thus $x^{-2}f(x) = ax^{-2} + bx^{-1} + g(x)$ where $g \in L^2(\mathbb{R}_+)$, so $\int_0^1 |x^{-2}f(x)|^2\mathrm{d}x < \infty$ if and only if $a = b = 0$, so that $f(x) = \int_0^x (x-y)f''(y)\mathrm{d}y$ and $f'(x) = \int_0^x f''(y)\mathrm{d}y$, hence $f \in \mathcal{H}_0^2(\mathbb{R}_+)$.

Reciprocally, if $f \in \mathcal{H}_0^2(\mathbb{R}_+)$ then $x^{-2}f \in L^2(\mathbb{R}_+)$ with $\|x^{-2}f\|_{L^2(\mathbb{R}_+)} \leq \frac{4}{3}\|f''\|_{L^2(\mathbb{R}_+)}$ by (3.16), hence $f \in \mathcal{D}(L_\alpha^{\max})$ for all α. □

4. Domain of Bessel Operators for $|\mathrm{Re}(m)| < 1$

Below we state the first main result of our paper (which is an extension of a result of [1]).

Theorem 4.1. *If $|\mathrm{Re}\sqrt{\alpha}| < 1$, then $\mathcal{D}(L_\alpha^{\min}) = \mathcal{H}_0^2(\mathbb{R}_+)$. Moreover,*

$$\{\alpha \in \mathbb{C} \mid |\mathrm{Re}\sqrt{\alpha}| < 1\} \ni \alpha \mapsto L_\alpha^{\min} \tag{4.1}$$

is a holomorphic family of closed operators.

The proof of this theorem is based on the following lemma.

Lemma 4.2. *Let $|\mathrm{Re}\sqrt{\alpha}| < 1$ and $f \in \mathcal{D}(L_\alpha^{\min})$. Then*

$$\|x^{-2}f\| \leq \frac{1}{\mathrm{dist}\big(\alpha, (1+\mathrm{i}\mathbb{R})^2\big)}\|L_\alpha^{\min}f\|. \tag{4.2}$$

Proof. Let $a > 0$. Set $g := L_\alpha^{\min}f$, $f^a := f\Big|_{[0,a]}$, $g^a := g\Big|_{[0,a]}$. Let $G_\alpha^{a\rightarrow}$ be as in (3.4). As in the proof of Prop. 3.1,

$$f^a = G_\alpha^{a\rightarrow}g^a. \tag{4.3}$$

So

$$\|x^{-2}f\| = \lim_{a\to\infty}\|x^{-2}f^a\| \tag{4.4}$$

$$= \lim_{a\to\infty}\|x^{-2}G_\alpha^{a\rightarrow}g^a\| = \|Q_\alpha g\| \leq \frac{1}{\mathrm{dist}\big(\alpha, (1+\mathrm{i}\mathbb{R})^2\big)}\|g\|. \tag{4.5}$$

□

Proof of Theorem 4.1. We can cover the region on the lhs of (4.1) by disks touching the boundary of this region, that is, (1.2). Inside each disk, we apply Thm A.1 and Lemma 4.2. We obtain in particular, that if $|\mathrm{Re}\sqrt{\alpha_i}| < 1$, $i = 1, 2$, then $\mathcal{D}(L_{\alpha_1}^{\min}) = \mathcal{D}(L_{\alpha_2}^{\min})$. But clearly $\mathcal{D}(L_{\frac{1}{4}}^{\min}) = \mathcal{H}_0^2(\mathbb{R}_+)$. □

Theorem 4.3. *We have*

$$\mathcal{D}(L_\alpha^{\max}) = \mathcal{H}_0^2 + \mathbb{C}x^{\frac{1}{2}+m}\xi + \mathbb{C}x^{\frac{1}{2}-m}\xi, \quad |\mathrm{Re}\sqrt{\alpha}| < 1,\ \alpha \neq 0; \tag{4.6}$$

$$\mathcal{D}(L_\alpha^{\max}) = \mathcal{H}_0^2 + \mathbb{C}x^{\frac{1}{2}}\xi + \mathbb{C}x^{\frac{1}{2}}\log(x)\xi, \quad \alpha = 0. \tag{4.7}$$

Besides,

$$\mathcal{D}(L_{\alpha_1}^{\max}) \neq \mathcal{D}(L_{\alpha_2}^{\max}), \qquad \alpha_1 \neq \alpha_2, \quad |\mathrm{Re}\sqrt{\alpha}_i| < 1, \quad i = 1, 2. \tag{4.8}$$

Furthermore,

$$\left\{\alpha \in \mathbb{C} \mid |\mathrm{Re}\sqrt{\alpha}| < 1\right\} \ni \alpha \mapsto L_\alpha^{\max} \tag{4.9}$$

is a holomorphic family of closed operators.

