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## ON THE DOUADY SPACE OF A COMPACT COMPLEX SPACE IN THE CATEGORY $\mathcal{C}$

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### Introduction

Let  $X$  be a complex space. Let  $D_x$  be the Douady space of compact complex subspaces of  $X$  [6] and  $\rho_x: Z_x \rightarrow D_x$  the corresponding universal family of subspaces of  $X$ . Thus there is a natural embedding  $Z_x \subseteq D_x \times X$  such that  $\rho_x$  is induced by the projection  $D_x \times X \rightarrow D_x$ . Let  $\pi_x: Z_x \rightarrow X$  be induced by the other projection  $D_x \times X \rightarrow X$ . For any irreducible component  $D_\alpha$  of  $D_{x, \text{red}}$  we denote by  $\rho_\alpha: Z_\alpha \rightarrow D_\alpha$  the universal family restricted to  $D_\alpha$ , and set  $\pi_\alpha = \pi_{x|Z_\alpha}: Z_\alpha \rightarrow X$ , where  $D_{x, \text{red}}$  is the underlying reduced subspace of  $D_x$ . On the other hand, we have introduced in [9] a category  $\mathcal{C}$  of compact complex spaces as follows (cf. also [10]). A compact complex space  $X$  is in  $\mathcal{C}$  if and only if there exist a compact Kähler manifold  $Y$  and a generically surjective meromorphic map  $h: Y \rightarrow X_{\text{red}}$ ,  $X_{\text{red}}$  being as above. Then the main purpose of this paper is to prove the following theorem: *Let  $X$  be a compact complex space in  $\mathcal{C}$ . Then for every irreducible component  $D_\alpha$  of  $D_{x, \text{red}}$  such that  $Z_\alpha$  is reduced,  $D_\alpha$  is compact and again belongs to  $\mathcal{C}$ .* The proof also shows that if  $X$  is Moishezon, then  $D_\alpha$  also is Moishezon, which is a special case of a theorem of Artin [1]. Moreover since the Barlet space  $B(X)$  of compact cycles of  $X$  [4] is a proper holomorphic image of the union of those irreducible components of  $D_{x, \text{red}}$  for which  $Z_\alpha$  are reduced and of pure fiber dimension, the result also implies that every irreducible component of  $B(X)$  is again in  $\mathcal{C}$  if  $X$  is in  $\mathcal{C}$ . Here we note that the same result as above was also obtained by Campana [5] independently.

The arrangement of this article is as follows. In § 1 and § 2 we define respectively the notion of a Moishezon morphism and of a morphism in the category  $\mathcal{C}/S$ , which is a relative version of the category  $\mathcal{C}$  above, and summarize some functorial properties of these morphisms. In § 3 we

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make some general study on the irreducibility of general fiber of a morphism, in part to be used in § 5. Then in § 4 we give the main ingredient of the proof of our theorem, Proposition 4, which states that if the general fiber of  $\rho_\alpha: Z_\alpha \rightarrow D_\alpha$  is reduced and irreducible, then  $\pi_\alpha$  defined above is Moishezon. In fact, combining this with the results in §§ 1 and 2 we obtain the theorem immediately in this special case. The reduction of the general case to this special case will then be given in § 5, thus completing the proof of the theorem. Actually our theorem is expected to be true for any irreducible component of  $D_{X, \text{red}}$ . Presupposing the future investigation of this problem along the line of [9] and in view of an application [11] also, we have developed our exposition in the relative form as in [9] so that the above theorem is also true in this generalized form (see Theorem in § 5 for the precise statement). Finally in the Appendix we give a direct proof of Lemma 2.

*Notation.* Let  $f: X \rightarrow S$  be a morphism of complex spaces. Then for any morphism  $\alpha: T \rightarrow S$  we often write  $X_T = X \times_S T$  and  $f_T: X_T \rightarrow T$  for the natural projection. For instance if  $U \subseteq S$  is open,  $f_U$  is the induced morphism  $X_U = f^{-1}(U) \rightarrow U$ . In particular if  $T = \{s\}$  is a point of  $S$  we write  $X_s$  instead of  $X_{\{s\}}$ . For a complex space  $X$ ,  $X_{\text{red}}$  denotes the underlying reduced analytic subspace.

## § 1. Moishezon morphisms

(1.1) We fix notation and terminology for meromorphic maps. Let  $f: X \rightarrow S$  and  $g: Y \rightarrow S$  be morphisms of reduced complex spaces. Then a *meromorphic S-map*  $\alpha: X \rightarrow Y$  from  $X$  to  $Y$  is a reduced analytic subspace  $\Gamma \subseteq X \times_S Y$  such that the natural projection  $p: \Gamma \rightarrow X$  is a proper bimeromorphic morphism in the sense that  $p$  is proper and that there is a dense Zariski open subset  $U$  (resp.  $V$ ) of  $\Gamma$  (resp.  $X$ ) such that  $p$  induces an isomorphism of  $U$  and  $V$ . We call  $\alpha$  a (proper) *bimeromorphic S-map*, or being *S-bimeromorphic*, if the natural projection  $q: \Gamma \rightarrow Y$  also is a proper bimeromorphic morphism. We say that  $f$  and  $g$  are bimeromorphic if there is a bimeromorphic S-map of  $X$  to  $Y$ .

If  $f$  is proper in the above definition,  $q(\Gamma)$  is an analytic subspace of  $Y$  and is called the image of  $X$  by  $\alpha$ . On the other hand,  $\alpha$  is called *generically surjective* (resp. *generically finite*) if  $q(\Gamma)$  contains a dense Zariski open subset of  $Y$  (resp.  $q$  is generically finite). When  $f$  is proper, the generic surjectivity is equivalent to saying that  $Y = q(\Gamma)$ .

Given a meromorphic  $S$ -map  $\alpha: X \rightarrow Y$  as above we often identify  $\alpha$  with the induced  $S$ -morphism  $\alpha' = qp^{-1}|_V: V \rightarrow Y$ . Then the subspace  $\Gamma$  above is recovered from  $\alpha'$  as the closure in  $X \times_S Y$  of the graph  $\Gamma_{\alpha'} \subseteq V \times_S Y$  of  $\alpha'$  and is called the graph of  $\alpha'$ . Then an  $S$ -morphism is nothing but the meromorphic  $S$ -map  $\alpha$  for which we can take  $V = X$ .

Let  $f: X \rightarrow S$  and  $g_i: Y_i \rightarrow S$ ,  $1 \leq i \leq m$ , be morphisms of complex spaces and  $\alpha_i: X \rightarrow Y_i$  be meromorphic  $S$ -maps. Then we can define naturally a meromorphic  $S$ -map  $\prod_i \alpha_i: X \rightarrow Y_1 \times_S \cdots \times_S Y_m$  called the product of  $\alpha_i$  over  $S$ ; one verifies readily that the closure of the graph of the  $S$ -morphism  $\alpha'_1 \times_S \cdots \times_S \alpha'_m$  is analytic in  $X \times_S Y_1 \times_S \cdots \times_S Y_m$  where  $\alpha'_i$  for  $\alpha_i$  has the same meaning as  $\alpha'$  for  $\alpha$  as above.

For later reference we recall here the analytic Chow lemma due to Hironaka [14], [15].

(1.1.1) Let  $\alpha: X \rightarrow Y$  be a meromorphic  $S$ -map as above. Then there exist a complex manifold  $X^*$ , and a projective bimeromorphic morphism  $h: X^* \rightarrow X$  such that the composition  $\alpha h: X^* \rightarrow Y$  is a morphism.

(1.2) Let  $f: X \rightarrow S$  be a proper morphism of complex spaces. We call  $f$  *locally projective* if for every relatively compact open subset  $Q$  of  $S$  there is an invertible sheaf  $\mathcal{L} = \mathcal{L}(Q)$  defined on  $X$  such that  $\mathcal{L}|_{X_Q}$  is  $f_Q$ -ample (cf. Notation). (In this case we simply say that  $\mathcal{L}$  is  $f_Q$ -ample.) Thus if  $f$  is locally projective, then  $f_Q$  is projective for every relatively compact open subset  $Q \subseteq S$ .

(1.2.1) A composition of two locally projective morphisms is again locally projective.

*Proof.* Let  $f: X \rightarrow Y$ ,  $g: Y \rightarrow Z$  be locally projective. Let  $h = gf: X \rightarrow Z$ . Let  $Q$  be any relatively compact open subset of  $Z$ . Take another relatively compact open subset  $Q'$  of  $Z$  with  $Q \subset Q'$ . Then  $\tilde{Q}' = g^{-1}(Q')$  is a relatively compact open subset of  $Y$ . Take an invertible sheaf  $\mathcal{L}$  on  $X$  (resp.  $\mathcal{F}$  on  $Y$ ) which is  $f_{\tilde{Q}'}$ -ample (resp.  $g_{Q'}$ -ample). Then it is easy to see that  $\mathcal{L} \otimes_{\mathcal{O}_X} f^* \mathcal{F}^m$  is  $h_Q$ -ample for all sufficiently large  $m$  (cf. EGA II, 4.6.13 (ii)). Thus  $h$  is locally projective.

(1.3) Let  $f: X \rightarrow S$  be a locally projective morphism.

(1.3.1) If  $X$  has only a finite number of irreducible components, then  $f$  is  $S$ -bimeromorphic to a projective morphism.

*Proof.* Let  $Q$  be any relatively compact open subset of  $S$  such that

$X_Q$  meets every irreducible component of  $X$ . Let  $\mathcal{L}$  be an invertible sheaf on  $X$  which is  $f_Q$ -ample. Restricting  $Q$  and replacing  $\mathcal{L}$  by its high power  $\mathcal{L}^n$ ,  $n \gg 0$ , we may assume that  $\mathcal{L}$  is even  $f_Q$ -very ample. Let  $\alpha: X \rightarrow P(f_* \mathcal{L})$  be the natural meromorphic  $S$ -map of  $X$  into the projective fiber space  $P(f_* \mathcal{L})$  over  $S$  associated to the coherent analytic sheaf  $f_* \mathcal{L}$  on  $S$  (cf. [3], [13]). By our assumption  $\alpha$  is an embedding on  $X_Q$ . Since  $X_Q$  meets every irreducible component of  $X$ , this implies that  $\alpha$  is bimeromorphic onto its image. Hence  $f$  is bimeromorphic to a projective morphism.

Q.E.D.

From the above proof follows also the following:

(1.3.2) Let  $f: X \rightarrow S$  be as in (1.3.1). Then there exist an invertible sheaf  $\mathcal{L}$  on  $X$  and a dense Zariski open subset  $W$  of  $S$  such that  $\mathcal{L}$  is  $f_w$ -(very) ample.

(1.4) DEFINITION. Let  $f: X \rightarrow S$  be a proper morphism of reduced complex spaces. We call  $f$  Moishezon if  $f$  is bimeromorphic to a locally projective morphism  $g: Y \rightarrow S$ . By (1.3.1) when  $X$  has only a finite number of irreducible components,  $f$  is Moishezon if and only if  $f$  is bimeromorphic to a projective morphism.

*Remark.* In [17] Moishezon introduced the notion of an  $A$ -space over another complex space, and stated some of their fundamental properties. From his definition it follows readily that for a proper morphism  $f: X \rightarrow S$  of reduced complex spaces  $X$  is an  $A$ -space over  $S$  if and only if  $f$  is locally Moishezon in the sense that for each point  $s \in S$  there is a neighborhood  $s \in U$  such that the induced morphism  $f_U: X_U \rightarrow U$  is Moishezon in the sense defined above.

