On the Dynamical Theory of Gratings.

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In the usual theory of gratings, upon the lines laid down by Fresnel, the various parts of the primary wave-front after undergoing influences, whether affecting the phase or the amplitude, are conceived to pursue their course as if they still formed the fronts of waves of large area. This supposition, justifiable as an approximation when the grating interval is large, tends to fail altogether when the interval is reduced so as to be comparable with the wave-length. A simple example will best explain the nature of the failure. Consider a grating of perfectly reflecting material whose alternate parts are flat and parallel and equally wide, but so disposed as to form a groove of depth equal to a quarter wave-length, and upon this let light be incident perpendicularly. Upon Fresnel's principles the central regularly reflected

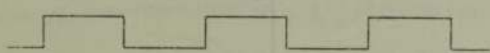

Fig. 1.

image must vanish, being constituted by the combination of equal and opposite vibrations. If the grating interval be large enough, this conclusion is approximately correct and could be verified by experiment. But now suppose that the grating interval is reduced until it is less than the wave-length of the light. The conclusion is now entirely wide of the mark. Under the circumstances supposed there are no lateral spectra and the *whole* of the incident energy is necessarily thrown into the regular reflection, which is accordingly total instead of evanescent. A closer consideration shows that the recesses in this case act as resonators in a manner not covered by Fresnel's investigations, and illustrates the need of a theory more strictly dynamical.

The present investigation, of which the interest is mainly optical, may be regarded as an extension of that given in 'Theory of Sound,'* where plane waves were supposed to be incident perpendicularly upon a regularly corrugated surface, whose form was limited by a certain condition of symmetry. Moreover, attention was there principally fixed upon the case where the wave-length of the corrugations was long in comparison with that

* Second edition, § 272*a*, 1896.

of the waves themselves, so that in the optical application there would be a large number of spectra. It is proposed now to dispense with these restrictions. On the other hand, it will be supposed that the *depth* of the corrugations is small in comparison with the length (λ) of the waves.

The equation of the reflecting surface may be taken to be $z = \zeta$, where ζ is a periodic function of x, whose mean value is zero, and which is independent of y. By Fourier's theorem we may write

$$\zeta = c_1 \cos px + c_2 \cos 2px + s_2 \sin 2px + \ldots + c_n \cos npx + s_n \sin npx + \ldots$$
$$= \tfrac{1}{2}c_1 (e^{ipx} + e^{-ipx}) + \tfrac{1}{2}(c_n - is_n)\, e^{inpx} + \tfrac{1}{2}(c_n + is_n)\, e^{-inpx} + \ldots, \quad (1)$$

the wave-length (ϵ) of the corrugation being $2\pi/p$. Formerly the s terms were omitted and attention was concentrated upon the case where c_1 was alone sensible. The omission of the s terms makes the grating symmetrical, so that at perpendicular incidence the spectra on the two sides are similar. It is known that this condition is often, and indeed advantageously, departed from in practice.

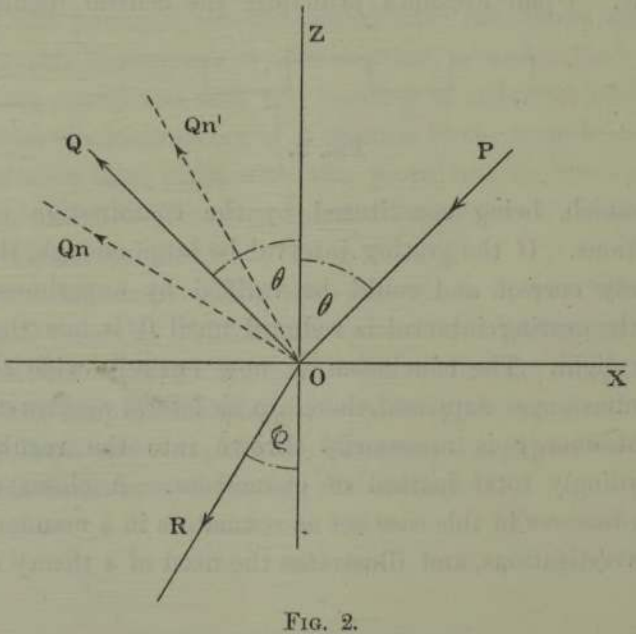

Fig. 2.

The vibrations incident at obliquity θ, POZ, fig. 2, are represented by

$$\psi = e^{ik(Vt + z\cos\theta + x\sin\theta)}, \quad (2)$$

where $k = 2\pi/\lambda$, and V is the velocity of propagation in the upper medium. Here ψ satisfies in all cases the same general differential equation, but its significance must depend upon the character of the waves. In the acoustical application, to which for the present we may confine our attention, ψ is the velocity-potential. In optics it is convenient to change the precise interpretation according to circumstances, as we shall see later.

The waves regularly reflected along OQ are represented by

$$\psi = \mathrm{A}_0 \, e^{ik(\mathrm{V}t - z\cos\theta + x\sin\theta)}, \tag{3}$$

in which A_0 is a (possibly complex) coefficient to be determined. In all the expressions with which we have to deal the time occurs only in the factor $e^{ik\mathrm{V}t}$, running through. For brevity this factor may be omitted.

In like manner the waves regularly refracted along OR into the lower medium have the expression

$$\psi_1 = \mathrm{B}_0 \, e^{ik_1(z\cos\phi + x\sin\phi)}, \tag{4}$$

ϕ being the angle of refraction; and, by the law of refraction,

$$k_1 : k = \mathrm{V} : \mathrm{V}_1 = \sin\theta : \sin\phi. \tag{5}$$

In addition to the incident and regularly reflected and refracted waves, we have to consider those corresponding to the various *spectra*. For the reflected spectra of the nth order we have

$$\psi = \mathrm{A}_n \, e^{ik(-z\cos\theta_n + x\sin\theta_n)} + \mathrm{A}'_n \, e^{ik(-z\cos\theta'_n + x\sin\theta'_n)}, \tag{5'}$$

where, by the elementary theory of these spectra,

$$\epsilon\sin\theta_n - \epsilon\sin\theta = \pm n\lambda, \quad \text{or} \quad \sin\theta_n - \sin\theta = \pm n\lambda/\epsilon = \pm np/k. \tag{6}$$

We shall choose the upper sign for θ_n and the lower for θ'_n. In virtue of (6) the complete expression for ψ in the upper medium takes the form

$$\begin{aligned}\psi \,.\, e^{-ikx\sin\theta} = {} & e^{ikz\cos\theta} + \mathrm{A}_0 \, e^{-ikz\cos\theta} + \ldots \\ & + \mathrm{A}_n \, e^{inpx} \, e^{-ikz\cos\theta_n} + \mathrm{A}'_n \, e^{-inpx} \, e^{-ikz\cos\theta'_n} + \ldots,\end{aligned} \tag{7}$$

where n has in succession the values 1, 2, 3, etc.

