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On the dynamics of polynomial-like mappings

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ON THE DYNAMICS OF POLYNOMIAL-LIKE MAPPINGS

BY ADRIEN DOUADY AND JOHN HAMAL HUBBARD

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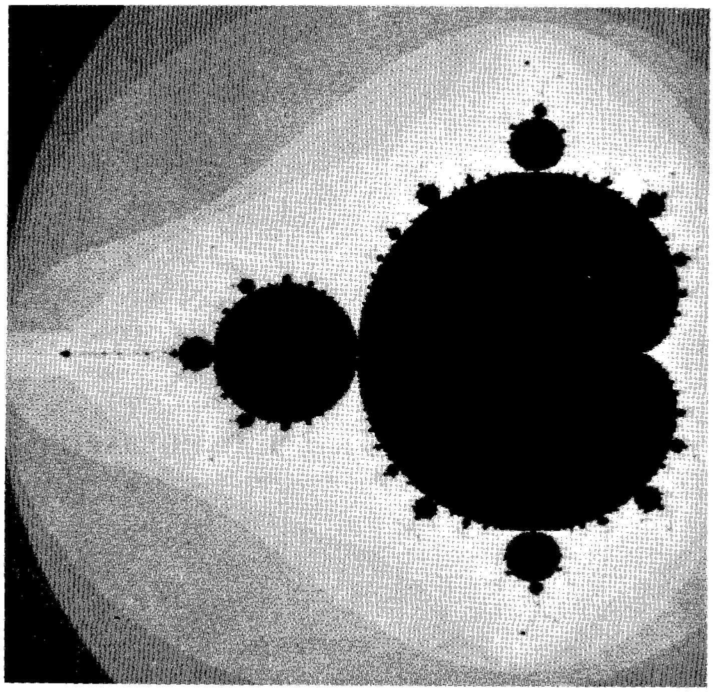

Fig. 1

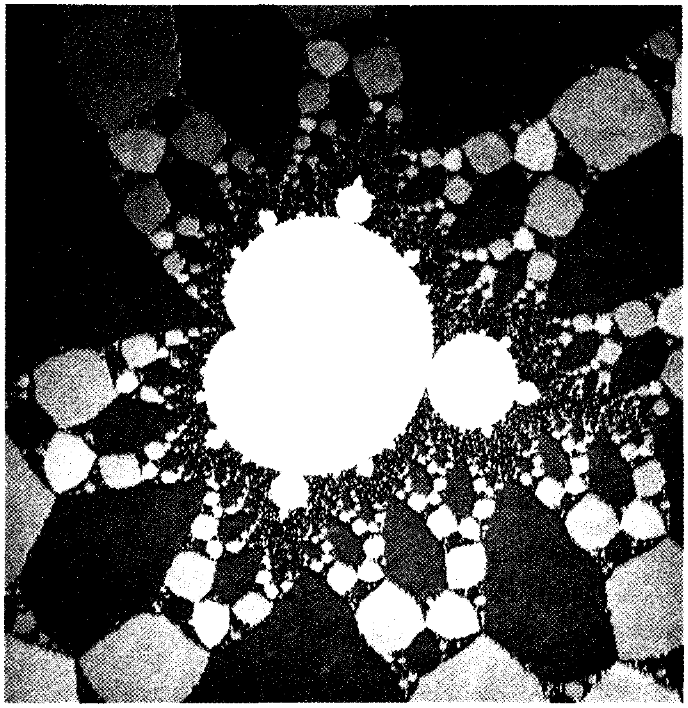

Fig. 4

INTRODUCTION

Figure 1 is a picture of the *standard Mandelbrot set* M. The article [M] by Benoit Mandelbrot, containing the first pictures and analyses of this set aroused great interest.