(1.5) Clearly the Moishezon property of a morphism is invariant under  $S$ -bimeromorphic equivalence. We now list some fundamental properties of Moishezon morphisms.

- 1) A composition of two Moishezon morphisms are again Moishezon.
- 2)  $f: X \rightarrow S$  is Moishezon if and only if for each irreducible component  $X_i$  of  $X$  the restriction  $f = f|_{X_i}: X_i \rightarrow S$  is Moishezon.
- 3) If  $f$  is Moishezon, there are a locally projective morphism  $g: X^* \rightarrow S$  with  $X^*$  nonsingular and a bimeromorphic  $S$ -morphism  $h: X^* \rightarrow X$ .
- 4) Suppose that there exist a locally projective morphism  $g: Y \rightarrow S$  and a generically finite meromorphic  $S$ -map  $h: X \rightarrow Y$ . Then  $f$  is Moishezon.

*Proof.* 1) and 3) follows from (1.1.1) and (1.2.1). Let  $\mu: \tilde{X} \rightarrow X$  be the

normalization of  $X$ . Since  $\mu$  is bimeromorphic,  $f$  is Moishezon if and only if  $f\mu$  is Moishezon. From this 2) follows readily. 4) Changing  $f$  under bimeromorphic equivalence we may assume that  $h$  is a morphism. Let  $h = h_2h_1$  with  $h_1: X \rightarrow X^*$  and  $h_2: X^* \rightarrow Y$  be the Stein factorization of  $h$ , where  $h_1$  is a bimeromorphic, and  $h_2$  is a finite,  $S$ -morphisms. Since a finite morphism is projective,  $gh_2: X^* \rightarrow S$  is locally projective by (1.2.1), and hence 4).

(1.6) Less trivial to prove is the following:

**PROPOSITION 1.** *Let  $f: X \rightarrow S$  be a Moishezon morphism, and  $g: Y \rightarrow S$  a proper morphism, of reduced complex spaces. Suppose that there is a generically surjective meromorphic  $S$ -map  $h: X \rightarrow Y$ . Then  $g$  also is Moishezon.*

*Proof.* By (1.5) 2) we may assume that  $Y$ , and then  $X$  and  $S$  also, are irreducible. By (1.1.1) and (1.5) 3) we may further assume that  $f$  is locally projective,  $X$  is nonsingular and  $h$  is a morphism. Then there is a dense Zariski open subset  $V_0$  of  $Y$  such that  $V_0$  is nonsingular and  $h_{V_0}: X_{V_0} \rightarrow V_0$  is smooth. Let  $\mathcal{L}$  be an invertible sheaf on  $X$  which is  $f_w$ -ample for some dense Zariski open subset  $W$  of  $S$  (1.3.2). Restricting  $V_0$  we may assume that  $V_0 \subseteq Y_w$ . Then if  $n$  is sufficiently large, say,  $n \geq n_0$  for some  $n_0 > 0$ , there is a dense Zariski open subset  $V_n$  of  $Y$  such that  $V_n \subseteq V_0$  and  $H^1(X_y, \mathcal{L}_y^n) = 0$  for all  $y \in V_n$  where  $\mathcal{L}_y^n = \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X_y}$ . Let  $\mathcal{E}_n = h_* \mathcal{L}^n$ . Then  $\mathcal{E}_n$  is a coherent analytic sheaf on  $Y$  which is locally free of rank, say  $r_n$ , on  $V_n$  (cf. [3, p. 122, Cor. 3.9]). Moreover taking  $n_0$  larger if necessary we may assume that  $r_n > 0$  for  $n \geq n_0$ . On the other hand, by [20] we can find a proper surjective bimeromorphic morphism  $\sigma_n: \tilde{Y}_n \rightarrow Y$  such that  $\tilde{\mathcal{E}}_n = \sigma_n^* \mathcal{E}/\mathcal{T}_n$ ,  $\mathcal{T}_n$  being the torsion part of  $\sigma_n^* \mathcal{E}$ , is locally free of rank  $r_n$  on  $\tilde{Y}_n$ . Moreover we can assume that  $\sigma_n$  gives an isomorphism of  $\tilde{V}_n = \sigma_n^{-1}(V_n)$  onto  $V_n$ . Let  $\tilde{g}_n = \sigma_n g: \tilde{Y}_n \rightarrow S$  and set  $\mathcal{M}_n = \bigwedge^{r_n} \tilde{\mathcal{E}}_n$ , where  $\bigwedge^{r_n}$  denotes the  $r_n$ -th exterior product. Then  $\mathcal{M}_n$  is an invertible sheaf on  $\tilde{Y}_n$ . Let  $\alpha_n: \tilde{Y}_n \rightarrow P(\tilde{g}_n)_* \mathcal{M}_n$  be the natural meromorphic  $S$ -map from  $\tilde{Y}_n$  to the projective fiber space  $P(\tilde{g}_n)_* \mathcal{M}_n$  over  $S$  associated to the coherent analytic sheaf  $\tilde{g}_n_* \mathcal{M}_n$  on  $S$  (cf. [3, IV, § 1]). Then we show that for a sufficiently large  $n$ ,  $\alpha_n$  is generically finite; then by (1.5) d) the proposition would follow.

For this purpose it is enough to show that for some  $n \geq n_0$ , for some  $\tilde{y} \in \tilde{Y}_n$  and for some neighborhood  $U$  of  $s = \tilde{g}_n(y)$  in  $S$ , the following holds

true; there are sections  $\varphi_1, \dots, \varphi_d \in \Gamma(\tilde{Y}_{n,U}, \mathcal{M}_n)$  such that the meromorphic  $U$ -map  $\Phi: \tilde{Y}_{n,U} \rightarrow U \times \mathbf{CP}^{d-1}$  associated to  $\varphi_\alpha$  is holomorphic and locally biholomorphic at  $\tilde{y}$ , where  $\tilde{Y}_n$  is over  $S$  by  $\tilde{g}_n$ . (Note that  $\tilde{Y}_n$  is irreducible.) First we take  $n, \tilde{y}$  and  $U$  in such a way that  $\tilde{y} \in \tilde{V}_n$ ,  $U$  is a sufficiently small Stein open neighborhood of  $s$ , and that  $H^1(X_U, m_y^2 \mathcal{L}^n) = 0$ , where  $y = \sigma_n(\tilde{y})$  and  $m_y$  is the maximal ideal of  $\mathcal{O}_Y$  at  $y$ . Clearly this is possible since  $\mathcal{L}$  is  $f_w$ -ample and  $y \in V_n \subseteq Y_w$ . Then in particular the restriction map  $\beta_n: \Gamma(X_U, \mathcal{L}^n) \rightarrow \Gamma(X_U, \mathcal{L}_{(2)}^n)$  is surjective, as follows from the long exact sequence associated to the short one

$$0 \longrightarrow m_y^2 \mathcal{L}^n \longrightarrow \mathcal{L}^n \longrightarrow \mathcal{L}_{(2)}^n \longrightarrow 0$$

where we have put  $\mathcal{L}_{(2)}^n = \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_X / m_y^2 \mathcal{O}_X$ . Further since

$$H^1(X_U, m_y \mathcal{L}^n / m_y^2 \mathcal{L}^n) \cong H^1(X_U, \mathcal{L}^n \otimes_{\mathcal{O}_Y} m_y / m_y^2) \cong H^1(X_y, \mathcal{L}_y^n) \otimes_{\mathcal{C}} m_y / m_y^2 = 0,$$

from the short exact sequence  $0 \rightarrow m_y \mathcal{L}^n / m_y^2 \mathcal{L}^n \rightarrow \mathcal{L}_{(2)}^n \rightarrow \mathcal{L}_y^n \rightarrow 0$  we have the exact sequence

$$0 \longrightarrow \Gamma(X_y, \mathcal{L}_y^n) \otimes_{\mathcal{C}} m_y / m_y^2 \xrightarrow{\gamma_n} \Gamma(X_U, \mathcal{L}_{(2)}^n) \xrightarrow{\delta_n} \Gamma(X_y, \mathcal{L}^n) \longrightarrow 0.$$

Fix  $n$  and write  $r = r_n$ . Then take and fix a base  $(\bar{\psi}_1^0, \dots, \bar{\psi}_r^0)$  of  $\Gamma(X_y, \mathcal{L}_y^n)$ . Let  $(y_1, \dots, y_m)$ ,  $m = \dim Y$ , be a local coordinate system around  $y$  of  $Y$  and  $\bar{y}_i$  the residue classes of  $y_i$  in  $\mathcal{O}_Y / m_y^2 \mathcal{O}_Y$ . Then we take any base  $(\bar{\psi}_1, \dots, \bar{\psi}_d)$ ,  $d = r(m+1)$ , of  $\Gamma(X_U, \mathcal{L}_{(2)}^n)$  satisfying the following conditions;  $\delta_n(\bar{\psi}_i) = \bar{\psi}_i^0$ ,  $1 \leq i \leq r$ , and  $\bar{\psi}_{k+r+j} = \gamma_n(\bar{y}_k \bar{\psi}_j)$ ,  $1 \leq k \leq m$ ,  $1 \leq j \leq r$ , where  $\bar{y}_k \bar{\psi}_j = \bar{\psi}_j \otimes \bar{y}_k \in \Gamma(X_y, \mathcal{L}_y^n) \otimes_{\mathcal{C}} m_y / m_y^2$ . For each  $1 \leq k \leq d$  take and fix  $\psi_k \in \Gamma(X_U, \mathcal{L}^n)$  with  $\beta_n(\psi_k) = \bar{\psi}_k$ . With respect to the natural identification  $\Gamma(X_U, \mathcal{L}^n) \cong \Gamma(Y_U, \mathcal{E}_n) \subseteq \Gamma(\tilde{Y}_{n,U}, \tilde{\mathcal{E}}_n)$ , we consider  $\psi_i$  naturally as sections of  $\tilde{\mathcal{E}}_n$  on  $\tilde{Y}_{n,U}$ . Then for any  $1 \leq i_1 \dots \leq i_r \leq d$  define  $\varphi_{i_1 \dots i_r} \in \Gamma(\tilde{Y}_{n,U}, \mathcal{M}_n)$  by  $\varphi_{i_1 \dots i_r} = \psi_{i_1} \wedge \dots \wedge \psi_{i_r}$ . We claim that these  $\varphi_\alpha = \varphi_{i_1 \dots i_r}$  have the desired properties.