Similarly, in the lower medium the spectra of the nth order are represented by

$$\psi_1 = \mathrm{B}_n \, e^{ik_1(z\cos\phi_n + x\sin\phi_n)} + \mathrm{B}'_n \, e^{ik_1(z\cos\phi'_n + x\sin\phi'_n)}, \tag{8}$$

where
$$\sin\phi_n - \sin\phi = \pm np/k_1. \tag{9}$$

Accordingly, for the complete expression of ψ_1, we have with use of (5),

$$\psi_1 \,.\, e^{-ikx\sin\theta} = \mathrm{B}_0 \, e^{ik_1 z\cos\phi} + \ldots + \mathrm{B}_n \, e^{inpx} \, e^{ik_1 z\cos\phi_n} + \mathrm{B}'_n \, e^{-inpx} \, e^{ik_1 z\cos\phi'_n}. \tag{10}$$

We must now introduce boundary conditions to be satisfied at the transition between the two media when $z = \zeta$. It may be convenient to commence with a very simple case determined by the condition that $\psi = 0$. The whole of the incident energy is then thrown back, and is distributed between the regularly reflected waves and the various reflected spectra.

We proceed by approximation depending on the smallness of ζ. Expanding the exponentials on the right side of (7), we get

$$(1+A_0)(1-\tfrac{1}{2}k^2\zeta^2\cos^2\theta+\ldots)+(1-A_0)(ik\zeta\cos\theta+\ldots)$$
$$+A_n e^{inpx}(1-ik\zeta\cos\theta_n+\ldots)+A'_n e^{-inpx}(1-ik\zeta\cos\theta'_n+\ldots)=0. \quad (11)$$

In this equation the value of ζ is to be substituted from (1), and then in accordance with Fourier's theorem the coefficients of the various exponential terms, such as e^{inpx}, e^{-inpx}, are to be separately equated to zero. As the first approximation, we get from the constant term (independent of x)

$$1+A_0=0, \quad (12)$$

and from the terms in e^{inpx}, e^{-inpx},

$$A_n = -ik\cos\theta\,(c_n-is_n), \quad A'_n = -ik\cos\theta\,(c_n+is_n). \quad (13)$$

Thus, as was to be expected, A_n, A'_n are of the first order in ζ, and if we stop at the second order inclusive, (11) may be written

$$1+A_0+2ik\zeta\cos\theta+A_n e^{inpx}(1-ik\zeta\cos\theta_n)+A'_n e^{-inpx}(1-ik\zeta\cos\theta'_n)=0. \quad (14)$$

For the second approximation to A_0 we get

$$1+A_0-\tfrac{1}{2}k^2\cos\theta\,\Sigma\,(c_n^2+s_n^2)(\cos\theta_n+\cos\theta'_n)=0. \quad (15)$$

By means of (13) and (15) we may verify the principle that the energies of the incident, and of all the reflected vibrations taken together, are equal. The energy corresponding to unit of wave-front of the incident waves may be supposed to be unity, and for the other waves $\text{mod}^2 A_0$, $\text{mod}^2 A_1$, $\text{mod}^2 A'_1$, etc. But what we have to consider are not equal areas of wave-front, but areas corresponding to the same extent of reflecting surface, *i.e.*, areas of wave-front proportional to $\cos\theta$, $\cos\theta_1$, $\cos\theta'_1$, etc. Hence,

$$\cos\theta\,.\,\text{mod}^2 A_0+\Sigma\,\cos\theta_n\,.\,\text{mod}^2 A_n+\Sigma\,\cos\theta'_n\,.\,\text{mod}^2 A'_n=\cos\theta, \quad (16)$$

with which the special approximate values already given are in harmony. In the formation of (16) only real values of $\cos\theta_n$, $\cos\theta'_n$ are to be included. If $p>k$, no real values exist, *i.e.*, there are no lateral spectra. The regular reflection is then total, and this without limitation upon the magnitude of the c's. The question is further considered in 'Theory of Sound,' § 272 *a*.

In pursuing a second approximation for the coefficients of the lateral spectra, we will suppose for the sake of brevity that the s terms in (1) are omitted. From the term involving e^{inpx} in (14), we get with use of (13),

$$A_n = -ik\cos\theta\, c_n + \tfrac{1}{2}k^2\cos\theta\cos\theta'_n\, c_n\, c_{2n}$$
$$+\tfrac{1}{2}k^2\cos\theta\,\{(c_1\cos\theta_{n-1}+c_{2n-1}\cos\theta'_{n-1})\,c_{n-1}$$
$$+(c_2\cos\theta_{n-2}+c_{2n-2}\cos\theta'_{n-2})\,c_{n-2}+\dots$$
$$+(c_1\cos\theta_{n+1}+c_{2n+1}\cos\theta'_{n+1})\,c_{n+1}$$
$$+(c_2\cos\theta_{n+2}+c_{2n+2}\cos\theta'_{n+2})\,c_{n+2}+\dots\}, \quad (17)$$

in which the first (descending) series is to terminate when the suffix in $\cos\theta_{n-r}$ is equal to unity.

The value of A'_n may be derived from (17) by interchange of θ' and θ in $\cos\theta_{n-r}$, $\cos\theta'_{n-r}$, $\cos\theta_{n+r}$, $\cos\theta'_{n+r}$, $\cos\theta$ remaining unchanged. As a particular case of (17), we have, for the spectra of the first order,

$$A_1 = -ikc_1\cos\theta + \tfrac{1}{2}k^2c_1c_2\cos\theta\cos\theta'_1$$
$$+\tfrac{1}{2}k^2\cos\theta\,\{c_2(c_1\cos\theta_2+c_3\cos\theta'_2)$$
$$+c_3(c_2\cos\theta_3+c_4\cos\theta'_3)+\dots\}. \quad (18)$$

$$A'_1 = -ikc_1\cos\theta + \tfrac{1}{2}k^2c_1c_2\cos\theta\cos\theta_1$$
$$+\tfrac{1}{2}k^2\cos\theta\,\{c_2(c_1\cos\theta'_2+c_3\cos\theta_2)$$
$$+c_3(c_2\cos\theta'_3+c_4\cos\theta_3)+\dots\}, \quad (19)$$

the descending series in (17) disappearing altogether.