The set M is defined as follows. Let $P_c : z \mapsto z^2+c$ and for each c let K_c be the set of $z \in \mathbb{C}$ such that the sequence z, $P_c(z)$, $P_c(P_c(z))$,... is bounded. A classical Theorem of Fatou [F] and Julia [J] asserts that K_c is connected if $0 \in K_c$ and a Cantor set otherwise (0 plays a distinguished role because it is the critical point of P_c). More recent introductions to the subject are [Br] and [Bl], where this Theorem in particular is proved. The set M is the set of c for which K_c is connected.

In addition to the main component and the components which derive from it by successive bifurcations, M consists of a mass of filaments (see *Figs.* 2 and 3) loaded with small droplets, which ressemble M itself. The combinatorial description of these filaments is complicated; it is sketched in [D-H] and will be the object of a later publication.

Figure 4 concerns an apparently unrelated problem. Let f_λ be the polynomial $(z-1)(z+1/2+\lambda)(z+1/2-\lambda)$, which we will attempt to solve by Newton's method, starting at $z_0=0$ and setting

$$z_{n+1}=N_\lambda(z_n)=z_n-f_\lambda(z_n)/f'_\lambda(z_n).$$

As we shall see, 0 is the worst possible initial guess. Color λ blue if $z_n \to 1$, red if $z_n \to -1/2+\lambda$ and green if $z_n \to -1/2-\lambda$; leave λ white if the sequence z_n does not converge to any of the roots. Figure 4 represents a small region in the λ-plane; a sequence of zooms leading to this picture is given in chapter VI. This family has been studied independantly by Curry, Garnett and Sullivan [C-G-S] who also obtained similar figures.

If two cubic polynomials have roots which form similar triangles, the affine map sending the roots of one to the roots of the other conjugates their Newton's methods. Up to this equivalence, the family f_λ contains all cubic polynomials with marked roots except z^3 and z^3-z^2 with 0 as first marked root. The point $z_0=0$ is a point of inflexion of f_λ and hence a critical point of the Newton's method; it is the only critical point besides the roots.