Since the problem is local around  $y$  and  $\sigma_n$  gives a natural isomorphism of  $\tilde{V}_n$  and  $V_n$ , in what follows we identify  $\tilde{V}_n$  and  $V_n$  by  $\sigma_n$  and therefore  $\tilde{y}$  with  $y$  and  $\tilde{\mathcal{E}}_n|_{V_n}$  with  $\mathcal{E}_n|_{V_n}$ . Further we consider  $\mathcal{M}_n|_{V_n}$  as an invertible sheaf on  $V_n$  and  $\psi_i$  as sections of  $\mathcal{E}_n$  on  $Y_U \cap V_n$ . Now  $\psi_1, \dots, \psi_r$  define a trivialization  $\mathcal{E}_n \cong \mathcal{O}_Y$  of  $\mathcal{E}_n$ , and hence also  $\mathcal{M}_n \cong \mathcal{O}_Y$  of  $\mathcal{M}_n$ , in some neighborhood  $N$  of  $y$ . In particular we may consider each  $\psi_i$  (resp.  $\varphi_{i_1 \dots i_r}$ ) as an  $r$ -tuple of holomorphic functions (resp. a holomorphic function) on  $N$ . Then we have by construction  $\psi_i = (0, \dots, 0, 1, 0, \dots, 0)$

for  $1 \leq i \leq r$  where 1 is on the  $i$ -th place and  $\psi_{kr+j} \equiv (0, \dots, 0, y_k, 0, \dots, 0)$  modulo  $m_y^2$ , where  $y_k$  is on the  $j$ -th place. Hence we have  $\varphi_{1\dots r}(y) = \psi_1 \wedge \dots \wedge \psi_r(0) \neq 0$  and  $\varphi_{1\dots k\dots r(kr+k)} = \psi_1 \wedge \dots \wedge \hat{\psi}_k \wedge \dots \wedge \psi_r \wedge \psi_{kr+k} \equiv y_k$  modulo  $m_y^2$  where  $\hat{u}$  implies the absense of  $u$ . The former implies that  $\Phi$  is holomorphic at  $y$  and the latter implies that  $\Phi$  is locally biholomorphic at  $y$ . Hence our claim is verified. Q.E.D.

- (1.7) Let  $f: X \rightarrow S$  be a Moishezon morphism. Then:
- 5) For every reduced analytic subspace  $X' \subseteq X$  the induced morphism  $f' = f|_{X'}: X' \rightarrow S$  is Moishezon.
- 6) Let  $\mu: \tilde{S} \rightarrow S$  be a morphism of reduced complex spaces. Then the induced map  $f_{\tilde{S}, \text{red}}: X_{\tilde{S}, \text{red}} \rightarrow \tilde{S}$  is Moishezon.
- 7) Let  $g: Y \rightarrow S$  be another Moishezon morphism. Then  $f \times_S g: X \times_S Y \rightarrow S$  also is Moishezon.

*Proof.* Let  $g: X^* \rightarrow S$  and  $h: X^* \rightarrow X$  be as in (1.5) 3). Let  $Z = h^{-1}(X')$  with reduced structure. Then  $g|_Z: Z \rightarrow S$  is locally projective and  $h|_Z: Z \rightarrow X'$  is surjective. Hence by the above proposition  $f$  is Moishezon. This proves 5). We show 6). Let  $g$  and  $h$  be as above. Then  $h$  induces a surjective morphism  $h_{\tilde{S}, \text{red}}: X_{\tilde{S}, \text{red}}^* \rightarrow X_{\tilde{S}, \text{red}}$  over  $S$ . Since  $g_{\tilde{S}, \text{red}}: X_{\tilde{S}, \text{red}}^* \rightarrow \tilde{S}$  is locally projective, 6) also follows from the above proposition. Since  $f \times_S g$  is the composition of the natural projection  $X \times_S Y \rightarrow Y$  and  $g$ , 7) follows from (1.5) 1) and 6) above.

## § 2. Morphisms in $\mathcal{C}/S$

(2.1) DEFINITION. Let  $g: Y \rightarrow S$  be a proper morphism of complex spaces. Then: 1) ([9, Def. 4.1])  $g$  is called *Kähler* if there exist an open covering  $\{U_\alpha\}$  of  $Y$  and a  $C^\infty$  function  $p_\alpha$  defined on each  $U_\alpha$  such that for each  $\alpha$ ,  $p_\alpha$  is strictly plurisubharmonic when restricted to each fiber of  $g|_{U_\alpha}: U_\alpha \rightarrow S$  and that  $p_\alpha - p_\beta$  is pluriharmonic on each  $U_\alpha \cap U_\beta$ . 2)  $g$  is called *locally Kähler* if for every relatively compact open subset  $Q$  of  $S$  there exist  $\{U_\alpha\}$  and  $\{p_\alpha\}$  satisfying the condition as above except that  $p_\alpha$  is assumed to be strictly plurisubharmonic only when restricted to each fiber of  $g|_{U_\alpha \cap g^{-1}(Q)}: U_\alpha \cap g^{-1}(Q) \rightarrow Q$ .

In the above definition the real closed  $(1, 1)$ -form  $\omega_\alpha = \sqrt{-1}\partial\bar{\partial}p_\alpha$ , each defined on  $U_\alpha$ , patch together to give a global real closed  $(1, 1)$ -form  $\omega$  on  $Y$ , which we call a relative Kähler form for  $g$  (resp. for  $g$  over  $Q$ ).

- (2.2) In the following all the morphisms considered are proper.

- 1) Every (locally) projective morphism is (locally) Kähler.
- 2) Let  $g: Y \rightarrow S$  be a (locally) Kähler morphism and  $\alpha: \tilde{S} \rightarrow S$  a morphism of complex spaces. Then the induced morphism  $g_{\tilde{S}}: Y_{\tilde{S}} \rightarrow \tilde{S}$  is (locally) Kähler.
- 3) Let  $f: X \rightarrow Y$  and  $g: Y \rightarrow S$  be locally Kähler morphism of complex spaces. Then the composition  $gf: X \rightarrow S$  is again locally Kähler. Conversely if  $gf$  is (locally) Kähler, then  $f$  also is (locally) Kähler.
- 4) Let  $f: X \rightarrow S$  and  $g: Y \rightarrow S$  be locally Kähler morphisms. Then  $f \times_s g: X \times_s Y \rightarrow S$  is locally Kähler.

*Proof.* See [9, Lemma 4.4] for 1). We show the former half of 3). Let  $Q \subset Q' \subset S$  and  $\tilde{Q} = g^{-1}(Q')$  be as in the proof of (1.2.1) (with  $Z$  replaced by  $S$ ). Let  $\omega_{Q'}$  (resp.  $\omega_{\tilde{Q}}$ ) be a relative Kähler form for  $g$  over  $Q'$  (resp.  $f$  over  $\tilde{Q}$ ). Then for all sufficiently large  $n > 0$ ,  $\omega_{\tilde{Q}} + nf^*\omega_{Q'}|_{(gf)^{-1}(Q)}$  gives a relative Kähler form for the morphism  $gf$  over  $Q$  (cf. the proof of [9, Lemma 4.4]). Hence  $gf$  is locally Kähler. Since  $f \times_s g$  is a composite of the natural projection  $X \times_s Y \rightarrow Y$  and  $g$ , 4) follows from this and 2). The other assertions follow immediately from the definition.

(2.3) **DEFINITION.** Let  $S$  be a reduced complex space. Then we define the category  $\mathcal{C}/S$  as follows: An object of  $\mathcal{C}/S$  is a proper morphism  $f: X \rightarrow S$  of reduced complex spaces for which there exist a proper and locally Kähler morphism  $g: Y \rightarrow S$  and a generically surjective meromorphic  $S$ -map  $h: Y \rightarrow X$  (Notation:  $f \in \mathcal{C}/S$ ); and a morphism in  $\mathcal{C}/S$  is a morphism  $u: X_1 \rightarrow X_2$  of complex spaces with  $f_2 u = f_1$  where  $f_i: X_i \rightarrow S \in \mathcal{C}/S$ ,  $i = 1, 2$ .

*Remark.* 1) Note the deviation from the notation adopted in [9, p. 51]; there we used the notation  $\mathcal{C}/S$  for the category  $\text{loc-}\mathcal{C}/S$  which is defined as follows: An object of  $\text{loc-}\mathcal{C}/S$  is a proper morphism  $f: X \rightarrow S$  of complex spaces for which there exists an open covering  $\{U_\alpha\}$  of  $S$  such that  $f_{U_\alpha}: X_{U_\alpha} \rightarrow U_\alpha \in \mathcal{C}/U_\alpha$  for each  $\alpha$ , with morphisms defined as above. 2) When  $S$  is a point, we write  $\mathcal{C}$  instead of  $\mathcal{C}/S$ . In this case the definition coincides with that given in [9, 4.3] except that we consider only reduced spaces here.

(2.4) We shall give some functorial properties of morphisms in  $\mathcal{C}/S$  analogous to Moishezon morphisms.

- 1) Every Moishezon morphism belongs to  $\mathcal{C}/S$ .
- Let  $f: X \rightarrow S$  and  $g: Y \rightarrow S$  be proper morphisms of reduced complex spaces. Suppose that  $g \in \mathcal{C}/S$ . Then:

- 2)  $f \in \mathcal{C}/S$  if and only if there exist a proper and locally Kähler morphism  $g^*: Y^* \rightarrow S$  with  $Y^*$  nonsingular, and a surjective  $S$ -morphism  $h^*: Y^* \rightarrow S$ .
- 3) For every analytic subspace  $Y'$  of  $Y$  the induced morphism  $g|_{Y'}: Y' \rightarrow S$  is again in  $\mathcal{C}/S$ .
- 4) Suppose that there is a generically surjective meromorphic  $S$ -map  $h: Y \rightarrow X$ . Then  $f \in \mathcal{C}/S$ .
- 5) Suppose that there is an  $S$ -morphism  $h: X \rightarrow Y$  with  $h \in \mathcal{C}/Y$ . Then  $f \in \mathcal{C}/S$ .
- 6) For any reduced complex space  $\tilde{S}$  over  $S$  the induced morphism  $g: Y_{\tilde{S}, \text{red}} \rightarrow \tilde{S}$  is in  $\mathcal{C}/\tilde{S}$ .
- 7) Suppose that  $f \in \mathcal{C}/S$ . Then  $f \times_S g: X \times_S Y \rightarrow S$  is again in  $\mathcal{C}/S$ .

*Proof.* 1) follows from (2.2) 1) and the definition of a Moishezon morphism. The proofs of 2), 3) and 4) are the same as those of 1), 2) and 3) of [9, Lemma 4.6] respectively, using (2.2) instead of [9, Lemma 4.4], and will be omitted.

5) By assumption and by 2) there exist a locally Kähler morphism  $\tilde{g}: \tilde{Y} \rightarrow S$  (resp.  $\tilde{h}: \tilde{X} \rightarrow Y$ ) and a surjective  $S$ -(resp.  $X$ -)morphism  $\alpha: \tilde{Y} \rightarrow Y$  (resp.  $\beta: \tilde{X} \rightarrow X$ ). Then the natural map  $\gamma: \tilde{X} \times_Y \tilde{Y} \rightarrow S$  is locally Kähler by (2.2) 4). Moreover there is a natural surjective  $S$ -morphism  $\tilde{X} \times_Y \tilde{Y} \rightarrow X$ , which proves 5). Let  $\tilde{g}: \tilde{Y} \rightarrow S$  and  $\alpha: \tilde{Y} \rightarrow Y$  be as above. Then  $\tilde{Y}_{\tilde{S}} \rightarrow \tilde{S}$  is locally Kähler by (2.2) 2) and there is a natural surjective  $\tilde{S}$ -morphism  $\tilde{Y}_{\tilde{S}} \rightarrow Y_{\tilde{S}}$ . This proves 6). 7) then follows from 5) and 6) as in the proof of (2.2) 4).