Proof. Using $\mathcal{D}(L_\alpha^{\min}) = \mathcal{H}_0^2$, (2.2) and (2.3) can be now rewritten as (4.6) and (4.7).

Clearly, $x^{\frac{1}{2}+m}\xi$ and $x^{\frac{1}{2}}\log(x)\xi$ do not belong to $\mathcal{H}_0^2(\mathbb{R}_+)$ (because their second derivatives are not square integrable). Therefore, $\mathcal{D}(L_\alpha^{\max}) \neq \mathcal{H}_0^2(\mathbb{R}_+)$. This together with Proposition 3.4 implies (4.8).

We have $(L_\alpha^{\min})^* = L_{\overline{\alpha}}^{\max}$. Therefore, to obtain the holomorphy we can use Proposition A.2. □

The most important holomorphic family of Bessel operators is

$$\{m \in \mathbb{C} \mid \mathrm{Re}(m) > -1\} \ni m \mapsto H_m. \tag{4.10}$$

Its holomorphy has been proven in [6]. Using arguments similar to those in the proof of Theorem 4.3, we obtain a closer description of this family in the region $|\mathrm{Re}(m)| < 1$.

Theorem 4.4. *We have*

$$\mathcal{D}(H_m) = \mathcal{H}_0^2 + \mathbb{C}x^{\frac{1}{2}+m}\xi, \quad |\mathrm{Re}(m)| < 1. \tag{4.11}$$

Besides, if $m_1 \neq m_2$ and $|\mathrm{Re}(m_i)| < 1$, $i = 1, 2$, then $\mathcal{D}(H_{m_1}) \neq \mathcal{D}(H_{m_2})$.

5. Two-Sided Green's Operator

For any $m \in \mathbb{C}$, $m \neq 0$, let us introduce the operator G_m with the kernel

$$G_m(x,y) := \frac{1}{2m}\left(x^{\frac{1}{2}+m}y^{\frac{1}{2}-m}\theta(y-x) + x^{\frac{1}{2}-m}y^{\frac{1}{2}+m}\theta(x-y)\right). \tag{5.1}$$

Recall that θ is the Heaviside function. (5.1) is one of Green's operators of L_{m^2} in the sense of (2.6), Following [8], we will call it the *two-sided Green's operator.*

The operator G_m is not bounded on $L^2(\mathbb{R}_+)$ for any $m \in \mathbb{C}$. However, at least for $\mathrm{Re}(m) > -1$, it is useful in the L^2 setting.

Proposition 5.1. *Let* $\mathrm{Re}(m) > -1$, $m \neq 0$ *and* $a > 0$.

1. *If $g \in L^2[0,a]$, then*

$$f(x) = G_m g(x) = \int_0^\infty G_m(x,y)g(y)\mathrm{d}y \tag{5.2}$$

is well defined, belongs to $\in AC^1]0,\infty[$ and $L_\alpha f = g$.

2. *Conversely, if* $f \in AC^1]0,\infty[$, $L_\alpha f = g \in L^2[0,a]$, *then there exist* c_+, c_- *such that*

$$f(x) = c_+ x^{\frac12+m} + c_- x^{\frac12-m} + G_m g(x), \quad m \neq 0. \tag{5.3}$$

Proof. Note first that $\mathrm{Re}(m) > -1$ implies $x^{\frac12+m}$ is locally in L^2. Using this, the proof of the first part of the proposition is a straightforward computation done, in a more general setting, in [8], see §2.7 and Definition 2.10 there. For the second part, note that $L_\alpha(f - G_m g) = 0$ by the first part of the proposition, and that the two functions $x^{\frac12\pm m}$ give a basis of the nullspace of L_α. □

Let us introduce the operator $Z_m := \frac{1}{x^2} G_m$ with the kernel

$$Z_m(x,y) = \frac{1}{2m}\Big(x^{-\frac32+m} y^{\frac12-m}\theta(y-x) + x^{-\frac32-m} y^{\frac12+m}\theta(x-y)\Big). \tag{5.4}$$

Proposition 5.2. *Let* $\mathrm{Re}(m) > 1$. *Then* Z_m *is bounded and*

$$\|Z_m\| = \frac{1}{\mathrm{dist}\big(m^2, (1+\mathrm{i}\mathbb{R})^2\big)} \tag{5.5}$$

Proof. If U is given by (3.10), then UZ_mU^{-1} has the kernel

$$\frac{1}{2m}\Big(\mathrm{e}^{-(m-1)(s-t)}\theta(s-t) + \mathrm{e}^{-(m+1)(s-t)}\theta(t-s)\Big). \tag{5.6}$$

If $\mathrm{Re}(m) > 1$, after the Fourier transformation (defined as in the proof of Proposition 3.2) it becomes the multiplication by the function

$$\omega \mapsto \frac{1}{2m}\left(\frac{1}{(m-1-\mathrm{i}\omega)} + \frac{1}{1+m+\mathrm{i}\omega)}\right) = \frac{1}{m^2-(1+\mathrm{i}\omega)^2}, \tag{5.7}$$

whose supremum is the right-hand side of (5.5). □

6. Domain of Bessel Operators for $\mathrm{Re}(m) > 1$

For $\mathrm{Re}(m) \geq 1$, there is a unique closed Bessel operator. We will see in the following theorem that its domain is again the minimal 2nd order Sobolev space, except at the boundary $\mathrm{Re}(m) = 1$, cf. Sect. 7.