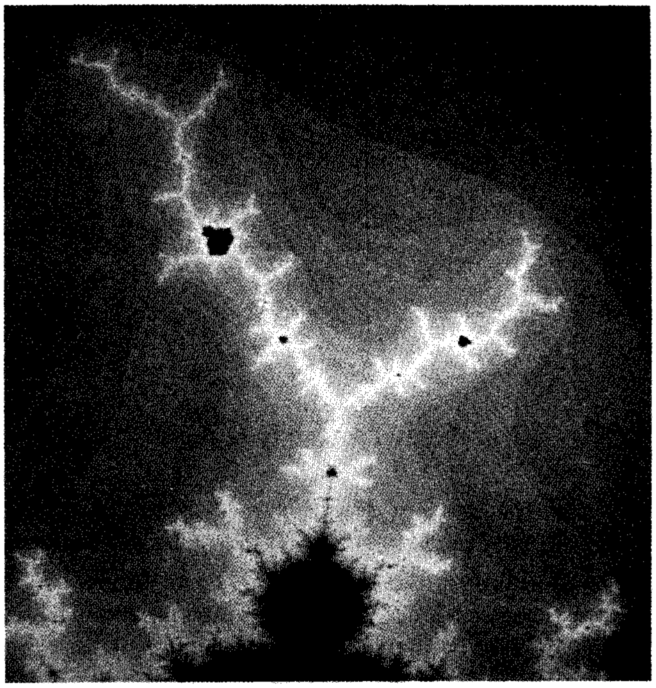

Fig. 2

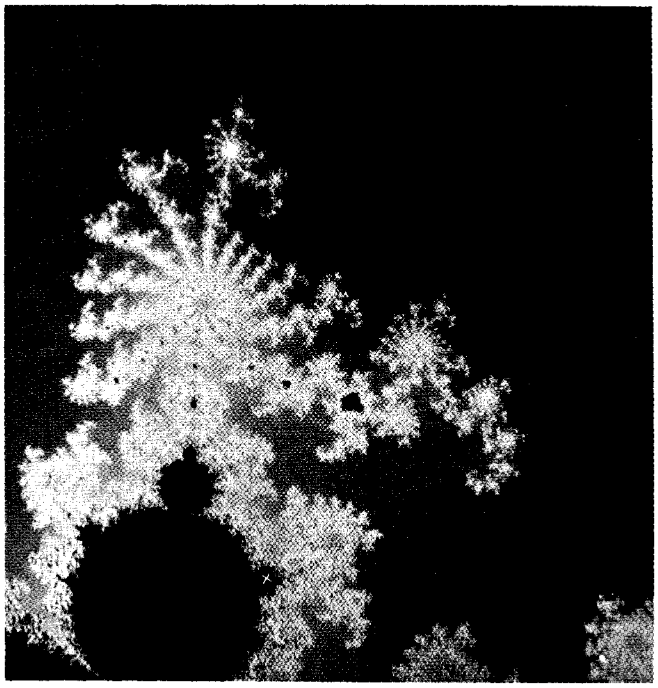

Fig. 3

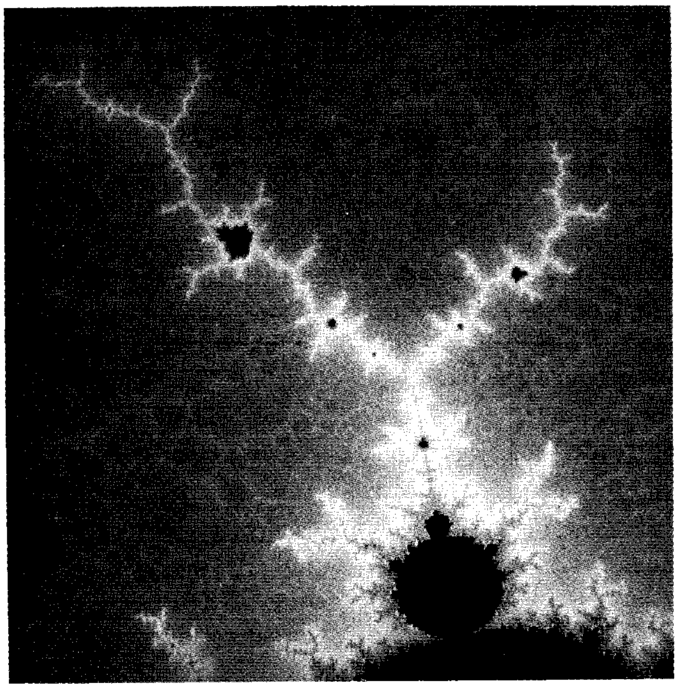

Fig. 5

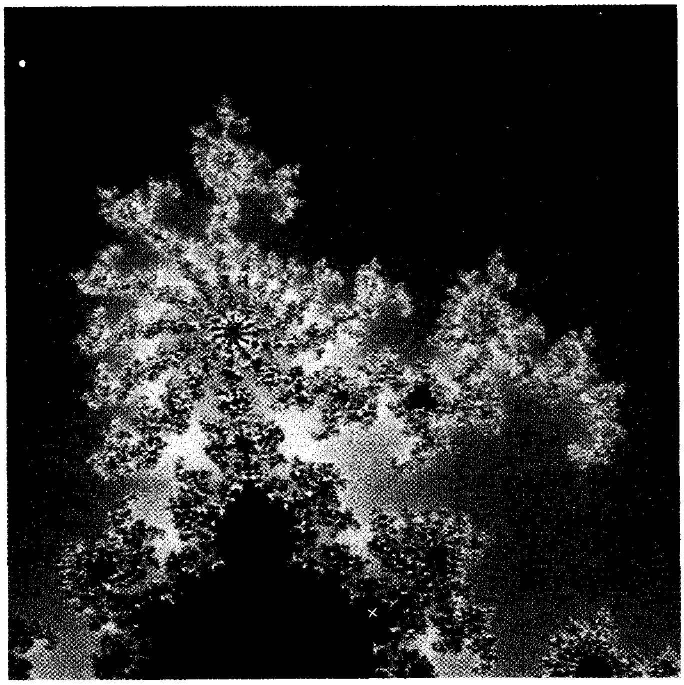

Fig. 6

If there is an attractive periodic cycle of the Newton's method other than the roots of f, by a Theorem of Fatou it attracts a critical point, which is necessarily 0; in that sense $z_0=0$ is the worst possible initial choice.