If the incidence is normal, $\cos\theta = 1$, $\cos\theta'_n = \cos\theta_n$, and thus A_n, A'_n become identical and assume specially simple forms. Referring to (7), we see that in this case

$$\psi = e^{ikz} + A_0\, e^{-ikz} + 2A_1 e^{-ikz\cos\theta_1}\cos px + \dots + 2A_n\, e^{-ikz\cos\theta_n}\cos npx + \dots, \quad (20)$$

in which, to the second order,

$$A_0 = -1 + k^2\,\Sigma c_n^2\cos\theta_n. \quad (21)$$

$$A_n = -ikc_n + \tfrac{1}{2}k^2\cos\theta_n\,.\,c_n\,c_{2n}$$
$$+\tfrac{1}{2}k^2\,\{(c_1+c_{2n-1})\,c_{n-1}\cos\theta_{n-1}+(c_2+c_{2n-2})\,c_{n-2}\cos\theta_{n-2}+\dots$$
$$+(c_1+c_{2n+1})\,c_{n+1}\cos\theta_{n+1}+(c_2+c_{2n+2})\,c_{n+2}\cos\theta_{n+2}+\dots\}. \quad (22)$$

If we suppose that in (1) only c_1 and c_2 are sensible, we have

$$A_0 = -1 + k^2c_1^2\cos\theta_1 + k^2c_2^2\cos\theta_2, \quad (23)$$
$$A_1 = -ikc_1 + \tfrac{1}{2}k^2c_1c_2(\cos\theta_1+\cos\theta_2), \quad (24)$$
$$A_2 = -ikc_2 + \tfrac{1}{2}k^2c_1^2\cos\theta_1, \quad (25)$$
$$A_3 = \tfrac{1}{2}k^2c_1c_2(\cos\theta_1+\cos\theta_2), \quad (26)$$

while A_4, A_5, etc., vanish to the second order of small quantities inclusive.

There is no especial difficulty in carrying the approximations further. As an example, we may suppose that c_1 is alone sensible in (1), so that we may write

$$\zeta = c \cos px, \tag{27}$$

and also that the incidence is perpendicular. For brevity we will denote $k\cos\theta_n$ or $k\cos\theta'_n$ by μ_n. The boundary condition ($\psi = 0$) becomes by (7) in this case,

$$e^{ik\zeta} - e^{-ik\zeta} + (A_0+1)\, e^{-ik\zeta} + 2A_1\, e^{-i\mu_1\zeta}\cos px + \ldots\ldots + 2A_n\, e^{-i\mu_n\zeta}\cos npx + \ldots\ldots = 0, \tag{28}$$

in which

$$e^{ik\zeta} - e^{-ik\zeta} = 4i\,\{J_1(kc)\cos px - J_3(kc)\cos 3px + J_5(kc)\cos 5px - \ldots\}, \tag{29}$$

$$e^{-ik\zeta} = J_0(kc) - 2J_2(kc)\cos 2px + \ldots - i\,\{2J_1(kc)\cos px - 2J_3(kc)\cos 3px + \ldots\}, \tag{30}$$

with similar expressions for $e^{-i\mu_1\zeta}$, $e^{-i\mu_2\zeta}$, etc. By Fourier's theorem the terms independent of x, in $\cos px$, $\cos 2px$, etc., must vanish separately. The first gives

$$(A_0+1)\, J_0(kc) - 2iA_1J_1(\mu_1c) - 2A_2J_2(\mu_2c) + 2iA_3J_3(\mu_3c) + 2A_4J_4(\mu_4c) + \ldots = 0. \tag{31}$$

The term in $\cos px$ gives

$$2i\, J_1(kc) - i\,(A_0+1)\, J_1(kc) + A_1\,\{J_0(\mu_1c) - J_2(\mu_1c)\} - iA_2\,\{J_1(\mu_2c) - J_3(\mu_2c)\} - A_3\{J_2(\mu_3c) - J_4(\mu_2c)\} + \ldots = 0. \tag{32}$$

The term in $\cos 2px$ gives

$$-(A_0+1)\, J_2(kc) - iA_1\,\{J_1(\mu_1c) - J_3(\mu_1c)\} + A_2\,\{J_0(\mu_2c) + J_4(\mu_2c)\} + \ldots\ldots = 0. \tag{33}$$

The term in $\cos 3px$ gives

$$-2i\, J_3(kc) + i\,(A_0+1)\, J_3(kc) - A_1\,\{J_2(\mu_1c) - J_4(\mu_2c)\} - iA_2\,\{J_1(\mu_2c) + J_5(\mu_2c)\} + A_3\,\{J_0(\mu_3c) - J_6(\mu_3c)\} + \ldots\ldots = 0. \tag{34}$$

We see from these that A_0+1 is of the second order in kc, that A_1 is of the first order, A_2 of the second order, A_3 of the third order, and so on. Expanding the Bessel's functions, we find, to the second order inclusive, as in (23), (24), (25), (26),

$$\left.\begin{aligned} A_0 &= -1 + k\mu_1c^2, & A_1 &= -ikc, \\ A_2 &= \tfrac{1}{2}k\mu_1c^2, & A_3 &= 0. \end{aligned}\right\}, \tag{35}$$

A_4, etc., vanishing. To the third order inclusive (34) now gives

$$A_3 = \tfrac{1}{24}\, ikc^3\,(k^2 - 3\mu_1^2 + 6\mu_1\mu_2). \tag{36}$$

From (33) to the same order we have still for A_2,

$$A_2 = \tfrac{1}{2}k\mu_1c^2, \tag{37}$$

and from (32)

$$A_1 = -ikc + \tfrac{1}{8}\, ikc^3\,(k^2 + 4k\mu_1 + 2\mu_1\mu_2 - 3\mu_1^2). \tag{38}$$

These are complete to the third order of kc inclusive. To this order A_4, A_5, etc., vanish.

So far as the third order of kc inclusive, A_0 remains as in (35); but it is worth while here to retain the terms of the fourth order. We find from (31)—

$$A_0 = -1 + k\mu_1 c^2 + \tfrac{1}{8}kc^4 (k^2\mu_1 - 4k\mu_1^2 + 2\mu_1^3 - 2\mu_1^2\mu_2 + \mu_1\mu_2^2). \qquad (39)$$

It is to be noted that k, μ_1, μ_2 are not independent. By (6), with $\theta = 0$,

$$\mu_n^2 = k^2 \cos^2\theta_n = k^2 - n^2p^2, \qquad (40)$$

so that

$$\mu_1^2 = k^2 - p^2, \qquad \mu_2^2 = k^2 - 4p^2,$$

and

$$3k^2 - 4\mu_1^2 + \mu_2^2 = 0. \qquad (41)$$

By use of (41) it may be verified to the fourth order that when μ_1, μ_2 are real, so that the spectra of the second order are actually formed,

$$\text{mod}^2 A_0 + \frac{2\mu_1}{k}\text{mod}^2 A_1 + \frac{2\mu_2}{k}\text{mod}^2 A_2 = 1, \qquad (42)$$

expressing the conservation of vibratory energy.