Theorem 6.1. *Let* $\mathrm{Re}(m) > 1$. *Then* $\mathcal{D}(H_m) = \mathcal{H}_0^2(\mathbb{R}_+)$.

Proof. We know that $\mathcal{H}_0^2(\mathbb{R}_+) \subset \mathcal{D}(L_{m^2}^{\max})$ for any m. But for $\mathrm{Re}(m) > 1$ we have $L_{m^2}^{\max} = H_m$. This proves the inclusion $\mathcal{H}_0^2(\mathbb{R}_+) \subset \mathcal{D}(H_m)$.

Let us prove the converse inclusion. Let $f \in \mathcal{D}(H_m)$. It is enough to assume that $f \in L^2[0,1]$. Let $g := H_m f$. Then $g \in L^2[0,1]$. By Prop. 5.1, we can write

$$f(x) = c_+ x^{\frac12+m} + c_- x^{\frac12-m} + \frac{x^{\frac12+m}}{2m}\int_x^1 y^{\frac12-m} g(y)\mathrm{d}y + \frac{x^{\frac12-m}}{2m}\int_0^x y^{\frac12+m} g(y)\mathrm{d}y. \tag{6.1}$$

For $x > 1$ we have

$$f(x) = c_+ x^{\frac12+m} + x^{\frac12-m}\left(c_- + \frac{1}{2m}\int_0^1 y^{\frac12+m} g(y)\mathrm{d}y\right), \tag{6.2}$$

hence $c_+ = 0$. We have, for $x \to 0$,

$$\left| x^{\frac{1}{2}+m} \int_x^1 y^{\frac{1}{2}-m} g(y)\mathrm{d}y \right| \le x \int_0^1 |g(x)|\mathrm{d}y \to 0; \tag{6.3}$$

$$\left| x^{\frac{1}{2}-m} \int_0^x y^{\frac{1}{2}+m} g(y) \right| \mathrm{d}y \le x \int_0^x |g(y)|\mathrm{d}y \to 0. \tag{6.4}$$

$x^{\frac{1}{2}-m}$ is not square integrable near zero. Hence $c_- = 0$. Thus

$$f(x) = \frac{x^{\frac{1}{2}+m}}{2m} \int_x^1 y^{\frac{1}{2}-m} g(y)\mathrm{d}y + \frac{x^{\frac{1}{2}-m}}{2m} \int_0^x y^{\frac{1}{2}+m} g(y)\mathrm{d}y. \tag{6.5}$$

By (6.3) and (6.4), $\lim_{x\to 0} f(x) = 0$. Now

$$f'(x) = \frac{(\frac{1}{2}+m)x^{-\frac{1}{2}+m}}{2m} \int_x^1 y^{\frac{1}{2}-m} g(y)\mathrm{d}y + \frac{(\frac{1}{2}-m)x^{-\frac{1}{2}-m}}{2m} \int_0^x y^{\frac{1}{2}+m} g(y)\mathrm{d}y, \tag{6.6}$$

$$\left| x^{-\frac{1}{2}-m} \int_0^x y^{\frac{1}{2}+m} g(y)\mathrm{d}y \right| \le \int_0^x |g(y)|\mathrm{d}y \to 0, \tag{6.7}$$

$$\left| x^{-\frac{1}{2}+m} \int_x^1 y^{\frac{1}{2}-m} g(y)\mathrm{d}y \right| \le \int_0^\epsilon |g(y)|\mathrm{d}y + x^{-\frac{1}{2}+\mathrm{Re}(m)} \int_\epsilon^1 y^{\frac{1}{2}-\mathrm{Re}(m)} |g(y)|\mathrm{d}y. \tag{6.8}$$

For any $\epsilon > 0$, the second term on the right of (6.8) goes to zero. The first, by making ϵ small, can be made arbitrarily small. Therefore (6.8) goes to zero. Thus $\lim_{x\to 0} f'(0) = 0$.