### § 3. Irreducibility of the general fiber of a morphism

(3.1) Let  $f: X \rightarrow Y$  be a finite surjective morphism of reduced complex spaces. Then we call  $f$  a *finite (ramified) covering* if each irreducible component of  $X$  is mapped surjectively onto some irreducible component of  $Y$ .

LEMMA 1. *Let  $\beta: X \rightarrow Y$  be a finite covering of reduced complex spaces with  $Y$  irreducible. Then there are a normal complex space  $\tilde{X}$  and a finite covering  $\gamma: \tilde{X} \rightarrow Y$  such that the induced morphism  $\beta_{\tilde{X}}^*: (X \times_Y \tilde{X})^* \rightarrow \tilde{X}$  is biholomorphic to the natural projection  $E \times \tilde{X} \rightarrow \tilde{X}$ , where  $(X \times_Y \tilde{X})^*$  is the normalization of  $X \times_Y \tilde{X}$ , and  $E$  is a finite set considered as a 0-dimensional reduced complex space.*

*Proof.* Replacing  $X$  and  $Y$  by their normalizations  $X'$  and  $Y'$  respectively, and then considering separately the finite coverings  $\beta_i: X_i \rightarrow Y'$  induced by  $\beta$  between the irreducible components  $X'_i$  of  $X'$  and  $Y'$ , we infer readily that we may assume that both  $X$  and  $Y$  are normal and irreducible. Then by the argument in [21, p. 62] we can find a normal complex space  $\tilde{X}$ , a finite group  $G$  of biholomorphic automorphisms of  $\tilde{X}$  and a subgroup  $H$  of  $G$  such that we have the natural isomorphisms  $h: X \cong \tilde{X}/H$  and  $g: Y \cong \tilde{X}/G$  with  $\beta = g^{-1}\pi h$ , where  $\tilde{X}/H$  and  $\tilde{X}/G$  are the quotients of  $\tilde{X}$  by  $H$  and  $G$  respectively endowed with their natural structures of normal complex spaces, and  $\pi: X/H \rightarrow X/G$  is the natural projection. Then identifying  $\pi$  with  $\beta$  by the above isomorphisms, this implies the lemma as follows. Let  $A$  be the diagonal of  $\tilde{X} \times_{\tilde{X}/G} \tilde{X}$  and let  $G$  act on  $\tilde{X} \times_{\tilde{X}/G} \tilde{X}$  by  $(x_1, x_2) \rightarrow (gx_1, x_2)$  for each  $g \in G$ . Then  $\tilde{X} \times_{\tilde{X}/G} \tilde{X} = \bigcup_{g \in G} gA$  so that  $(\tilde{X} \times_{\tilde{X}/G} \tilde{X})^* \cong \coprod_{g \in G} gA$  and each  $gA$  is mapped isomorphically onto  $\tilde{X}$  by the second projection. Accordingly, we have  $(\tilde{X}/H \times_{\tilde{X}/G} \tilde{X})^* \cong \coprod_{g \in E} (\hat{\pi} \times \text{id}_{\tilde{X}})(gA) \cong \coprod_{g \in E} gA \cong E \times \tilde{X}$  where  $\hat{\pi}: \tilde{X} \rightarrow \tilde{X}/H$  is the natural projection and  $E$  is any complete set of representatives of  $G/H$  in  $G$ .

Q.E.D.

(3.2) Let  $f: X \rightarrow S$  be a proper surjective morphism of reduced complex spaces. In what follows the ‘general’ fiber of  $f$  is always considered with respect to the Zariski topology of  $S$ . For example ‘the general fiber of  $f$  is reduced and irreducible’ means that  $X_s$  is reduced and irreducible for every  $s \in U$  for some dense Zariski open subset  $U$  of  $S$ .

**PROPOSITION 2.** *Let  $f: X \rightarrow S$  be as above. Then there exist a finite surjective morphism  $\beta: \tilde{S} \rightarrow S$  with  $\tilde{S}$  reduced, and a reduced analytic subspace  $\tilde{X}$  of  $X \times_S \tilde{S}$  such that if  $\tilde{f}: \tilde{X} \rightarrow \tilde{S}$  and  $\alpha: \tilde{X} \rightarrow X$  are the naturally induced morphisms, then 1) the irreducible components of  $\tilde{X}$  are mutually disjoint, 2)  $\alpha$  is bimeromorphic, and in particular every irreducible component of  $\tilde{X}$  is mapped bimeromorphically onto an irreducible component of  $X$  and 3) the general fiber of  $\tilde{f}$  is reduced and irreducible. Moreover if  $f$  is flat, then we can take  $\beta$  to be a finite covering.*

*Proof.* Let  $\nu: X' \rightarrow X$  be the normalization of  $X$  and let  $f\nu = \beta g$  with  $g: X' \rightarrow \tilde{S}$  and  $\beta: \tilde{S} \rightarrow S$  be the Stein factorization of  $f\nu: X' \rightarrow S$ . Then we set  $\tilde{X} = (\nu \times g)(X') \cong X \times_S \tilde{S}$ , and define  $\alpha$  and  $\tilde{f}$  as above. Then clearly  $\beta$  is finite surjective and 2) is satisfied. We shall show 1). Suppose that  $\tilde{X}_i \cap \tilde{X}_j \neq \emptyset$  for some distinct irreducible components  $\tilde{X}_i$  and  $\tilde{X}_j$  of  $\tilde{X}$ .

Let  $\tilde{x} \in \tilde{X}_i \cap \tilde{X}_j$  be any point and  $\tilde{s} = g(\tilde{x})$ . Then if  $X'_i$  and  $X'_j$  are the irreducible components of  $X'$  with  $(\nu \times g)(X'_i) = \tilde{X}_i$  and  $(\nu \times g)(X'_j) = \tilde{X}_j$  respectively, then we have  $X'_{i,\tilde{s}} \neq \emptyset$  and  $X'_{j,\tilde{s}} \neq \emptyset$ . Since  $X'_{\tilde{s}}$  is connected by the definition of Stein factorization, this implies that there is some irreducible component  $X'_k \neq X'_i$  of  $X'$  such that  $X'_{i,\tilde{s}} \cap X'_{k,\tilde{s}} \neq \emptyset$ . This is a contradiction since  $X'$  is normal. Hence 1) is proved. Then the reducedness of the general fiber of  $\tilde{f}$  follows from [9, Lemma 1.5]. So it remains to show that the general fiber of  $\tilde{f}$  is irreducible. This in turn follows from that of  $g$ , and the latter can be seen as follows. Let  $r: X^* \rightarrow X'$  be a resolution of  $X'$  and  $g^* = gr: X^* \rightarrow S$ . Then there is a dense Zariski open subset  $V$  of  $S$  such that  $g_V^*: X_V^* \rightarrow V$  is smooth, and hence irreducible since each fiber of  $g^*$  is connected as well as that of  $g$ ,  $X'$  being normal. Hence  $X_s' = r(X_s^*)$  are also irreducible for all  $s \in V$ . Q.E.D.

*Remark.* In the above proof, to show the irreducibility of the general fiber of  $g$ , instead of resolution we can also use the fact that if  $h: X \rightarrow S$  is a proper morphism with  $X$  normal, then the set  $\{s \in S; X_s \text{ is normal and } f \text{ is flat at each point of } X_s\}$  is dense and Zariski open in  $S$ , which can be shown as in [9, Lemmas 1.4, 1.5] starting from a result of [2].

(3.3) We shall show that a general fiber of a proper flat morphism is irreducible if at least one fiber is reduced and irreducible. Though the result is not absolutely necessary for the proof of Theorem, it provides us with a useful criterion for the applicability of Proposition 4 in §4. First we need some lemmas.

**LEMMA 2.** *Let  $f: X \rightarrow Y$  be a proper morphism of complex spaces and  $y \in Y$ . Then  $f$  is flat at each point of  $X$  if and only if for any morphism  $h: D \rightarrow Y$  with  $h(0) = y$ , the induced morphism  $f_D: X_D \rightarrow D$  is flat at each point of  $X_{D,0}$  where  $D = \{t \in C; |t| < 1\}$  is the unit disc and  $0 \in D$  is the origin.*

*Proof.* This is an immediate consequence of the existence of ‘platificateur’ in [16, Th. 1] (cf. also [14, Th. 2.4]). We shall also give a direct proof of the lemma in the Appendix.

**COROLLARY.** *Let  $f: X \rightarrow Y$  be a proper surjective morphism of reduced and irreducible complex spaces. Let  $y \in Y$ . Suppose that  $X_y$  is reduced and irreducible, and that  $\dim X_y = \dim X - \dim Y$ . Then  $f$  is flat at every point of  $X_y$ .*

*Proof.* It suffices to show that for any  $h: D \rightarrow Y$  with  $h(0) = y, f_D: X_D \rightarrow D$  is flat along  $X_{D,0}$ . Since  $X_{D,0} \cong X_y$  is reduced and irreducible, by Nakayama we may assume that  $X_D$  is reduced. Let  $X_{Di}, 1 \leq i \leq m$ , be the irreducible components of  $X_D$ . Restricting  $D$  smaller we may further assume that  $f_D(X_{Di}) = D$  or  $\{0\}$  for each  $i$ . Since  $X_{D,0}$  is reduced and irreducible, if  $f_D(X_{Di}) = \{0\}$  for some  $i$ , we must have  $X_{Di} = X_{D,0}$ , and  $i$  is unique, say  $i = m$ . Note that since  $f_D$  is surjective,  $m > 1$ . Then for  $1 \leq i < m$  we have  $\dim X_{Di,t} \leq \dim X_{Di,0} < \dim X_{D,0}$ , and hence  $\dim X_{D,t} < \dim X_{D,0}$ , or  $\dim X_{h(t)} \leq \dim X_y$ , for  $t \neq 0$ . On the other hand, our dimensional assumption implies that  $\dim X_y = \dim X_{y'}$  for all  $y'$  sufficiently near to  $y$  since  $X$  is irreducible. This is a contradiction. Hence  $f(X_{Di}) = D$  for all  $i$  so that  $f_D$  is flat along  $X_{D,0}$ . Q.E.D.

**LEMMA 3.** *Let  $f: X \rightarrow S$  be a proper flat morphism of complex spaces. Suppose that  $S$  is reduced and irreducible. Suppose further that for some  $o \in S, X_o$  is reduced and pure dimensional. Then  $X$  also is reduced and pure dimensional.*

*Proof.* Since  $X_o$  is reduced, by Nakayama and the flatness of  $f$  we infer readily that  $X$  is reduced (cf. the proof of [9, Lemma 1.4]). To show the pure dimensionality it suffices to show that there is no irreducible component, say  $X_1$ , of  $X$  such that if  $q_1$  is the dimension of the general fiber of the induced map  $X_1 \rightarrow S$ , then  $q_1 < q_o = \dim X_o$ . Suppose that such an  $X_o$  exists. Let  $S_k(f) = \{x \in X; \text{codh}_x X_{f(x)} \leqq k\}$ . Then  $S_k(f)$  is an analytic subset of  $X$  by [2]. Hence  $S_{q_1}(f) \supseteq X_1, X_1$  being reduced, and so  $\dim S_{q_1}(f) \geqq q_1$ . Since  $S_{q_1}(f)_o = \{x \in X_o; \text{codh}_x X_o \leqq q_1\}$ , this implies that on  $X_o$  there is a nonzero holomorphic function  $\varphi$  with support of dimension  $\leqq q_1$  (cf. [3, p. 76, Cor. 5.2 d)  $\rightarrow$  b]) applied to  $\mathcal{F} = \mathcal{O}_X$  and  $d = q_1$ ). This is a contradiction to the reducedness and pure dimensionality together of  $X_o$ . Hence  $X$  is pure dimensional. Q.E.D.