When μ_1 is real, but not μ_2, we may write $\mu_2 = -i\nu_2$, where ν_2 is positive. In this case

$$A_0 = -1 + k\mu_1 c^2 + \tfrac{1}{8}kc^4 (k^2\mu_1 - 4k\mu_1^2 + 2\mu_1^3 - \mu_1\nu_2^2) + \tfrac{1}{4}ikc^4\mu_1^2\nu_2,$$

$$A_1 = -ikc + \tfrac{1}{8}ikc^3 (k^2 + 4k\mu_1 - 3\mu_1^2) + \tfrac{1}{4}kc^3\mu_1\nu_2;$$

and in virtue of (41) to the fourth order,

$$\text{mod}^2 A_0 + \frac{2\mu_1}{k}\text{mod}^2 A_1 = 1. \qquad (43)$$

Again, if μ_1, μ_2 are both imaginary, equal, say, to $-i\nu_1$, $-i\nu_2$, we have from (39) with separation of real and imaginary parts,

$$A_0 = -1 + \tfrac{1}{2}k^2\nu_1^2c^4 - i\,(k\nu_1 c^2 + \text{terms in } c^4),$$

so that, to the fourth order,

$$\text{mod}^2 A_0 = 1, \qquad (44)$$

expressing that the regular reflection is now total.

In the acoustical interpretation for a gaseous medium ψ represents the velocity-potential, and the boundary condition ($\psi = 0$) is that of constant pressure. In the electrical and optical interpretation the waves are incident from air, or other dielectric medium, upon a perfectly conducting and, therefore, perfectly reflecting corrugated substance. Here ψ represents the electromotive intensity Q parallel to y, that is parallel to the lines of the grating, the boundary condition being the evanescence of Q.

We now pass on to the boundary condition next in order of simplicity, which ordains that $d\psi/dn = 0$, where dn is drawn normally at the surface of separation. Since the surfaces $z-\zeta = 0$, $\psi =$ constant, are to be perpendicular, the condition expressed in rectangular co-ordinates is

$$\frac{d\psi}{dz} - \frac{d\psi}{dx}\frac{d\zeta}{dx} = 0, \tag{45}$$

ψ being given by (7) and ζ by (1).

For the purposes of the first approximation, we require in $d\psi/dx$ only the part independent of the c's and s's, since $d\zeta/dx$ is already of the first order. Thus at the surface

$$\frac{d\psi}{dx} = ik \sin\theta\, e^{ikx\sin\theta}(1+A_0).$$

Also, correct to the first order,

$$\frac{d\psi}{dz} = ik\, e^{ikx\sin\theta}[\cos\theta\,\{1-A_0+(1+A_0)\, ik\zeta\cos\theta\}$$

$$-\ldots\ldots - \cos\theta_n A_n\, e^{inpx} - \cos\theta'_n A'_n\, e^{-inpx}].$$

Thus (45) gives

$$\cos\theta\,(1-A_0)+\cos^2\theta\,(1+A_0)\, ik\zeta - \cos\theta_n A_n\, e^{inpx}$$

$$-\cos\theta'_n A'_n\, e^{-inpx} - \ldots\ldots - \sin\theta\,(1+A_0)\frac{d\zeta}{dx} = 0. \tag{46}$$

From the term independent of x we see that, as was to be expected,

$$A_0 = 1. \tag{47}$$

Also

$$A_n \cos\theta_n = i\,(c_n - is_n)\,\{k\cos^2\theta - np\sin\theta\}, \tag{48}$$

$$A'_n \cos\theta'_n = i\,(c_n + is_n)\,\{k\cos^2\theta + np\sin\theta\}. \tag{49}$$

When $n = 1$ in (48), (49), we may put $s_1 = 0$. These equations constitute the complete solution to a first approximation.

For the second approximation we must retain the terms of the first order in $d\psi/dx$. Thus from (5), (7)

$$e^{-ikx\sin\theta}\frac{d\psi}{dx} = ik\,[\sin\theta\,\{1+A_0+(1-A_0)\, ik\cos\theta\}$$

$$+\sin\theta_n A_n\, e^{inpx} + \sin\theta'_n A'_n\, e^{-inpx}]$$

$$= ik\,\{2\sin\theta + \sin\theta_n A_n\, e^{inpx} + \sin\theta'_n A'_n\, e^{-inpx}\}, \tag{50}$$

since to the first order inclusive $A_0 = 1$. Also

$$e^{-ikx\sin\theta}\frac{d\psi}{dz} = ik\cos\theta\,\{1-A_0+2ik\zeta\cos\theta\}$$

$$-ik\cos\theta_n A_n\, e^{inpx}(1-ik\zeta\cos\theta_n) - ik\cos\theta'_n A'_n\, e^{-inpx}(1-ik\zeta\cos\theta'_n). \tag{51}$$

Thus by (45) the boundary condition is

$$\cos\theta\,(1-A_0)+2ik\zeta\cos^2\theta-2\sin\theta\frac{d\zeta}{dx}$$

$$-A_n e^{inpx}\left\{\cos\theta_n-ik\cos^2\theta_n\,\zeta+\sin\theta_n\frac{d\zeta}{dx}\right\}$$

$$-A'_n e^{-inpx}\left\{\cos\theta'_n-ik\cos^2\theta'_n\,\zeta+\sin\theta'_n\frac{d\zeta}{dx}\right\}=0. \qquad (52)$$

In the small terms we may substitute for A_n, A'_n their approximate values from (48), (49).