Finally

$$\begin{aligned} f''(x) + g(x) &= \frac{(m^2 - \frac{1}{4})x^{-\frac{3}{2}+m}}{2m} \int_x^1 y^{\frac{1}{2}-m} g(y)\mathrm{d}y \\ &\quad + \frac{(m^2 - \frac{1}{4})x^{-\frac{3}{2}-m}}{2m} \int_0^x y^{\frac{1}{2}+m} g(y)\mathrm{d}y \qquad (6.9) \\ &= \left(m^2 - \frac{1}{4}\right) Z_m g(x). \qquad (6.10) \end{aligned}$$

By Proposition 5.2 Z_m is bounded. Hence $f'' \in L^2(\mathbb{R}_+)$. Therefore, $f \in \mathcal{H}_0^2(\mathbb{R}_+)$. □

7. Domain of Bessel Operators for $\mathrm{Re}(m) = 1$

In this section, we treat the most complicated situation concerning the domain of H_m, namely the case $\mathrm{Re}(m) = 1$. By (1.3), we then have $H_m = L_{m^2}^{\min} = L_{m^2}^{\max}$. We prove the following theorem.

Theorem 7.1. *Let* $\mathrm{Re}(m) = 1$.

1. $\mathcal{H}_0^2(\mathbb{R}_+)$ *is a dense subspace of* $\mathcal{D}(H_m)$ *of infinite codimension.*
2. *If* ξ *is a* $C_c^2[0, \infty[$ *function equal* 1 *near zero, then* $x^{\frac{1}{2}+m}\xi \in \mathcal{D}(H_m)$ *but* $x^{\frac{1}{2}+m}\xi \notin \mathcal{H}_0^2(\mathbb{R}_+)$.

3. If $\mathrm{Re}(m') = 1$ *and* $m \neq m'$*, then* $\mathcal{D}(H_m) \cap \mathcal{D}(H_{m'}) = \mathcal{H}_0^2(\mathbb{R}_+)$.

By (1.3), it is clear that $\mathcal{H}_0^2(\mathbb{R}_+) \subset \mathcal{D}(H_m)$ and $x^{\frac{1}{2}+m}\xi \in \mathcal{D}(H_m)$. The density of $\mathcal{H}_0^2(\mathbb{R}_+)$ in $\mathcal{D}(H_m)$ is a consequence of $H_m = L_{m^2}^{\min}$. The last assertion of the theorem is a special case of Proposition 3.4. In the rest of this section, we construct an infinite dimensional vector subspace $\mathcal{V}$ of $\mathcal{D}(H_m)$ such that $\mathcal{V} \cap \big(\mathcal{H}_0^2(\mathbb{R}_+) + \mathbb{C}x^{\frac{1}{2}+m}\xi\big) = \{0\}$, which will finish the proof of the theorem.

Let us study the behavior at zero of the functions in $\mathcal{D}(H_m)$. For functions in the subspace $\mathcal{H}_0^2(\mathbb{R}_+) + \mathbb{C}x^{\frac{1}{2}+m}\xi$ this is easy, cf. the next lemma, but this is not so trivial for the other functions.

Lemma 7.2. *If* $f = f_0 + cx^{\frac{1}{2}+m}\xi \in \mathcal{H}_0^2(\mathbb{R}_+) + \mathbb{C}x^{\frac{1}{2}+m}\xi$ *then*

$$c = \lim_{x\to 0} x^{-\frac{1}{2}-m} f(x). \tag{7.1}$$

Proof. If $f_0 \in \mathcal{H}_0^2(\mathbb{R}_+)$ then $f_0(x) = \int\limits_0^x (x-y) f_0''(y)\mathrm{d}y$. Therefore, $\sqrt{3}|f_0(x)| \leq x^{\frac{3}{2}}\|f_0''\|_{L^2[0,x]}$ and since $\mathrm{Re}(m+\frac{1}{2}) = \frac{3}{2}$ we get $\lim_{x\to 0} x^{-m-\frac{1}{2}} f_0(x) = 0$, which implies (7.1). □

Let $a > 0$. Let G_m^a be the operator G_m compressed to the interval $[0, a]$. Its kernel is

$$G_m^a(x, y) = 1\!\!1_{[0,a]}(x) G_m(x, y) 1\!\!1_{[0,a]}(y). \tag{7.2}$$

We will write $L_\alpha^{a,\max}$ for the maximal realization of operator L_α on $L^2[0, a]$.