**PROPOSITION 3.** *Let  $f: X \rightarrow S$  be a proper flat and surjective morphism of complex spaces. Suppose that  $S$  is reduced and irreducible. Suppose further that for some  $o \in S$  the fiber  $X_o$  is reduced and irreducible. Then the general fiber of  $f$  is irreducible.*

*Proof.* By Lemma 3  $X$  is reduced and pure dimensional. Apply Proposition 2 to  $f$  and obtain a proper surjective morphism  $\tilde{f}: \tilde{X} \rightarrow \tilde{S}$  and finite coverings  $\alpha: \tilde{X} \rightarrow X$  and  $\beta: \tilde{S} \rightarrow S$  with  $\beta \circ \tilde{f} = f \circ \alpha$  satisfying the properties stated in the proposition. Let  $\beta^{-1}(o) = \{\tilde{o}_1, \dots, \tilde{o}_m\}$ . Then it suffices to

show that  $\beta$  is locally biholomorphic at each  $\tilde{o}_k$  and that  $m = 1$ . In fact, then  $\beta$  must be bimeromorphic and hence the irreducibility of the general fiber of  $f$  follows from that of  $\tilde{f}$  together with the surjectivity of  $\alpha$ . Now to prove the above assertion first we note that since  $f$  is flat,  $X$  is pure dimensional and  $S$  is irreducible, every fiber of  $f$  is pure dimensional of dimension  $q = \dim X_o$ , and, further, since  $X_o$  is reduced and irreducible, every irreducible component of  $X$  contains  $X_o$ . Combining this with 2) of Proposition 2 and the fact that  $\alpha|_{\tilde{X}_{\tilde{s}}}: \tilde{X}_{\tilde{s}} \rightarrow X_{\beta(\tilde{s})}$  is an embedding for each  $\tilde{s} \in \tilde{S}$ , we get that  $\alpha$  induces the isomorphisms  $\tilde{X}_{\tilde{o}_k} \cong X_o$  for all  $k$ . This then implies that  $\beta$  is locally biholomorphic at each  $\tilde{o}_k$ , for otherwise  $\tilde{X}_{\tilde{o}_k}$  is nonreduced at each of its points since so is  $\tilde{S}$  at  $\tilde{o}_k$  already. By Corollary above it also follows from  $\tilde{X}_{\tilde{o}_k} = X_o$ , that  $\tilde{f}$  is flat in a neighborhood of  $\tilde{X}_{\tilde{o}_k}$  for each  $k$ . Now we need the following result from [9, Cor. 3.3]; let  $g: Y \rightarrow Z$  be a proper flat morphism of reduced complex spaces. Suppose that every fiber of  $g$  has pure dimension  $q$  which is independent of  $z$ . For any  $z \in Z$  let  $Y_{z,i}$ ,  $i = 1, \dots, n = n(z)$ , be the irreducible components of  $Y_{z,\text{red}}$  and  $m_{z,i}$  the multiplicities of  $Y_z$  along  $Y_{z,\text{red}}$  (cf. [9, 3.1]). Then for any continuous  $(q, q)$ -form  $\chi$  on  $X$  the function

$$\lambda_\chi(z) = \sum_{i=1}^n m_{z,i} \int_{Y_{z,i}} \chi$$

is a continuous function on  $Z$ . Using this we shall now show that  $m = 1$ . Let  $\omega$  be any Hermitian  $(1, 1)$ -form on  $X$  (cf. [9, Def. 1.2]) and set  $\chi = \omega \wedge \cdots \wedge \omega$  ( $q$ -times) and  $\tilde{\chi} = \alpha^* \chi$ . Then  $\lambda_\chi(s)$  (resp.  $\lambda_{\tilde{\chi}}(\tilde{s})$ ) are functions which are defined on  $S$  (resp.  $\tilde{S}$ ) and continuous in a neighborhood of  $o$  (resp.  $\beta^{-1}(o)$ ) by the result quoted above. Let  $U$  (resp.  $\tilde{U}_k$ ) be a neighborhood of  $o$  (resp.  $\tilde{o}_k$ ) such that  $\beta$  induces isomorphisms  $\beta_k: \tilde{U}_k \cong U$  for each  $k$ . For any  $s \in U$  we write  $\tilde{s}_k = \beta_k^{-1}(s)$ . On the other hand, since  $\alpha$  is bimeromorphic, there is a dense Zariski open subset  $V$  of  $U$  such that for each  $s \in V$ ,  $\tilde{X}_{\tilde{s}_i}$ ,  $1 \leq i \leq m$ , are reduced and irreducible and  $\alpha(\tilde{X}_{\tilde{s}_k}) \neq \alpha(\tilde{X}_{\tilde{s}_\ell})$  if  $k \neq \ell$ . Hence noting that  $X_s = \bigcup_k \tilde{X}_{\tilde{s}_k}$  we have  $\lambda_\chi(s) = \sum_{k=1}^m \lambda_{\tilde{\chi}}(\tilde{s}_k)$  for every  $s \in V$ . Now take a sequence  $\{s^{(i)}\}$  of points of  $V$  converging to  $o$  in  $U$ . Then since  $\lambda_\chi$  (resp.  $\lambda_{\tilde{\chi}}$ ) is continuous at  $o$  (resp.  $\tilde{o}_k$ ), we get that  $\lambda_\chi(o) = \lim_i \lambda_\chi(s^{(i)}) = \lim_i \sum_{k=1}^m \lambda_{\tilde{\chi}}(\tilde{s}_k^{(i)}) = \sum_{k=1}^m \lambda_{\tilde{\chi}}(\tilde{o}_k)$ . Since

$$\lambda_\chi(o) = \int_{X_o} \chi = \int_{\tilde{X}_{\tilde{o}_k}} \tilde{\chi} = \lambda_{\tilde{\chi}}(\tilde{o}_k),$$

this implies that  $m = 1$ , for  $\int_{X_o} \chi > 0$ .

Q.E.D.

#### § 4. Moishezonness of $\pi_A$ in a special case

(4.1) Let  $f: X \rightarrow S$  be a proper morphism of complex spaces. Let  $\beta_{X/S}: D_{X/S} \rightarrow S$  be the relative Douady space of  $X$  over  $S$  parametrizing analytic subspaces of  $X$  contained in the fibers of  $f$  (cf. [17], [9]). Let  $\rho_{X/S}: Z_{X/S} \rightarrow D_{X/S}$  be the corresponding universal family, so that there is a natural embedding  $Z_{X/S} \subseteq D_{X/S} \times_S X$  with  $\rho_{X/S}$  induced by the natural projection  $p_1: D_{X/S} \times_S X \rightarrow D_{X/S}$ . We denote by  $\pi_{X/S}$  the natural morphism  $Z_{X/S} \rightarrow X$  induced by the projection  $p_2: D_{X/S} \times_S X \rightarrow X$ . Then  $\pi_{X/S}$ , restricted to each fiber of  $\rho_{X/S}$ , is an embedding. Let  $\alpha: \tilde{S} \rightarrow S$  be a morphism of complex spaces with  $\tilde{S}$  reduced and  $Z \subseteq \tilde{S} \times_S X$  a subspace. Let  $\rho: Z \rightarrow \tilde{S}$  be the natural projection. If  $\rho$  is flat, then we call  $\rho$  a flat family of subspaces of  $X$  over  $S$  parametrized by  $\tilde{S}$ . In the general case, by Frisch [8] there is a dense Zariski open subset  $W$  of  $\tilde{S}$  such that  $\rho_w: Z_w \rightarrow W$  is flat. (In what follows we use this result of Frisch without further reference.) Then there is a unique  $S$ -morphism  $\tau: W \rightarrow D_{X/S}$  such that  $\rho_w$  is isomorphic to the map induced from  $\rho_{X/S}$  via  $\tau$ , where  $W$  is over  $S$  by  $\alpha|_W$ . We call such a map  $\tau$  simply the *universal S-map associated to  $\rho_w$* .

Now we recall the following consequence of Hironaka's flattening theorem [14] which is of frequent use in the sequel.

LEMMA 4. *The universal S-map  $\tau$  extends to a meromorphic S-map  $\tau^*: \tilde{S} \rightarrow D_{X/S, \text{red}}$ . In particular if  $\alpha$  is proper, then the closure of  $\tau(W)$  in  $D_{X/S, \text{red}}$  is an analytic subspace of  $D_{X/S, \text{red}}$  which is proper over  $S$ .*

*Proof.* See [9, Lemma 5.1].

(4.2) In the case of a projective morphism a special way of constructing  $D_{X/S}$  is available by Grothendieck [12], [13]; what we need here from his construction is the following:

LEMMA 5. *Let  $f: X \rightarrow S$  be a projective morphism and  $\beta_{X/S}: D_{X/S} \rightarrow S$  be the relative Douady space of  $X$  over  $S$ . Let  $Q$  be any relatively compact open subset of  $S$  and  $A$  any connected component of  $\beta_{X/S}^{-1}(Q)_{\text{red}}$ . Then the induced morphism  $h: A \rightarrow Q$  is projective.*

*Proof.* Let  $Q'$  be any relatively compact open subset of  $S$  with  $Q \subset Q'$ . Let  $\mathcal{L}$  be an  $f_{Q'}$ -very ample invertible sheaf on  $X$  such that  $f_* \mathcal{L}$  is locally free on  $Q'$ . So we have an  $Q'$ -embedding  $j: X_{Q'} \rightarrow P(f_* \mathcal{L})_{Q'}$  with  $\mathcal{L} \cong j^* \mathcal{O}_P(1)$ ,  $P = P(f_* \mathcal{L})$ . Then replacing  $S$  by  $Q'$  we may assume that  $X = P(\mathcal{E})$  for some locally free coherent analytic sheaf  $\mathcal{E}$  on  $S$ . Now for

$d \in D_{X/S}$  write  $Z_d = Z_{X/S, d}$  and consider  $Z_d \subseteq X_{\beta(d)} \cong \mathbf{CP}^{r-1}$  by  $\pi_{X/S}$ , where  $\beta = \beta_{X/S}$  and  $r = \text{rank } \mathcal{E}$ . For every  $d \in D_{X/S}$  define a polynomial  $P_d = P_d(n)$  in  $n$  by  $P_d = \sum_{i \geq 0} (-1)^i H^i(Z_d, \mathcal{O}_{Z_d}(n))$ . Then  $P_d$  is independent of  $a \in A$  (cf. [3]) and we set  $P_A = P_a$  for any  $a \in A$ . Set  $\tilde{A} = \{d \in D_{X/S, \text{red}}; P_d = P_A\}$ . Then  $A$  is a connected component of  $\tilde{A}_Q$ . Hence it suffices to show that  $\tilde{A}_Q$  is projective over  $Q$ . In fact, the proof of [13, IX, Théorème 1.1] (and [12, 221, § 3]) shows that for each point  $s \in S$  there exists a neighborhood  $s \in U$  in  $S$  and an integer  $\nu_0 = \nu_0(s)$  such that for all  $\nu \geq \nu_0$  the natural map  $\beta^* f_* \mathcal{O}_X(\nu) \cong p_{1*} p_2^* \mathcal{O}_X(\nu) \rightarrow \rho_{X/S*} \mathcal{O}_{Z_{X/S}}(\nu)$ ,  $\mathcal{O}_{Z_{X/S}}(\nu) = \pi_{X/S}^* \mathcal{O}_X(\nu)$ , is surjective on  $\tilde{A}_U$  and the corresponding morphism  $\tilde{A}_U \rightarrow \text{Grass}_m(f_* \mathcal{O}_X(\nu))_U$  is a closed embedding over  $U$ , where  $\text{Grass}_m(f_* \mathcal{O}_X(\nu))$  is the Grassmann variety of locally free quotients of  $f_* \mathcal{O}_X(\nu)$  of rank  $m$ , where  $m = m(\nu) = \text{rank}(\rho_{X/S*} \mathcal{O}_{Z_{X/S}}(\nu))$  [13]. Hence for all sufficiently large  $\nu$ ,  $\tilde{A}_Q$  can be embedded in  $\text{Grass}_m(f_* \mathcal{O}_X(\nu))_Q$  over  $Q$  and hence is projective over  $Q$ . Q.E.D.