In (52) the coefficients of the various terms in e^{inpx}, e^{-inpx} must vanish separately. In pursuing the approximation we will write for brevity

$$\zeta=\zeta_1 e^{ipx}+\zeta_{-1}e^{-ipx}+\ldots+\zeta_n e^{inpx}+\zeta_{-n}e^{-inpx}, \qquad (53)$$

where

$$\zeta_1=\zeta_{-1}=\tfrac{1}{2}c_1,$$

and

$$\zeta_n=\tfrac{1}{2}(c_n-is_n), \qquad \zeta_{-n}=\tfrac{1}{2}(c_n+is_n). \qquad (54)$$

The term independent of x gives A_0 to the second approximation. Thus

$$\cos\theta\,(1-A_0)+iA_n(k\cos^2\theta_n+np\sin\theta_n)\,\zeta_{-n}$$

$$+iA'_n(k\cos^2\theta'_n-np\sin\theta'_n)\,\zeta_n=0. \qquad (55)$$

In (55), as follows from (6),

$$k\cos^2\theta_n+np\sin\theta_n=k\cos^2\theta-np\sin\theta,$$

and

$$k\cos^2\theta'_n-np\sin\theta'_n=k\cos^2\theta+np\sin\theta.$$

Hence with introduction of the values of A_n, A'_n from (48), (49),

$$\cos\theta\,(1-A_0)=\ldots\ldots+\frac{c_n^2+s_n^2}{2\cos\theta_n}(k\cos^2\theta-np\sin\theta)^2$$

$$+\frac{c_n^2+s_n^2}{2\cos\theta'_n}(k\cos^2\theta+np\sin\theta)^2+\ldots\ldots, \qquad (56)$$

as might also be inferred from (48), (49) alone, with the aid of the energy equation—

$$\cos\theta=\cos\theta\,\mathrm{mod}^2A_0+\ldots+\cos\theta_n\,\mathrm{mod}^2A_n+\cos\theta'_n\,\mathrm{mod}^2A'_n. \qquad (57)$$

From the term in e^{inpx} in (52) we get

$$\cos\theta_n A_n=2i(k\cos^2\theta-np\sin\theta)\,\zeta_n$$

$$+iA'_n(k\cos^2\theta'_n-2np\sin\theta'_n)\,\zeta_{2n}+\ldots\ldots$$

$$+i\,A_{n-r}(k\cos^2\theta_{n-r}-rp\sin\theta_{n-r})\,\zeta_r$$

$$+i\,A_{n+r}(k\cos^2\theta_{n+r}+rp\sin\theta_{n+r})\,\zeta_{-r}$$

$$+i\,A'_{n-r}(k\cos^2\theta'_{n-r}-(2n-r)p\sin\theta'_{n-r})\,\zeta_{2n-r}$$

$$+iA'_{n+r}(k\cos^2\theta'_{n+r}-(2n+r)p\sin\theta'_{n+r})\,\zeta_{2n+r}. \qquad (58)$$

In (58) r is to assume the values 1, 2, 3, etc., the descending series terminating when $n-r=1$.

The corresponding equation for A'_n may be derived from (58) by changing the sign of n, with the understanding that

$$A_{-m}=A'_m,\qquad A'_{-m}=A_m;\qquad \theta_{-m}=\theta'_m,\qquad \theta'_{-m}=\theta_m. \qquad (59)$$

If the incidence be perpendicular, so that $\theta'_m=-\theta_m$, and if $\zeta_{-m}=\zeta_m$, which requires that $s_m=0$, the values of A'_n and A_n become identical.

If $n=1$, the descending series in (58) make no contribution. We have

$$\cos\theta_1 A_1 = 2i\,(k\cos^2\theta-p\sin\theta)\,\zeta_1+iA'_1\,(k\cos^2\theta'_1-2p\sin\theta'_1)\,\zeta_2$$
$$+iA_2\,(k\cos^2\theta_2+p\sin\theta_2)\,\zeta_{-1}+iA_3\,(k\cos^2\theta_2+2p\sin\theta_2)\,\zeta_{-2}+\ldots$$
$$+iA'_2\,(k\cos^2\theta'_2-3p\sin\theta'_2)\,\zeta_3+iA'_3\,(k\cos^2\theta'_3-4p\sin\theta'_3)\,\zeta_4+\ldots \qquad (60)$$

We will now introduce the simplifying suppositions that $\theta=0$, $s_m=0$, making $A'_n=A_n$, and also that only c_1 and c_2 are sensible, so that $\zeta_3=\zeta_4=\ldots\ldots=0$. We will also, as before, denote $k\cos\theta_n$ or $k\cos\theta'_n$ by μ_n. Accordingly (60) gives, with use of (6), (48), (49),

$$A_1=\frac{ik^2c_1}{\mu_1}-\frac{k^2c_1c_2}{2\mu_1^{\,2}}(\mu_1^2+2p^2)-\frac{k^2c_1c_2}{2\mu_1\mu_2}(\mu_2^2+2p^2). \qquad (61)$$

In like manner, we get from (58)

$$A_2=\frac{ik^2c_2}{\mu_2}-\frac{k^2c_1^2}{2\mu_1\mu_2}(\mu_1^2-p^2), \qquad (62)$$

$$A_3=-\frac{k^2c_1c_2(\mu_1+\mu_2)}{2\mu_3}\left\{1-\frac{2p^2}{\mu_1\mu_2}\right\}, \qquad (63)$$

$$A_4=-\frac{k^2c_2^2}{2\mu_2\mu_4}(\mu_2^2-4p^2), \qquad (64)$$

after which A_5, A_6, etc., vanish to this order of approximation. In any of these equations we may replace μ_n^2 by its value from (6), that is $k^2-n^2p^2$.

The boundary condition of this case, *i.e.*, $d\psi/dn=0$, is realised acoustically when aerial waves are incident upon an immovable corrugated surface. In the interpretation for electrical and luminous waves, ψ represents the magnetic induction (b) paralled to y, so that the electric vector is perpendicular to the lines of the grating, the boundary condition at the surface of a perfect reflector being $db/dn=0$.

We have thus obtained the solutions for the two principal cases of the incidence of polarised light upon a perfect corrugated reflector. In comparing the results for the first order of approximation as given in (13) for the first case and in (48), (49) for the second, we are at once struck with the fact that in the second case, though not in the first, the intensity of a spectrum may become infinite through the evanescence of $\cos\theta_n$ or $\cos\theta'_n$, which occurs when the spectrum is just disappearing from the field of view. But the effect is not limited to the particular spectrum which is on the point of disappearing. Thus in (61) A_1, giving the spectrum of the *first* order, becomes infinite as the spectrum of the *second* order disappears ($\mu_2 = 0$). Regarded from a mathematical point of view, the method of approximation breaks down. The problem has no definite solution, so long as we maintain the suppositions of perfect reflection, of an infinite train of simple waves, and of a grating infinitely extended in the direction perpendicular to its ruling. But under the conditions of experiment, we may at least infer the probability of abnormalities in the brightness of any spectrum at the moment when one of higher order is just disappearing, abnormalities limited, however, to the case where the electric displacement is perpendicular to the ruling.* It may be remarked that when the incident light is unpolarised, the spectrum about to disappear is polarised in a plane parallel to the ruling.