Lemma 7.3. *Let* $\mathrm{Re}(m) > -1$*. Then* G_m^a *is a bounded operator on* $L^2[0, a]$*. If* $g \in L^2[0, a]$*, then* $G_m^a g \in \mathcal{D}(L_{m^2}^{a,\max})$ *and* $L_{m^2}^{a,\max} G_m^a g = g$*. Consequently,* G_m^a *is injective.*

Proof. We check that (7.2) belongs to $L^2\big([0, a] \times [0, a]\big)$. This proves that G_m^a is Hilbert Schmidt, hence bounded. G_m^a is a right inverse of $L_{m^2}^{a,\max}$, because G_m is a right inverse of L_{m^2} (see Proposition 5.1). □

Lemma 7.4. *Let* $\mathrm{Re}(m) = 1$*. Let* $g \in L^2[0, a]$ *and* $f = G_m^a g$*. Then*

$$\lim_{x\to 0}\left(2mx^{-\frac{1}{2}-m} f(x) - \int_x^a y^{\frac{1}{2}-m} g(y)\mathrm{d}y\right) = 0. \tag{7.3}$$

Therefore, if

$$\lim_{x\to 0}\int_x^a y^{\frac{1}{2}-m} g(y)\mathrm{d}y \tag{7.4}$$

does not exist, then $f = G_m^a g \notin \mathcal{H}_0^2(\mathbb{R}_+) + \mathbb{C}x^{\frac{1}{2}+m}\xi$.

Proof. We have

$$2mx^{-\frac{1}{2}-m} f(x) = \int_x^a y^{-\frac{1}{2}-m} g(y)\mathrm{d}y + x^{-2m}\int_0^x y^{\frac{1}{2}+m} g(y)\mathrm{d}y. \tag{7.5}$$

Since $\mathrm{Re}(m) = 1$, the absolute value of the second term on the right hand side is less than

$$x^{-\frac{1}{2}}\int_0^x (y/x)^{\frac{3}{2}} |g(y)|\mathrm{d}y \leq x^{-\frac{1}{2}}\int_0^x |g(y)|\mathrm{d}y \leq \|g\|_{L^2[0,x]}$$

This proves (7.3). If $f = G_m^a g \in \mathcal{H}_0^2(\mathbb{R}_+) + \mathbb{C}x^{\frac{1}{2}+m}\xi$, then by (7.3) and (7.1) there exists (7.4). This proves the second statement of the lemma. $\square$

Lemma 7.5. *Let* $\mathrm{Re}(m) = 1$. *There exists an infinite dimensional subspace* $\mathcal{V} \subset \mathcal{D}(H_m)$ *such that*

$$\mathcal{V} \cap \big(\mathcal{H}_0^2(\mathbb{R}_+) + \mathbb{C}x^{\frac{1}{2}+m}\xi\big) = \{0\}. \tag{7.6}$$

Proof. For each $\tau \in]\frac{1}{2}, 1[$ let $g_\tau \in C^2(]0,1])$, for $0 < x < \frac{1}{2}$ given by

$$g_\tau(x) = x^{-\frac{3}{2}+m}\big(\ln(1/x)\big)^{-\tau}$$

and arbitrary on $[\frac{1}{2}, 1]$. Then for $x < \frac{1}{2}$, we have

$$|g_\tau(x)|^2 = x^{-1}\big(\ln(1/x)\big)^{-2\tau} = (2\tau-1)^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\big(\ln(1/x)\big)^{1-2\tau}.$$

Hence

$$\int_0^{\frac{1}{2}} |g_\tau(x)|^2 \mathrm{d}x = (2\tau-1)^{-1}(\ln 2)^{1-2\tau},$$

and $g_\tau \in L^2[0,1]$. Moreover, if $x < \frac{1}{2}$ then

$$x^{\frac{1}{2}-m} g_\tau(x) = x^{-1}\big(\ln(1/x)\big)^{-\tau} = (\tau-1)^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\big(\ln(1/x)\big)^{1-\tau}.$$

Hence

$$\int_x^{\frac{1}{2}} y^{-\frac{1}{2}} g_\tau(y)\mathrm{d}y = (\tau-1)^{-1}(\ln 2)^{1-\tau} + (1-\tau)^{-1}\big(\ln(1/x)\big)^{1-\tau} \to \infty \quad \text{as } x \to 0.$$

Let $\mathcal{G}$ be the vector subspace of $L^2[0,1]$ generated by the functions g_τ with $\frac{1}{2} < \tau < 1$. Note that each finite set $\{g_\tau \mid \tau \in A\}$ with $A \subset]\frac{1}{2}, 1[$ finite is linearly independent. Indeed, if $\sum_{\tau \in A} c_\tau g_\tau = 0$ and $\sigma = \min A$ and $\tau \neq \sigma$ then $\frac{g_\tau(x)}{g_\sigma(x)} = \big(\ln(1/x)\big)^{\sigma-\tau} \to 0$ as $x \to 0$ so we get $c_\sigma = 0$, etc. Moreover, for each not zero $g = \sum_{\tau \in A} c_\tau g_\tau \in \mathcal{G}$ (with $c_\tau \neq 0$) we have $\lim_{x\to 0}\left|\int_x^1 y^{-\frac{1}{2}} g(y)\mathrm{d}y\right| = \infty$. Indeed, we may assume $c_\sigma = 1$, and then,

$$\int_x^{\frac{1}{2}} y^{-\frac{1}{2}} g(y)\mathrm{d}y = (1-\sigma)^{-1}\big(\ln(1/x)\big)^{1-\sigma} + \sum_{\tau \in A} c_\tau(\tau-1)^{-1}(\ln 2)^{1-\tau} + \sum_{\tau \neq \sigma} c_\tau(1-\tau)^{-1}\big(\ln(1/x)\big)^{1-\tau},$$

and the first term on the right-hand side tends to $+\infty$ more rapidly than all the other, hence

$$\left|\int_x^{\frac{1}{2}} y^{-\frac{1}{2}} g(y)\mathrm{d}y\right| \geq \frac{1}{2(1-\sigma)}\big(\ln(1/x)\big)^{1-\sigma}$$

if x is small enough.