The white region in Figure 4 looks remarkably like M. Figures 5 and 6 show that the ressemblance extends to the finest detail. Here the coloring corresponds to speed of convergence to the roots.

We interpret this fact as follows. For a certain value λ_0 of λ, at the "center" of the white region, the sequence z_n is periodic of some period k (in this case 3) for Newton's method. Since 0 is critical for Newton's method, the periodic cycle containing 0 is superattractive, and on a neighborhood of 0 the function N_λ^k will behave like $z \mapsto z^2$. If V is a small neighborhood of 0, then $N_{\lambda_0}^k(V) \subset V$, but for a slightly larger neighborhood U we will have $N_{\lambda_0}^k(U) \supset U$. For λ close to λ_0 we will still have $N_\lambda^k(U) \supset U$ but now N_λ^k will behave on U like some polynomial $z \mapsto z^2+c$ for some $c=\chi(\lambda)$.

Figure 8 represents basins of attraction of the roots for the polynomial f_λ, $\lambda=-.0100605+i.220311$ (that value of λ is indicated with a cross on Figure 6). The shading corresponds to speed of attaction by the appropriate root; the black region is not attracted by any root. Figure 7 is a picture of K_c for $c=-.714203+i.245052$ (again indicated with a cross on Figure 3). The reader will agree that the words "behave like", although still undefined, are not excessive.

If $\chi(\lambda) \in M$, then $z_n \in V$ for all n and the sequence cannot converge to one of the roots. The remarkable fact is that χ induces a homeomorphism of $\chi^{-1}(M)$ onto M.

The above phenomenon is very general. If f_λ is any family of analytic functions depending analytically on λ, then an attempt to classify values of λ according to the dynamical properties of f_λ will often produce copies of M in the λ-plane.

The object of this paper is to explain this phenomenon, by giving a precise meaning to the words "behave like", which appear in the heuristic description above. The notion of polynomial-like mapping was invented for this purpose. It was motivated by the observation that when studying iteration of polynomials, one uses the theory of analytic functions continually, but the rigidity of polynomials only rarely.

In the real case, something analoguous happens. Many results extend from quadratic polynomials to convex functions with negative Schwarzian curvature. But there is a major difference: in the real case, one is mainly interested in functions which send an interval into itself. An analytic function which sends a disc into itself is always contracting for the Poincaré metric, and there is not much to say.

In the complex case, the appropriate objects of study are maps $f: U \to \mathbb{C}$ where $U \subset \mathbb{C}$ is a simply connected bounded domain. f extends to the boundary of U and sends ∂U into a curve which turns several times around U, staying outside $\overline{U}$. Our definition is slightly different from the above, mainly for convenience.

Many dynamic properties of polynomials extend to the framework of polynomial-like mappings. In fact, you can often simply copy the proof in the new setting. For the two following statements, however, that procedure does not go through.