In both the cases of boundary conditions hitherto treated, the circumstances are especially simple in that the aggregate reflection is perfect, the whole of the incident energy being returned into the upper medium. We now pass on to more complicated conditions, which we may interpret as those of two gaseous media of densities σ and σ_1. Equality of pressures at the interface requires that

$$\sigma\psi = \sigma_1\psi_1, \tag{65}$$

and we have also to satisfy the continuity of normal velocity expressed by

$$d\psi/dn = d\psi_1/dn, \tag{66}$$

or, as in (45),

$$\frac{d(\psi-\psi_1)}{dz} - \frac{d(\psi-\psi_1)}{dx}\frac{d\zeta}{dx} = 0, \tag{67}$$

ψ and ψ_1 being given by (7), (10). We must content ourselves with a solution to the first approximation, at least for general incidence.

From (65),

$$\frac{\sigma}{\sigma_1}\{1 + A_0 + (1 - A_0)\, ik\zeta\cos\theta + A_n e^{inpx} + A'_n e^{-inpx}\}$$

$$= B_0(1 + ik_1\zeta\cos\phi) + B_n e^{inpx} + B'_n e^{-inpx}. \tag{68}$$

* See a "Note on the Remarkable Case of Diffraction Spectra described by Professor Wood," recently communicated to the 'Philosophical Magazine,' vol. 14, p. 60, 1907.

Distinguishing the various components in ζ as in (53), we find

$$\frac{\sigma}{\sigma_1}(1+A_0) = B_0, \tag{69}$$

$$\frac{\sigma}{\sigma_1}A_n - B_n = i\zeta_n\left\{k_1\cos\phi\, B_0 - \frac{\sigma}{\sigma_1}(1-A_0)\,k\cos\theta\right\}, \tag{70}$$

$$\frac{\sigma}{\sigma_1}A'_n - B'_n = i\zeta_{-n}\left\{k_1\cos\phi\, B_0 - \frac{\sigma}{\sigma_1}(1-A_0)\,k\cos\theta\right\}. \tag{71}$$

In forming the second boundary condition (67) we require in $d(\psi-\psi_1)/dx$ only the part independent of ζ. Thus

$$\frac{d(\psi-\psi_1)}{dx} = ik\sin\theta\, e^{ikx\sin\theta}\{1+A_0-B_0\}.$$

Also

$$e^{-ikx\sin\theta}\frac{d\psi}{dz} = ik\cos\theta\,\{1-A_0+(1+A_0)\,ik\zeta\cos\theta\}$$
$$-ik\cos\theta_n A_n\, e^{inpx} - ik\cos\theta'_n A'_n\, e^{-inpx},$$

$$e^{-ikx\sin\theta}\frac{d\psi_1}{dz} = ik_1\cos\phi\, B_0\,(1+ik_1\zeta\cos\phi)$$
$$+ik_1\cos\phi_n\, B_n\, e^{inpx} + ik_1\cos\phi'_n B'_n\, e^{-inpx}.$$

Thus (67) takes the form

$$\begin{aligned} &ik\cos\theta\,(1-A_0) - ik_1\cos\phi\, B_0 \\ &-k^2\cos^2\theta\,(1+A_0)\,\zeta + k_1^2\cos^2\phi\, B_0\,\zeta \\ &-e^{inpx}\{ik\cos\theta_n A_n + ik_1\cos\phi_n B_n\} \\ &-e^{-inpx}\{ik\cos\theta'_n A'_n + ik_1\cos\phi'_n B'_n\} \\ &= ik\sin\theta\,(1+A_0-B_0)\,d\zeta/dx. \end{aligned} \tag{72}$$

From the part independent of x we get

$$k\cos\theta\,(1-A_0) - k_1\cos\phi\, B_0 = 0, \tag{73}$$

and from the parts in e^{inpx}, e^{-inpx}

$$\begin{aligned} k\cos\theta_n A_n + k_1\cos\phi_n B_n = i\zeta_n\{&k^2\cos^2\theta\,(1+A_0) - k_1^2\cos^2\phi\, B_0 \\ &-npk\sin\theta\,(1+A_0-B_0)\}, \end{aligned} \tag{74}$$

and a similar equation involving A'_n, B'_n.

From (69), (73) we find

$$A_0 = \frac{\dfrac{\sigma_1}{\sigma} - \dfrac{\cot\phi}{\cot\theta}}{\dfrac{\sigma_1}{\sigma} + \dfrac{\cot\phi}{\cot\theta}}, \qquad B_0 = \frac{2}{\dfrac{\sigma_1}{\sigma} + \dfrac{\cot\phi}{\cot\theta}}. \tag{75}$$

Again, eliminating B_n between (70), (74), we get, with use of (5),

$$A_n \{k \cos \theta_n + k_1 \cos \phi_n . \sigma/\sigma_1\}$$
$$= \frac{2ik^2\zeta_n}{D}\left[\frac{\sigma_1}{\sigma}\cos^2\theta - \frac{\sigma_1-\sigma}{\sigma}\frac{np}{k}\sin\theta - \frac{\sin^2\theta\cos\phi}{\sin^2\phi}\left\{\cos\phi - \cos\phi_n + \frac{\sigma}{\sigma_1}\cos\phi_n\right\}\right], \quad (76)$$

D denoting the denominators in (75).

The equations (75) for the waves regularly reflected and refracted are those given (after Green) in 'Theory of Sound,' § 270. They are sufficiently general to cover the case where the two gaseous media have different constants of compressibility (m, m_1) as well as of density (σ, σ_1). The velocities of wave propagation are connected with these quantities by the relation, see (5),

$$k_1^2 : k^2 = \sin^2\theta : \sin^2\phi = V^2 : V_1^2 = m/\sigma : m_1/\sigma_1. \quad (77)$$

In ideal gases the compressibilities are the same, so that

$$\sigma_1 : \sigma = \sin^2\theta : \sin^2\phi. \quad (78)$$

In this case (75) gives

$$A_0 = \frac{\sin 2\theta - \sin 2\phi}{\sin 2\theta + \sin 2\phi} = \frac{\tan(\theta-\phi)}{\tan(\theta+\phi)}, \quad (79)$$

Fresnel's expression for the reflection of light polarised in a plane perpendicular to that of incidence. In accordance with Brewster's law the reflection vanishes at the angle of incidence whose tangent is V/V_1.