Finally, let $\varphi \in C_c^\infty[0,\infty[$ equal 1 on $[0,1]$. Let us define $\mathcal{V}$ as the space of functions on $\mathbb{R}_+$ of the form $f = \varphi G_m g$ with $g \in \mathcal{G}$. By Lemma 7.3, $G_m \mathbb{1}_{[0,1]}(x)$ is injective. Hence $\mathcal{V}$ is infinite dimensional and it satisfies (7.6) by Lemma 7.4. $\square$

8. Bilinear Forms Associated with Bessel Operators

As noted in the introduction, in this section we will avoid complex conjugation. Thus in the place of the usual sesquilinear scalar product

$$(f|g) := \int_0^\infty \overline{f(x)} g(x) \mathrm{d}x, \tag{8.1}$$

we will prefer to use the bilinear product

$$\langle f|g\rangle := \int_0^\infty f(x) g(x) \mathrm{d}x, \tag{8.2}$$

Clearly, (8.2) is well defined for $f, g \in L^2(\mathbb{R}_+)$. Instead of the usual adjoint T^*, we will use the transpose $T^\#$, defined with respect to (8.2), see [8].

An important role will be played by the first-order operators given by the formal expression

$$A_\rho := \partial_x - \frac{\rho}{x}. \tag{8.3}$$

A detailed analysis of (8.3) has been done in [6], where the notation was slightly different: $A_\rho := -\mathrm{i}(\partial_x - \frac{\rho}{x})$. Let us recall the main points of that analysis.

In the usual way, we define two closed realizations of A_ρ: the minimal and the maximal one, denoted $A_\rho^{\min}$, resp. $A_\rho^{\max}$. The following theorem was (mostly) proven in Section 3 of [6]. For the proof of the infinite codimensionality assertion in 6 see the proof of Lemma 3.9 there (where γ is arbitrary $> \frac{1}{2}$).

Theorem 8.1. *1. $A_\rho^{\min} \subset A_\rho^{\max}$.*

2. $A_\rho^{\min\,\#} = -A_{-\rho}^{\max}$, $\quad A_\rho^{\max\,\#} = -A_{-\rho}^{\min}$.

3. $A_\rho^{\min}$ and $A_\rho^{\max}$ are homogeneous of degree -1.

4. $A_\rho^{\min} = A_\rho^{\max}$ iff $|\mathrm{Re}(\rho)| \geq \frac{1}{2}$. If this is the case, we will often denote them simply by A_ρ

5. If $\mathrm{Re}(\rho) \neq \frac{1}{2}$, then $\mathcal{D}(A_\rho^{\min}) = \mathcal{H}_0^1$.

6. If $\mathrm{Re}(\rho) = \frac{1}{2}$, then $\mathcal{H}_0^1 + \mathbb{C}x^\rho\xi$ is a dense subspace of $\mathcal{D}(A_\rho)$ of infinite codimension.

7. If $|\mathrm{Re}(\rho)| < \frac{1}{2}$, then $\mathcal{D}(A_\rho^{\max}) = \mathcal{H}_0^1 + \mathbb{C}x^\rho\xi \neq \mathcal{H}_0^1$.

8. If $\mathrm{Re}(\rho), \mathrm{Re}(\rho') \in]-\frac{1}{2}, \frac{1}{2}]$ and $\rho \neq \rho'$ then $\mathcal{D}(A_\rho^{\max}) \neq \mathcal{D}(A_{\rho'}^{\max})$.

Now let us describe possible factorizations of H_m into operators of the form $A_\rho^{\min}$ and $A_\rho^{\max}$. On the formal level, they correspond to one of the factorizations (1.10) and (1.11).