(4.3) Let  $f: X \rightarrow S$  be a morphism of complex spaces. Let  $\beta_{X/S}: D_{X/S} \rightarrow S$ ,  $\rho_{X/S}: Z_{X/S} \rightarrow D_{X/S}$  and  $\pi_{X/S}: Z_{X/S} \rightarrow X$  be as in (4.1). For any locally closed analytic subspace  $A$  of  $D_{X/S, \text{red}}$  we shall denote by  $\rho_A: Z_A \rightarrow A$  the restriction of  $\rho_{X/S}$  to  $Z_A = \rho_{X/S}^{-1}(A)$  and  $\pi_A: Z_A \rightarrow X$  the  $S$ -morphism induced by  $\pi_{X/S}$ , where  $Z_A$  is over  $S$  by  $\beta_{X/S} \rho_A$ .

**PROPOSITION 4.** *Let  $f: X \rightarrow S$  be a proper morphism of complex spaces and  $Q$  a relatively compact open subset of  $S$ . Let  $A$  be a reduced and irreducible analytic subspace of  $\beta_{X/S}^{-1}(Q)$  which is proper over  $Q$  and for which the general fiber of  $\rho_A$  is reduced and irreducible. Then  $\pi_A$  is Moishezon.*

*Proof.* Changing the notation we set  $S = Q$ ,  $X = X_Q$  and  $f = f_Q$  so that  $A$  is an analytic subspace of  $D_{X/S}$ . (The original  $X, S$  and  $f$  do not appear explicitly in the following, so no confusion may arise). We first note that from our assumption it follows immediately that  $Z_A$  is reduced and irreducible. Consider the  $Z_A$ -embedding  $j: Z_A \times_A Z_A \subseteq Z_A \times_X (X \times_S X)$  defined by  $j(z_1, z_2) = (z_1, \pi_A(z_1), \pi_A(z_2))$  where  $Z_A \times_A Z_A$  (resp.  $X \times_S X$ ) is over  $Z_A$  (resp.  $X$ ) with respect to the projection to the first factor. Note that  $j$  is in fact obtained by the composition  $Z_A \times_A Z_A \subseteq Z_A \times_A (A \times_S X) \cong Z_A \times_S X \cong Z_A \times_X (X \times_S X)$  where the isomorphisms are all natural ones. On the other hand, let  $\Delta = \Delta_{X/S}$  be the diagonal of  $X \times_S X$  and  $\mathcal{I}$  the sheaf of ideals of  $\Delta$  in  $X \times_S X$ . Let  $\Delta_{(n)} = (\Delta, \mathcal{O}_{X \times_S X}/\mathcal{I}^{n+1})$  be the  $n$ -th infinitesimal neighborhood of  $\Delta$  in  $X \times_S X$ , and  $\beta_n: \Delta_{(n)} \rightarrow X$  be induced by the projection  $X \times_S X \rightarrow X$  to the first factor. Then  $\beta_n$  are finite, and

hence projective, morphisms. Let  $\delta_n: D_{(n)} \rightarrow X$  with  $D_{(n)} = D_{A_{(n)}/X}$  be the relative Douady spaces associated to  $\beta_n$ . Then for any connected component  $D_{(n),k}$  of  $D_{(n)}$  the induced morphism  $D_{(n),k} \rightarrow X$  is projective by Lemma 5 since  $f$  is proper.

Let  $Y_{(n)} = (Z_A \times_A Z_A) \cap (Z_A \times_X A_{(n)}) \subseteq Z_A \times_X A_{(n)}$  and  $\gamma_n: Y_{(n)} \rightarrow Z_A$  be the natural  $S$ -morphisms induced by the projections  $Z_A \times_X A_{(n)} \rightarrow Z_A$ , where the intersection is taken in  $Z_A \times_X (X \times_S X)$  considering  $Z_A \times_A Z_A$  as a subspace of  $Z_A \times_X (X \times_S X)$  via  $j$ . Then  $\gamma_n$  are finite surjective morphisms, the fibers over  $z \in Z_A$  being naturally identified with the subspace  $B_{z,n} = \pi_A(Z_{A,a}) \cap x_{(n)}$  of  $x_{(n)} = (x, \mathcal{O}_X/m_x^{n+1})$  where  $a = \rho_A(z)$ ,  $x = \pi_A(z)$  and  $m_x$  is the maximal ideal of  $\mathcal{O}_X$  at  $x$ . Now for each  $n$  there is a dense Zariski open subset  $U_n$  of  $Z_A$  such that  $\gamma_{n,U_n}: Y_{(n),U_n} \rightarrow U_n$  is flat, so that it may be considered as a flat family of subspaces of  $A_{(n)}$  over  $X$  parametrized by  $U_n$ . Let  $\tau_n: U_n \rightarrow D_{(n)}$  be the universal  $X$ -map associated to  $\gamma_{n,U_n}$  (cf. (4.1)). Then by Lemma 4  $\tau_n$  extends to a meromorphic  $X$ -map  $\tau_n^*: Z_A \rightarrow D_{(n)}$  and the closure  $E_n$  of  $\tau_n(U_n)$  in  $D_{(n)}$  is analytic in  $D_{(n)}$  and is proper over  $S$  as well as  $Z_A$ .

Now we shall show that (\*)  $\tau_n^*$  are bimeromorphic  $X$ -maps onto its image for all sufficiently large  $n$ . Then since  $\pi_A$  is proper and the images of  $\tau_n^*$  are contained in some  $D_{(n),k}$ ,  $Z_A$  being irreducible, this would imply that  $\pi_A: Z_A \rightarrow X$  is Moishezon by (1.7) 5), completing the proof of the proposition. To show (\*) we first observe the following: (') If  $z, z' \in U_m \cap U_n$  and if  $m \leq n$ , then  $\tau_n(z) = \tau_n(z')$  implies that  $\tau_m(z) = \tau_m(z')$ . In fact, for  $z \in U_n$ ,  $\tau_n(z)$  is the point of  $D_{(n)}$  corresponding to the subspace  $B_{z,n}$  of  $x_{(n)}$  defined above, and that  $B_{z,n} = B_{z',n}$  clearly implies that  $B_{z,m} = B_{z',m}$ . This shows (''). Now since  $Z_A$  is irreducible, for each  $n$  there exist an integer  $d_n \geq 0$  and a dense Zariski open subset  $V_n$  of  $U_n$  such that  $\dim_z \tau_n^{-1}\tau_n(z) = d_n$  for all  $z \in V_n$ . Then we see that  $d_n \leq d_m$  for  $m \leq n$  by (''). Hence there are integers  $n_0 > 0$  and  $d \geq 0$  such that  $d_n = d$  for all  $n \geq n_0$ .

Next we show that  $d = 0$ . Let  $W$  be a dense Zariski open subset of  $A$  such that  $Z_{A,a}$  is reduced and irreducible for all  $a \in W$ . Let  $V = \bigcap_n V_n$ . Then  $V$  is everywhere dense in  $Z_A$ . Suppose now that  $d > 0$ . Then there exist points  $z, z' \in V \cap \rho_A^{-1}(W)$ ,  $z \neq z'$ , such that  $z'$  belongs to an irreducible component  $C$  of  $\tau_{n_0}^{-1}\tau_{n_0}(z)$  containing  $z$ . (In particular  $\pi_A(z) = \pi_A(z')$ .) Then since both  $Z_{A,\rho_A(z)}$  and  $Z_{A,\rho_A(z')}$  are reduced and irreducible, there is an integer  $n_1 \geq n_0$  such that  $B_{z,n_1} \neq B_{z',n_1}$ , or equivalently,  $\tau_{n_1}(z) \neq \tau_{n_1}(z')$ . Hence

$\tau_{n_1, C} = \tau_{n_1}|_{C \cap U_{n_1}}$  is nontrivial, i.e., the fibers of  $\tau_{n_1, C}$  have dimension  $< d = \dim C$ . Note here that  $C \cap U_{n_1} \supseteq C \cap V \neq \emptyset$ . On the other hand, by ('')  $\tau_{n_1, C}^{-1}\tau_{n_1}(z)$  is one of the irreducible components of  $\tau_{n_1}^{-1}\tau_{n_1}(z)$  at  $z$  and hence there is an irreducible component  $C' \subseteq C$  of  $\tau_{n_1}^{-1}\tau_{n_1}(z)$  containing  $z$ , so that  $\dim_{z''} \tau_{n_1}^{-1}\tau_{n_1}(z'') < d$  for some  $z'' \in C'$ . This implies that  $d_{n_1} < d$  by the upper semi-continuity of the function  $\dim_z \tau_{n_1}^{-1}\tau_{n_1}(z)$ ,  $z \in U_{n_1}$ , which is a contradiction. Hence we get that  $\tau_n$  is generically finite for  $n \geq n_0$ .

Thus for each  $n \geq n_0$ , there exist an integer  $k_n > 0$  and a dense Zariski open subset  $Q_n$  of  $E_n$  contained in  $\tau_n(U_n)$  such that  $\tau_{n, Q_n}: \tau_n^{-1}(Q_n) \rightarrow Q_n$  is an unramified covering of degree  $k_n$ . Here one needs to recall that  $\tau_n$  extends to a meromorphic  $X$ -map from  $Z_A$  to  $E_n$  which are both proper over  $X$ . Then again by (''),  $k_n \leq k_m$  if  $n \geq m$  so that  $k_n = k$  for all  $n \geq n_2$  for some  $k \geq 1$  and  $n_2 \geq n_0$ . We show that  $k = 1$ . Let  $\tilde{Q} = \bigcap_n \tau_n^{-1}(Q)$  which is everywhere dense in  $Z_A$ . For  $\tilde{q} \in \tilde{Q}$ ,  $\tau_n^{-1}\tau_n(\tilde{q})$ , as a set, is independent of  $n \geq n_2$ . Suppose that  $k > 1$ . Then there are points  $z, z' \in \tilde{Q} \cap \rho_A^{-1}(W)$ ,  $z \neq z'$ , such that  $\tau_{n_2}(z) = \tau_{n_2}(z')$ . Then by the same argument as above we can find  $n > n_2$  such that  $\tau_n(z) \neq \tau_n(z')$ , implying that  $k_n < k$ , since  $z, z' \in \tilde{Q}$ . This contradicts our choice of  $n_2$ . Hence  $k = 1$ , i.e.,  $\tau_n^*$  is  $X$ -bimeromorphic onto its image for all  $n \geq n_2$ , and (\*) is proved. Q.E.D.