In like manner the introduction of (78) into (76) gives, after reduction,

$$A_n \{k \cos \theta_n + k_1 \cos \phi_n . \sigma/\sigma_1\}$$
$$= 2ik^2\zeta_n \cot\theta \tan(\theta-\phi) \{\cos(\theta+\phi)\cos(\theta-\phi) - \cos\phi(\cos\phi - \cos\phi_n) - np/k . \sin\theta\}. \quad (80)$$

If the wave-length of the corrugations be very long, $p = 0$, $\cos\phi_n$ becomes identical with $\cos\phi$, and thus A_n vanishes when $\cos(\theta+\phi) = 0$, that is at the same (Brewsterian) angle of incidence for which $A_0 = 0$, as was to be expected. In general $A_n = 0$, when

$$\cos(\theta+\phi)\cos(\theta-\phi) = \cos\phi(\cos\phi - \cos\phi_n) + np/k . \sin\theta. \quad (81)$$

If we suppose that np/k is somewhat small, we may obtain a second approximation to the value of $\cos(\theta+\phi)$. Thus, setting in the small terms $\theta+\phi = \frac{1}{2}\pi$, we get

$$\cos(\theta+\phi) = \tfrac{1}{2}\sec\theta\{\cos\phi - \cos\phi_n + np/k\}.$$

Here $\cos\phi_n = \cos\phi - np/k_1 \, . \tan\phi = \cos\phi - np/k \, . \cot^2\theta$,

so that
$$\cos(\theta+\phi) = \frac{np}{2k\sin^2\theta\cdot\cos\theta}. \tag{82}$$

This determines the angle of incidence at which to a second approximation (in np/k) the reflected vibration vanishes in the nth spectrum.

Since according to (82) with n positive $\theta+\phi < \frac{1}{2}\pi$, and $\theta_n > \theta$, it seemed not impossible that (82) might be equivalent to $\cos(\theta_n+\phi) = 0$, forming a kind of extension of Brewster's law. It appears, however, from (6) that

$$\cos(\theta_n+\phi) = \frac{np}{k\cos\theta}\left(\frac{1}{2\sin^2\theta} - 1\right), \tag{83}$$

so that the suggested law is not observed, although the departure from it would be somewhat small in the case of moderately refractive media.

For the other spectrum of the nth order we have only to change the sign of n in (82), (83).

When np/k is not small, we must revert to the original equation (81). Even this, it must be remembered, depends upon a first approximation, including only the first powers of the ζs.

Another special case of interest occurs when $\sigma_1 = \sigma$, so that in the acoustical application the difference between the two media is one of compressibility only. The introduction of this condition into (75) gives

$$A_0 = \frac{\tan\phi - \tan\theta}{\tan\phi + \tan\theta} = \frac{\sin(\phi-\theta)}{\sin(\phi+\theta)}, \tag{84}$$

the other Fresnel's expression.

Again, from (76),

$$A_n\{k\cos\theta_n + k_1\cos\phi_n\} = \frac{2ik^2\zeta_n}{D}\left\{\cos^2\theta - \frac{\sin^2\theta\cos^2\phi}{\sin^2\phi}\right\},$$

whence
$$A_n = \frac{2ik\zeta_n\cos\theta\sin(\phi-\theta)}{\sin\phi\cos\theta_n + \sin\theta\cos\phi_n}. \tag{85}$$

In this case the vibration in the nth spectrum does not vanish at any angle of incidence.

We have now to consider the application of our solutions to electromagnetic vibrations, such as constitute light, the polarisation being in one or other principal plane. In the usual electrical notation,

$$V^2 = 1/K\mu, \qquad V_1^2 = 1/K_1\mu_1,$$

K, K_1 being the specific inductive capacities, and μ, μ_1 the magnetic permeabilities; while in the acoustical problem,

$$V^2 = m/\sigma, \qquad V_1^2 = m_1/\sigma_1.$$

The boundary conditions are also of the same general form. For instance, the acoustical conditions

$$\sigma\psi = \sigma_1\psi_1, \qquad d\psi/dn = d\psi_1/dn,$$

may be written

$$(\sigma\psi) = (\sigma_1\psi_1), \qquad \sigma^{-1}d(\sigma\psi)\,dn = \sigma_1{}^{-1}d(\sigma_1\psi_1)/dn;$$

and in the upper medium where σ is constant it makes no difference whether we deal with ψ or $\sigma\psi$. Thus if in the case of light we identify ψ with β, the component of magnetic force parallel to y, the conditions to be satisfied at the surface are the continuity of β and of $K^{-1}d\beta/dn$.*

Comparing with the acoustical conditions, we see that K replaces σ, and consequently (by the value of V^2) μ replaces $1/m$. Hence, in the general solution (75), (76), it is only necessary to write K in place of σ. For optical purposes we may usually treat μ as constant. This corresponds to the special supposition (78), so that (79), (80) apply to light for which the magnetic force is parallel to the lines of the grating, or the electric force perpendicular to the lines, *i.e.*, in the plane of incidence.

From (76) we may fall back upon (48) by making $K_1 = \infty$, $\mu_1 = 0$, in such a way that V_1, and therefore ϕ, remains finite.

The other optical application depends upon identifying ψ with Q, the electromotive intensity parallel to y, *i.e.*, parallel to the lines of the grating. The conditions at the surface are now the continuity of Q and of $\mu^{-1}dQ/dn$. Equations (75), (76) become applicable if we replace σ by μ. If μ be invariable, this is the special case of (84), (85); so that these equations are applicable to light when the electric vibration is parallel to the lines of the grating, or perpendicular to the plane of incidence. The associated Fresnel's expression (79) or (84) suffices in each case to remind us of the optical circumstances.

In order to pass back from (76) to (13), we are to suppose $K_1 = \infty$, μ_1 (or σ_1) $= 0$, so that ϕ remains finite. Thus $D = \cot\phi/\cot\theta$, and the only terms to be retained in (76) are those which include the factor σ/σ_1.

The polarisation of the spectra reflected from glass gratings was noticed by Fraunhofer:—"Sehr merkwürdig ist es, dass unter einem gewissen Einfallswinkel ein Theil eines durch Reflexion entstandenen Spectrums aus *vollständig polarisirtem Lichte* besteht. Dieser Einfallswinkel ist für die verschiedenen Spectra sehr verschieden, und selbst noch sehr merklich für die verschiedenen Farben ein und desselben Spectrums. Mit dem Glasgitter $\epsilon = 0{\cdot}0001223$ ist polarisirt: E (+ I), *d.i.*, der *grüne* Theil dieses ersten

* See 'Phil. Mag.,' vol. 12, p. 81, 1881; 'Scientific Papers,' vol. 1, p. 520.