Theorem 8.2. *1. For $\mathrm{Re}(m) > -1$ we have*

$$\langle f|H_m g\rangle = \langle A_{\frac{1}{2}+m}^{\max} f | A_{\frac{1}{2}+m}^{\max} g\rangle, \quad f \in \mathcal{D}(A_{\frac{1}{2}+m}^{\max}), \quad g \in \mathcal{D}(A_{\frac{1}{2}+m}^{\max}) \cap \mathcal{D}(H_m). \tag{8.4}$$

Moreover,

$$\mathcal{D}(H_m) = \left\{ f \in \mathcal{D}(A^{\max}_{\frac12+m}) \mid A^{\max}_{\frac12+m} f \in \mathcal{D}(A^{\min}_{-\frac12-m}) \right\}, \tag{8.5}$$

$$H_m f = -A^{\min}_{-\frac12-m} A^{\max}_{\frac12+m} f, \qquad f \in \mathcal{D}(H_m). \tag{8.6}$$

2. *For* $\mathrm{Re}(m) > 0$ *we have*

$$\langle f|H_m g\rangle = \langle A^{\min}_{\frac12-m} f | A^{\min}_{\frac12-m} g\rangle, \quad f \in \mathcal{D}(A^{\min}_{\frac12-m}), \quad g \in \mathcal{D}(A^{\min}_{\frac12-m}) \cap \mathcal{D}(H_m). \tag{8.7}$$

Moreover,

$$\mathcal{D}(H_m) = \left\{ f \in \mathcal{D}(A^{\min}_{\frac12-m}) \mid A^{\min}_{\frac12-m} f \in \mathcal{D}(A^{\max}_{-\frac12+m}) \right\}, \tag{8.8}$$

$$H_m f = -A^{\max}_{-\frac12+m} A^{\min}_{\frac12-m} f, \qquad f \in \mathcal{D}(H_m). \tag{8.9}$$

The factorizations described in Theorem 8.2 can be used to define bilinear forms corresponding to H_m. For details of the proof, we refer again to [6], especially pages 571–574 and 577.

Theorem 8.3. *The following bilinear forms are extensions of*

$$\langle f|H_m g\rangle = \langle H_m f|g\rangle, \qquad f, g \in \mathcal{D}(H_m), \tag{8.10}$$

to larger domains:

1. *For* $1 \le \mathrm{Re}(m)$,

$$\langle A_{\frac12+m} f | A_{\frac12+m} g\rangle = \langle A_{\frac12-m} f | A_{\frac12-m} g\rangle, \qquad f, g \in \mathcal{H}^1_0. \tag{8.11}$$

2. *For* $0 < \mathrm{Re}(m) < 1$,

$$\langle A_{\frac12+m} f | A_{\frac12+m} g\rangle = \langle A^{\min}_{\frac12-m} f | A^{\min}_{\frac12-m} g\rangle, \qquad f, g \in \mathcal{H}^1_0. \tag{8.12}$$

3. *For* $\mathrm{Re}(m) = 0$,

$$\langle A_{\frac12+m} f | A_{\frac12+m} g\rangle, \qquad f, g \in \mathcal{D}(A_{\frac12+m}) \supset \mathcal{H}^1_0 + \mathbb{C}x^{\frac12+m}\xi, \tag{8.13}$$

$$\langle A_{\frac12-m} f | A_{\frac12-m} g\rangle, \qquad f, g \in \mathcal{D}(A_{\frac12-m}) \supset \mathcal{H}^1_0 + \mathbb{C}x^{\frac12-m}\xi. \tag{8.14}$$

4. *For* $-1 < \mathrm{Re}(m) < 0$,

$$\langle A^{\max}_{\frac12+m} f | A^{\max}_{\frac12+m} g\rangle, \qquad f, g \in \mathcal{H}^1_0 + \mathbb{C}x^{\frac12+m}\xi. \tag{8.15}$$

Note that for $\mathrm{Re}(m) > 0$ both factorizations yield the same quadratic form. This is no longer true for $\mathrm{Re}(m) = 0$, $m \neq 0$, when there are two distinct quadratic forms with distinct domains corresponding to H_m. Finally, for $-1 < m < 0$, and also for $m = 0$, we have a unique factorization.

Let us finally specialize Theorem 8.3 to real m. The following theorem is essentially identical with Thm 4.22 of [6].

Theorem 8.4. *For real* $-1 < m$, *the operators* H_m *are positive and self-adjoint. The corresponding sesquilinear form can be factorized as follows:*

1. *For* $1 \leq m$,

$$(\sqrt{H_m} f | \sqrt{H_m} g) = (A_{\frac{1}{2}+m} f | A_{\frac{1}{2}+m} g) = (A_{\frac{1}{2}-m} f | A_{\frac{1}{2}-m} g),$$
$$f, g \in \mathcal{Q}(H_m) = \mathcal{H}_0^1. \tag{8.16}$$

H_m *is essentially self-adjoint on* $C_c^\infty(\mathbb{R}_+)$.

2. *For* $0 < m < 1$,

$$(\sqrt{H_m} f | \sqrt{H_m} g) = (A_{\frac{1}{2}+m} f | A_{\frac{1}{2}+m} g) = (A^{\min}_{\frac{1}{2}-m} f | A^{\min}_{\frac{1}{2}-m} g),$$
$$f, g \in \mathcal{Q}(H_m) = \mathcal{H}_0^1. \tag{8.17}$$

H_m *is the Friedrichs extension of* L_{m^2} *restricted to* $C_c^\infty(\mathbb{R}_+)$.