*Remark.* A meromorphic map  $g: Y \rightarrow Y'$  of reduced complex spaces is called generically light if there is a dense Zariski open subset  $U \subseteq \Gamma$  such that  $\dim_\gamma q^{-1}q(\Gamma) = 0$  for every  $\gamma \in U$  where  $\Gamma$  is the graph of  $g$  and  $q: \Gamma \rightarrow Y'$  is the natural projection (cf. (1.1)). Then the above proof shows that even in the general case where  $A$  may not be proper over  $S$ , there is a generically light meromorphic  $X$ -map  $\lambda: Z_A \rightarrow B$  of complex spaces over  $X$  with  $B$  projective over  $X$ .

## § 5. Reduction of the general case and proof of Theorem

(5.1) We use the notation of (4.3).

**PROPOSITION 5.** *Let  $f: X \rightarrow S$  be a morphism of complex spaces. Let  $A$  be an irreducible component of  $D_{X/S, \text{red}}$  which is proper over  $S$  and for which  $Z_A$  is reduced. Then there exist 1) reduced and irreducible analytic subspaces  $B_i$ ,  $i = 1, \dots, n$ , of  $D_{X/S}$  such that  $B_i$  is proper over  $S$  and the general fiber of  $\rho_{B_i}: Z_{B_i} \rightarrow B_i$  is reduced and irreducible, 2) a reduced and irreducible analytic subspace  $B$  of  $\hat{B} = B_1 \times_S \dots \times_S B_n$  and 3) a generically*

*surjective meromorphic S-map*  $h: B \rightarrow A$ .

*Proof.* We write  $Z = Z_A$  and  $\rho = \rho_A$ . Let  $\tilde{\rho}: \tilde{Z} \rightarrow \tilde{A}$ ,  $\alpha: \tilde{Z} \rightarrow Z$  and  $\beta: \tilde{A} \rightarrow A$  be as in Proposition 2 applied to  $f = \rho$ . In particular  $\tilde{Z} \subseteq \tilde{A} \times_A Z$  with  $\tilde{\rho}$  induced by the natural projection  $\tilde{A} \times_A Z \rightarrow \tilde{A}$ . Further since  $\rho$  is flat, we may assume that  $\beta$  is a finite covering. Moreover for each  $s \in A$ ,  $Z_s = \bigcup_{\tilde{s} \in \beta^{-1}(s)} \alpha(\tilde{Z}_{\tilde{s}})$  by 2) of the proposition. Then we apply Lemma 1 to  $\beta$  and obtain a normal complex space  $A'$ , a finite covering  $\gamma: A' \rightarrow A$  and an  $A'$ -isomorphism  $\lambda: (\tilde{A} \times_A A')^* \cong E \times A'$ , where  $(\tilde{A} \times_A A')^*$  is the normalization of  $\tilde{A} \times_A A'$  and  $E$  is a finite set considered as a zero dimensional analytic space. Write  $A^* = (\tilde{A} \times_A A')^*$ . Let  $\rho^*: Z^* \rightarrow A^*$  be the pull-back of  $\tilde{\rho}$  to  $A^*$  with respect to the natural projection  $A^* \rightarrow \tilde{A}$ . Identifying  $E$  with  $\{1, \dots, n\}$ ,  $n = \#E$ , in a certain fixed way and  $A^*$  with  $E \times A'$  via  $\lambda$ , we write for each  $i$ ,  $Z_i^* = \rho^{*-1}(\{i\} \times A')$ , and  $\rho_i^* = \rho^*|_{Z_i^*}: Z_i^* \rightarrow \{i\} \times A' = A'$  and define  $\pi_i^*: Z_i^* \rightarrow X$  to be the natural map. We thus get the following commutative diagram

$$\begin{array}{ccccc} \coprod_i Z_i^* = Z^* & \longrightarrow & \tilde{Z} & & \\ \downarrow \coprod \rho_i^* & \downarrow \rho^* & \downarrow \tilde{\rho} & \searrow \alpha & \\ & & Z & \xrightarrow{\pi_A} & X \\ \coprod_i (A' \times \{i\}) = A^* & \longrightarrow & \tilde{A} & & \\ \downarrow \delta & & \downarrow \beta & \searrow \rho & \\ A' & \xrightarrow{\gamma} & A & & . \end{array}$$

Let  $U$  be any dense Zariski open subset of  $A'$  such that  $\rho_{i,U}^*: Z_{i,U}^* \rightarrow U$  are flat for all  $i$ . Then by the definition of  $\tilde{Z}$  we may consider  $\rho_{i,U}^*$  naturally as a flat family of subspaces of  $X$  over  $S$  parametrized by  $U$ . Let  $\tau_i: U \rightarrow D_{X/S}$  be the universal S-map associated to  $\rho_{i,U}^*$  and  $\tau = \prod_i \tau_i: U \rightarrow D_{X/S} \times_S \cdots \times_S D_{X/S}$  ( $n$ -times). Let  $B$  (resp.  $B_i$ ) be the closure of  $\tau(U)$  (resp.  $\tau_i(U)$ ) in  $\hat{D} = D_{X/S} \times_S \cdots \times_S D_{X/S}$  (resp.  $D_{X/S}$ ). Then by Lemma 4 (cf. also (1.1))  $B$  and  $B_i$  are reduced analytic subspaces of  $\hat{D}$  and  $D_{X/S}$  respectively which are proper over  $S$ . They are irreducible since so is  $U$ , and we have  $B \subseteq \hat{B} = B_1 \times_S \cdots \times_S B_n$  and  $\dim B \leq \dim A$ . Moreover since by 3) of Proposition 2 together with the definition of  $\tau_i$ ,  $Z_{X/S,i}$  is reduced and irreducible for each  $d \in \tau_i(U)$  (after restricting  $U$  if necessary), the general fiber of  $\rho_{B_i}: Z_{B_i} \rightarrow B_i$  is reduced and irreducible. (For instance, since  $\tau_i(U)$  is everywhere dense in  $B_i$  it follows that  $Z_{B_i}$  is reduced and irreducible. Then we can apply Proposition 3.)

Now let  $\rho_B^{(i)}: Z_B^{(i)} \rightarrow B$  be the pull-back of  $\rho_{B_i}$  with respect to the map

$B \rightarrow B_i$  induced by the natural projection  $\hat{B} \rightarrow B_i$ . Take the union  $\check{Z}_B = \bigcup_i Z_B^{(i)}$  in  $X \times_S B$  ( $\check{Z}_B = Z_{B_i} \times_{B_i} B$ ). Let  $\psi: \check{Z}_B \rightarrow B$  be the natural projection and take any dense Zariski open subset  $V$  of  $B$  such that  $\psi_V: \psi^{-1}(V) \rightarrow V$  is flat. Let  $\tau': V \rightarrow D_{X/S}$  be the universal  $S$ -map associated with  $\psi_V$ . Let  $\pi: \check{Z}_B \rightarrow X$  be the natural map induced by the projection  $X \times_S B \rightarrow X$ . Then from the construction above we have in  $X$  the equality  $\pi(\check{Z}_{B,b}) = \pi_A(Z_{A,\tau(u)})$  for each  $b \in \tau(U)$  where  $u \in U$  is any point with  $\tau(u) = b$ . In fact by 2) of Proposition 2 we have  $\pi_A(Z_{A,\tau(u)}) = \bigcup_{a \in \beta^{-1}\tau(u)} \tilde{\pi}(\tilde{Z}_a) = \bigcup_{a^* \in \delta^{-1}(u)} \pi^*(Z_{a^*}^*) = \bigcup_i \pi_i^*(Z_{i,u}^*) = \pi(\check{Z}_{B,b})$  in  $X$ , where  $\tilde{\pi} = \pi_A \alpha$  and  $\pi^*$  is the composite of  $\tilde{\pi}$  and the natural map  $Z^* \rightarrow \tilde{Z}$ . This implies that  $\tau'\tau|_{\tau^{-1}(V)} = j\gamma|_{\tau^{-1}(V)}$  where  $j: A \rightarrow D_{X/S}$  is the natural inclusion. In particular  $\tau'(V)$  contains  $\gamma(\tau^{-1}(V))$  and hence a nonempty Zariski open subset of  $A$  since  $\tau^{-1}(V) \neq \emptyset$ . Thus the closure  $\tau'(V)^-$  of  $\tau'(V)$  in  $D_{X/S}$ , which is an analytic subset of  $D_{X/S}$  by Lemma 4, contains  $A$  so that  $\dim B \geq \dim A$ . Combining with the opposite inequality noted above we have  $\dim B = \dim A$ , and thus  $\tau'(V)^- = A$ . Hence  $h = \tau'$  is a generically surjective meromorphic  $S$ -map from  $B$  onto  $A$ . Q.E.D.

*Remark.* In fact the above  $h$  is bimeromorphic as one shows readily.

(5.2) THEOREM. *Let  $f: X \rightarrow S$  be a proper morphism of reduced complex spaces, and  $Q$  a relatively compact open subset of  $S$ . Let  $\beta: D_{X/S} \rightarrow S$  be the relative Douady space of  $X$  over  $S$ . Suppose that  $f \in \mathcal{C}/S$  (resp. is Moishezon). Then for any irreducible component  $A$  of  $\beta^{-1}(Q)_{\text{red}}$  such that  $Z_A$  is reduced, the induced morphism  $\beta|_A: A \rightarrow Q$  is proper and again belongs to  $\mathcal{C}/S$  (resp. is Moishezon).*

*Proof.* We shall write  $f \in \mathcal{M}/S$  if  $f$  is Moishezon. First we show that  $A$  is proper over  $S$ . Since  $f \in \mathcal{C}/S$  (resp.  $\mathcal{M}/S$ ), there is a proper and locally Kähler (resp. locally projective) morphism  $g: Y \rightarrow S$  of complex spaces and a surjective  $S$ -morphism  $h: Y \rightarrow X$  (cf. (2.4) 2) and (1.5) 3)). Let  $X' = f^{-1}(Q)$ ,  $f' = f|_{X'}: X' \rightarrow Q$ ,  $Y' = g^{-1}(Q)$  and  $g' = g|_{Y'}: Y' \rightarrow Q$ . Then  $g'$  is Kähler (cf. (2.2) 1)). Hence by [9, Theorem 4.3]  $g'$  has property  $BP$ , i.e., every irreducible component of the relative Barlet space  $B(Y'/Q)$  (cf. [9]) is proper over  $Q$ . Then by [9, Prop. 4.8]  $f'$  also has property  $BP$ , which in turn implies that  $f'$  has property  $\bar{D}P$ , i.e., every irreducible component of  $\bar{D}_{X'/Q}$  is proper over  $Q$ , by [9, Prop. 3.4], where  $\bar{D}_{X'/Q}$  is the union of those irreducible components  $D_r$  of  $D_{X'/Q, \text{red}}$  such that  $Z_r = Z_{D_r}$  are reduced and pure dimensional. Then by [9, Lemma 3.5] and the remark

following it (where  $Z_\alpha$  and  $D_\alpha$  should read  $D_\alpha$  and  $S$  respectively), this further implies that the given  $A$  is proper over  $S$  since  $Z_A$  is reduced.