Spectrums, wenn $\sigma = 49°$ ist; E(+II), oder der grüne Theil in dem zweiten auf derselben Seite der Axe liegenden Spectrum, wenn $\sigma = 40°$ beträgt; endlich E (−I), oder der grüne Theil des ersten auf der entgegengesetzten Seite der Axe liegenden Spectrums, wenn $\sigma = 69°$. Wenn E(+I) vollständig polarisirt ist, sind es die übrigen Farben dieses Spectrums noch unvollständig."*

In Fraunhofer's notation σ is the angle of incidence, here denoted by θ, and λ (E) = 0·00001945 in the same measure (the Paris inch) as that employed for ϵ, so that $p/k = \lambda/\epsilon = 0·159$. If we suppose that the refractive index of the glass was 1·5, we get

Order.	θ.	Sin θ.	Sin ϕ.	ϕ.	$\theta+\phi$.
	°			° ′	° ′
E (+I)	49	0·755	0·503	30 11	79 11
E (+II)	40	0·643	0·429	25 25	65 25
E (−I)	69	0·934	0·623	107 33	107 33

On the other hand, from (82) we get for E(+I) $\theta+\phi = 77° \; 44'$, for E(+II) 59° 48′, and E(−I) 104° 45′, a fair agreement between the two values of $\theta+\phi$, except in the case of E (+II).

It appears, however, that the neglect of p^2 upon which (82) is founded is too rough a procedure. By trial and error I calculate from (81) for E(+I) $\theta = 48° \; 52'$; for E(+II) $\theta = 42° \; 17'$; for E(−I) $\theta = 65° \; 46'$. These agree perhaps as closely as could be expected with the observed values, considering that they are deduced from a theory which neglects the square of the depth of the ruling. The ordinary polarising angle for this index (1·5) is 56° 19′.

It would be of interest to extend Fraunhofer's observations; but the work should be in the hands of one who is in a position to rule gratings himself. On old and deteriorated glass surfaces polarisation phenomena are liable to irregularities.

In the hope of throwing light upon the remarkable observation of Professor Wood,† that a frilled collodion surface shows an enhanced reflection, I have pursued the calculation of the regularly reflected light to the second order in ζ, the depth of the groove, limiting myself, however, to the case of perpendicular incidence and to the supposition that ζ_1 and its equal

* Gilbert's 'Ann. d. Physik,' vol. 74, p. 337 (1823); 'Collected Writings,' Munich, 1888, p. 134.

† 'Physical Optics,' p. 145.

ζ_{-1} are alone sensible. Although the results are not what I had hoped, it may be worth while to record the principal steps.

Retaining only the terms independent of x, we get from the first condition (65),

$$\sigma/\sigma_1 \,.\, \{(1+A_0)(1-k^2\zeta_1^2)-2ik\zeta_1 \cos\theta_1 \, A_1\} = B_0(1-k_1^2\zeta_1^2)+2ik_1\zeta_1\cos\phi_1 \, B_1, \tag{86}$$

and from the second condition (67),

$$k(1-A_0)(1-k^2\zeta_1^2)+2ik^2\zeta_1 A_1\cos^2\theta_1 - k_1 B_0(1-k_1^2\zeta_1^2)-2ik_1^2\zeta_1 B_1\cos^2\phi_1 = -2ip^2\zeta_1(A_1-B_1). \tag{87}$$

Eliminating $B_0(1-k_1^2\zeta_1^2)$, and remembering that

$$k^2\cos^2\theta_1+p^2 = k^2, \qquad k_1^2\cos^2\phi_1+p^2 = k_1^2,$$

we get

$$k(1-A_0)-\sigma/\sigma_1 \,.\, k_1(1+A_0)+2ik^2\zeta_1 A_1$$
$$+\sigma/\sigma_1 \,.\, 2ikk_1\zeta_1 A_1\cos\theta_1+2ik_1^2\zeta_1 B_1\cos\phi_1-2ik_1^2\zeta_1 B_1 = 0, \tag{88}$$

in which we are to substitute the values of A_1, B_1 from (70), (74). From this point it is, perhaps, more convenient to take the principal suppositions separately.

Let, as in (78), $\sigma_1 : \sigma = \sin^2\theta : \sin^2\phi = k_1^2 : k^2$;

we have

$$A_0 = \frac{k_1-k}{k_1+k}, \qquad B_0 = \frac{2k^2}{k_1(k_1+k)},$$

and accordingly, from (70), (74),

$$k^2 A_1 - k_1^2 B_1 = 2ik^2\zeta_1(k_1-k), \quad k\cos\theta_1 A_1 + k_1\cos\phi_1 B_1 = 0;$$

so that $$A_1\{k\cos\phi_1+k_1\cos\theta_1\} = 2ik\zeta_1(k_1-k)\cos\phi_1.$$

Hence, from (88),

$$\frac{1-A_0}{1+A_0} = \frac{k}{k_1} + \frac{2k\zeta_1^2(k_1^2-k^2)}{k_1}\left\{1-\frac{(k_1^2-k^2)\cos\theta_1\cos\phi_1}{k_1(k\cos\phi_1+k_1\cos\theta_1)}\right\}. \tag{89}$$

Again, when $\sigma_1 = \sigma$,

$$A_0 = \frac{k-k_1}{k+k_1}, \qquad B_0 = \frac{2k}{k+k_1}$$

and from (70), (74),

$$A_1 = B_1 = \frac{2ik\zeta_1(k-k_1)}{k\cos\theta_1+k_1\cos\phi_1}. \tag{90}$$

The introduction of these into (88) gives

$$\frac{1-A_0}{1+A_0} = \frac{k_1}{k} - \frac{2(k_1^2-k^2)\zeta_1^2}{k}\left\{k_1 - \frac{k_1^2-k_2^2}{k\cos\theta_1+k_1\cos\phi_1}\right\}. \tag{91}$$

The question is whether the numerical value of A_0 is increased or diminished by the term in ζ_1^2. In (89) it is easy to recognise that in the standard case of k_1 greater than k (air to glass in optics) the term in ζ_1^2 is positive, θ_1 and ϕ_1 being supposed real. The effect of the second term is thus to bring the right-hand member nearer to unity than it would otherwise be, and thus to *diminish* the reflection. Again, in (91), the second term is negative, even when $\cos\theta_1 = 0$, as we may see by introducing the appropriate value of $\cos\phi_1$, viz., $\sqrt{(1-k^2/k_1^2)}$. The effect is therefore to subtract something from k_1/k, which is greater than unity, and thus again to diminish the reflection.

If in (89), (91) we neglect the terms in $(k_1^2-k^2)^2\zeta_1^2$, which will be specially small when the two media do not differ much, the formulæ become independent of the angles θ_1 and ϕ_1. In both cases the effect is the same as if the refractive index, supposed greater than unity, were diminished in the ratio $1-2(k_1^2-k^2)\zeta_1^2:1$. It appears then that the present investigation gives no hint of the enhanced reflection observed in certain cases by Professor Wood.