3. *For* $m = 0$,

$$(\sqrt{H_0} f | \sqrt{H_0} g) = (A_{\frac{1}{2}} f | A_{\frac{1}{2}} g), \qquad f, g \in \mathcal{Q}(H_0) = \mathcal{D}(A_{\frac{1}{2}}) \supsetneq \mathcal{H}_0^1 + \mathbb{C} x^{\frac{1}{2}} \xi. \tag{8.18}$$

H_0 *is both the Friedrichs and Krein extension of* L_0 *restricted to* $C_c^\infty(\mathbb{R}_+)$.

4. *For* $-1 < m < 0$,

$$(\sqrt{H_m} f | \sqrt{H_m} g) = (A^{\max}_{\frac{1}{2}+m} f | A^{\max}_{\frac{1}{2}+m} g), \qquad f, g \in \mathcal{Q}(H_m) = \mathcal{H}_0^1 + \mathbb{C} x^{\frac{1}{2}+m} \xi. \tag{8.19}$$

H_m *is the Krein extension of* L_{m^2} *restricted to* $C_c^\infty(\mathbb{R}_+)$.

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A Holomorphic Families of Closed Operators and the Kato-Rellich Theorem

In this appendix, we describe a few general concepts and facts from the operator theory, which we use in our paper.

The definition (or actually a number of equivalent definitions) of a *holomorphic family of bounded operators* is quite obvious and does not need to be recalled. In the case of unbounded operators, the situation is more subtle.

Suppose that Θ is an open subset of $\mathbb{C}$, $\mathcal{H}$ is a Banach space, and $\Theta \ni z \mapsto H(z)$ is a function whose values are closed operators on $\mathcal{H}$. We say that this is a *holomorphic family of closed operators* if for each $z_0 \in \Theta$ there exists a neighborhood Θ_0 of z_0, a Banach space $\mathcal{K}$ and a holomorphic family of bounded operators $\Theta_0 \ni z \mapsto A(z) \in B(\mathcal{K}, \mathcal{H})$ such that $\mathrm{Ran} A(z) = \mathcal{D}(H(z))$ and

$$\Theta_0 \ni z \mapsto H(z)A(z) \in B(\mathcal{K}, \mathcal{H})$$

is a holomorphic family of bounded operators.

The following theorem is essentially a version of the well-known Kato–Rellich Theorem generalized from self-adjoint to closed operators:

Theorem A.1. *Suppose that A is a closed operator on a Hilbert space $\mathcal{H}$. Let B be an operator $\mathcal{D}(A) \to \mathcal{H}$ such that*

$$\|Bf\| \leq c\|Af\|, \quad f \in \mathcal{D}(A). \tag{A.1}$$

Then for $|z| < \frac{1}{c}$ the operator $A + zB$ is closed on $\mathcal{D}(A)$ and

$$\left\{z \in \mathbb{C} \mid |z| < c^{-1}\right\} \ni z \mapsto A + zB \tag{A.2}$$

is a holomorphic family of closed operators.

Proof. We easily check that the norms $\sqrt{\|f\|^2 + \|Af\|^2}$ and $\sqrt{\|f\|^2 + \|(A + zB)f\|^2}$ are equivalent for $|z| < \frac{1}{c}$. Let $\mathcal{H}_0$ be the closure of $\mathcal{D}(A)$ in $\mathcal{H}$. The restriction of A to $\mathcal{H}_0$ is densely defined, so that we can define A^*. The operator $(A^*A + \mathbb{1})^{-\frac{1}{2}}$ is unitary from $\mathcal{H}_0$ to $\mathcal{D}(A)$. Clearly, it is bounded in the sense of $\mathcal{H}_0$. Now

$$\mathbb{C} \ni z \mapsto (A + zB)(A^*A + \mathbb{1})^{-\frac{1}{2}} \tag{A.3}$$

is obviously a polynomial of degree 1 with values in bounded operators (hence obviously a holomorphic family). □

Let us also quote the following fact proven by Bruk [5], see also [10]:

Proposition A.2. *If $z \mapsto A(z)$ is a holomorphic family of closed operators, then so is $z \mapsto A(\overline{z})^*$.*

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Jan Dereziński
Department of Mathematical Methods in Physics, Faculty of Physics
University of Warsaw
Pasteura 5
02-093 Warszawa
Poland
e-mail: `jan.derezinski@fuw.edu.pl`

Vladimir Georgescu
Laboratoire AGM
UMR 8088 CNRS
CY Cergy Paris Université
F-95000 Cergy
France
e-mail: vladimir.georgescu@math.cnrs.fr

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