Now apply Proposition 5 to our  $A$  and obtain  $B \subseteq B_1 \times_Q \cdots \times_Q B_n$  as in that proposition (with  $S$  replaced by  $Q$ ). In particular,  $B_i$  are proper over  $Q$ , the general fiber of  $\rho_{B_i}: Z_{B_i} \rightarrow B_i$  is reduced and irreducible, and there is a generically surjective meromorphic  $S$ -map  $B \rightarrow A$ . The first two facts, together with Proposition 4, shows that  $\pi_{B_i}: Z_{B_i} \rightarrow X'$  is Moishezon,  $1 \leq i \leq n$ . Hence  $f' \rho_{B_i}: Z_{B_i} \rightarrow Q \in \mathcal{C}/Q$  (resp.  $\mathcal{M}/Q$ ) by (2.4) 5) (resp. (1.5) 1)). Then by (2.4) 3) 4) and 7) (resp. (1.6) and (1.7) 5) 7)) the natural map  $B_i \rightarrow Q$ , and hence  $B \rightarrow Q$  also, belong to  $\mathcal{C}/Q$  (resp. are Moishezon). Finally by (2.4) 4) (resp. (1.6))  $\beta|_A: A \rightarrow Q \in \mathcal{C}/Q$  (resp.  $\mathcal{M}/Q$ ).

Q.E.D.

*Remark.* Taking  $S$  to be a point and then setting  $S = Q$ , we obtain the theorem stated in the introduction.

## Appendix

We shall give a direct proof of Lemma 2, in § 3.

Let  $D = \{t \in \mathbf{C}; |t| < 1\}$  be the unit disc. For any complex space  $Y$  and  $y \in Y$  we denote by  $S(Y, y)$  the set of morphisms  $h: D \rightarrow Y$  with  $h(0) = y$ . Let  $f: X \rightarrow Y$  be a morphism of complex spaces and  $y \in Y$ . Then for any  $h \in S(Y, y)$  we write  $X_h$  for  $X \times_Y D$  and  $f_h$  (resp.  $p_h$ ) for the natural projection  $X_h \rightarrow D$  (resp.  $X_h \rightarrow X$ ). Further for any coherent analytic sheaf  $\mathcal{F}$  on  $X$  we denote by  $\mathcal{F}_h$  the  $\mathcal{O}_{X_h}$ -module  $p_h^* \mathcal{F}$ . Then Lemma 2 is a special case of the following:

**PROPOSITION.** *Let  $f: X \rightarrow Y$  be a morphism of complex spaces and  $\mathcal{F}$  a coherent analytic sheaf on  $X$ . Let  $x \in X$  and  $y = f(x)$ . Suppose that  $Y$  is reduced. Then the following conditions are equivalent: 1)  $\mathcal{F}$  is  $f$ -flat at  $x$ . 2) For every  $h \in S(Y, y)$ ,  $\mathcal{F}_h$  is  $f_h$ -flat at  $x_h = (x, 0)$ .*

*Proof.* 2) is clearly a consequence of 1). So suppose that 2) is true. We use an analytic analogue of the technique due to Raynaud and Gruson [19, 2.1]. Let  $S(\mathcal{F})$  be the support of  $\mathcal{F}$  considered as the analytic subspace of  $X$  defined by the ideal sheaf of annihilators of  $\mathcal{F}$ . Then we proceed by induction on  $n = \dim_x(X_y \cap S(\mathcal{F}))$ . First replacing  $X$  by  $S(\mathcal{F})$  if necessary we may assume that  $X = S(\mathcal{F})$ , so that  $n = \dim_x X_y$ . Then there is a neighborhood  $U$  of  $x$  in  $X$  and a commutative diagram of complex spaces

$$\begin{array}{ccc} U & \xrightarrow{\tau} & Y \times \mathbf{C}^n \\ f \searrow & & \downarrow p_1 \\ & & Y \end{array}$$

such that  $\tau$  is finite at  $x$  (cf. [7, 3.3]). Then since  $\mathcal{F}_x$  and  $(\tau_*\mathcal{F})_{\tau(x)}$  are isomorphic as  $\mathcal{O}_{Y,y}$ -modules, we can replace  $f$  and  $\mathcal{F}$  by  $p_1$  and  $\tau_*\mathcal{F}$  respectively. Thus we may assume that  $X = Y \times V$  with  $Y$  Stein and  $V$  a polydisc in  $\mathbf{C}^n$  containing the origin  $0$ ,  $x = (y, 0) \in Y \times \mathbf{C}^n$  and  $f: X \rightarrow Y$  is the natural projection.

SUBLEMMA.  $\mathcal{F}$  is locally free at some point of  $X_y$ .

*Proof.* For any  $a \in X$  we set  $d(a) = \dim_{\mathbf{C}} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X/m_a \mathcal{O}_X$  where  $m_a$  is the maximal ideal of  $\mathcal{O}_X$  at  $a$ . Then  $d(a)$  is upper semicontinuous with respect to the Zariski topology. In particular, if we set  $d_0 = \min \{d(a); a \in X\}$ , then the set  $U_0 = \{a \in A; d(a) = d_0\}$  is Zariski open in  $X$  and  $\mathcal{F}$  is locally free on  $U_0$ . We may assume that  $x \in \overline{U_0}$ , the closure of  $U_0$ . Similarly if we put  $d_{y0} = \min \{d(a); a \in X_y\}$ , then  $U_{y0} = \{a \in X_y; d(a) = d_{y0}\}$  is dense and Zariski open in  $X_y \cong V$ . We show that  $d_0 = d_{y0}$ . Take  $h \in S(Y, y)$  in such a way that  $p_h^{-1}(U_0) \neq \emptyset$ . By our assumption  $\mathcal{F}_h$  is  $f_h$ -flat at  $x_h$  and hence  $f_h$ -flat in some neighborhood  $W$  of  $x_h$ . Since  $D$  is smooth of dimension 1, this is equivalent to saying that  $\mathcal{H}_{X_{h,0}}^0(\mathcal{F}_h) = 0$  on  $W$ . On the other hand, the latter implies that  $\dim S_n(\mathcal{F}_h) < n$  (a special case of a theorem of Trautmann [3, p. 66]) where  $S_n(\mathcal{F}) = \{u \in X; \text{codh}_u \mathcal{F} \leq n\}$ , codh denoting the cohomological dimension. Hence for the general point  $w \in X_{h,0}$ ,  $\text{codh}_w \mathcal{F}_h = n + 1$ , i.e.,  $\mathcal{F}_h$  is locally free at  $w$ . Thus if  $U_h$  is the maximal dense Zariski open subset of  $X_h$  on which  $\mathcal{F}_h$  is locally free, then  $U_h \cap X_y \neq \emptyset$ . Hence if  $r$  is the rank of  $\mathcal{F}_h$  on  $U_h$ , then taking any  $a' \in U_h \cap p_h^{-1}(U_{y0}) \subseteq X_{h,0}$  and  $w' \in U_h \cap p_h^{-1}(U_0)$  we have  $d_{y0} = d(p_h(a')) = d_h(a') = r = d_h(w') = d_0$ , where  $d_h$  is defined for  $\mathcal{F}_h$  in the same way as  $d(a)$ . Hence  $U_{y0} \subseteq U_0$  and  $\mathcal{F}$  is locally free at each point of  $U_{y0} \subseteq X_y$ .

Q.E.D.

By Sublemma there exists a  $v \in V$  such that  $\mathcal{F}$  is free of rank, say  $r$ , as an  $\mathcal{O}_X$ -module at  $x' = (y, v)$ . We take  $e_1, \dots, e_r \in \Gamma(X, \mathcal{F})$  which give free generators of  $\mathcal{F}$  at  $x'$ . This is possible since  $X$  is Stein. Let  $\alpha: \mathcal{O}_X^{\oplus r} \rightarrow \mathcal{F}$  be the map defined by  $e_i$ , and  $\mathcal{K}$  (resp.  $\mathcal{P}$ ) the kernel (resp. cokernel) of  $\alpha$ . Since  $\alpha$  is isomorphic in a neighborhood of  $x'$ ,  $\mathcal{K} = \mathcal{P} = 0$  at  $x'$ . In particular they are torsion  $\mathcal{O}_X$ -modules. Hence as a subsheaf of a free

sheaf on the reduced space  $X, \mathcal{K}$  must vanish identically on  $X$ . Thus we get an exact sequence

$$(*) \quad 0 \longrightarrow \mathcal{O}_X^{\oplus r} \xrightarrow{\alpha} \mathcal{F} \longrightarrow \mathcal{P} \longrightarrow 0$$

on  $X$ . Now we show that 2) is satisfied also for  $\mathcal{P}$ . For any  $h \in S(Y, y)$ , pulling back  $(*)$  to  $X_h$  we obtain the following exact sequence on  $X$

$$0 \longrightarrow \mathcal{O}_X^{\oplus r} \xrightarrow{\alpha_h} \mathcal{F}_h \longrightarrow \mathcal{P}_h \longrightarrow 0.$$

In fact, by the same reasoning as above, firstly  $\alpha_h$  is isomorphic at  $x'_h = (x', 0) \in X_h$  and then injective on the whole  $X_h$  since  $X_h \cong V \times D$  is reduced. Thus to show the flatness of  $\mathcal{P}_h$  it is enough to show that for every integer  $k \geq 1$  the natural map  $\alpha_h^{(k)} : \mathcal{O}_{X_h}/n^k \mathcal{O}_{X_h} \rightarrow \mathcal{F}_h/n^k \mathcal{F}_h$  induced by  $\alpha_h$  is injective, where  $n$  is the maximal ideal of  $\mathcal{O}_{D,0}$ . In fact, by the flatness of  $\mathcal{F}_h$  this implies that  $\text{Tor}_1^R(\mathcal{P}, R/a) = 0$  for all ideals  $a$  of  $R = \mathcal{O}_{D,0}$ . Since  $\alpha_h$  is isomorphic at  $x'_h$ , so are  $\alpha_h^{(k)}$  for all  $k > 0$ . Thus if  $\mathcal{K}_k = \text{Ker } \alpha_h^{(k)}$ ,  $\mathcal{K}_k = 0$  at  $X'_h$ . Thus the support of  $\mathcal{K}_k$  is a proper analytic subset of  $X_{h,0}$ . Since  $\mathcal{K}_k \subseteq \mathcal{O}_{X_h}/n^k \mathcal{O}_{X_h}$ , it follows that  $\mathcal{K}_k = 0$ . Hence  $\mathcal{P}_h$  is  $f_h$ -flat, and 2) is verified for  $\mathcal{P}$ .

Now we finish the proof as follows. Recall that  $\mathcal{P}_{x'} = 0$  so that  $\dim_x(X_y \cap S(\mathcal{P})) < n$ . If  $n = 0$ , then  $\mathcal{P} = 0$  at  $x$  so that  $\mathcal{F}$  is free at  $x$ . So suppose that  $n > 0$ . Then by induction and 2) for  $\mathcal{P}$ ,  $\mathcal{P}$  is  $f$ -flat at  $x$ . Then the flatness of  $\mathcal{F}$  follows from  $(*)$ . Q.E.D.

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