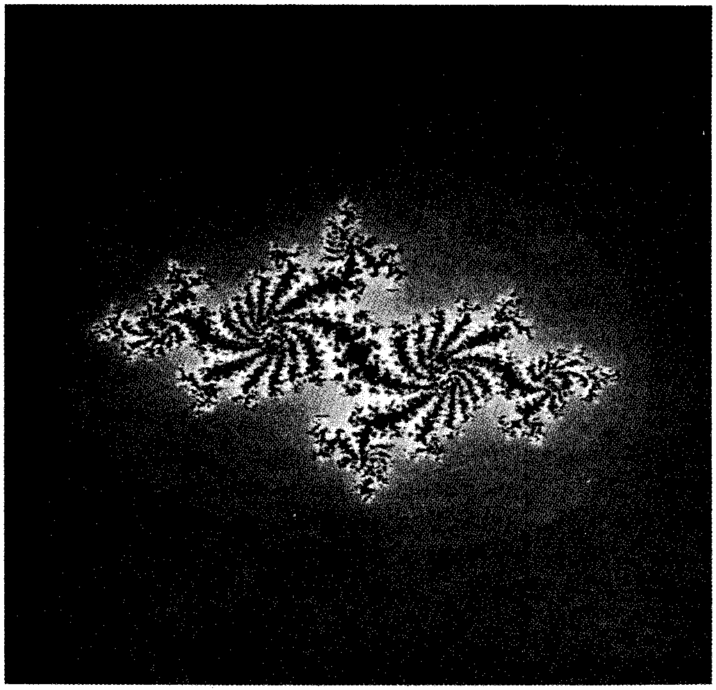

Fig. 7

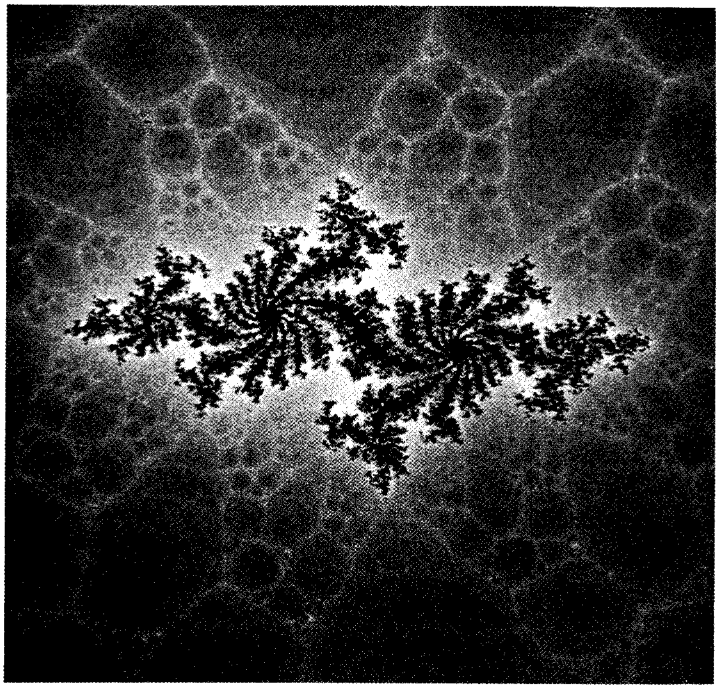

Fig. 8

(*a*) *The density of periodic points in* $J_f = \partial K_f$, proved by Fatou for polynomials (and rational mappings) using Picard's theorem.

(*b*) *The eventual periodicity of components of* $\mathring{K}_f$, proved by Sullivan [S1] using the fact that a polynomial (or a rational function) depends only on a finite number of parameters.

The Straightening Theorem (I,1) allows one to deduce various properties of polynomial-like mappings from the analogous properties of polynomials. In particular, this applies to statements (*a*) and (*b*) above.

Its proof relies on the "measurable Riemann mapping Theorem" of Morrey-Ahlfors-Bers ([A-B], [L], [A]). A review of this Theorem, as well as a dictionary between the languages of Beltrami forms and of complex structures, will be given in (I,3).

Chapter I is centered on the Straightening Theorem.

Chapters II to IV are devoted to the introduction of parameters in the situation.

Chapters V and VI give two applications. Chapter V explains the appearance of small copies of M in M. The computation is similar to one made by Eckmann and Epstein [E-E] in the real case. They were able, using our characterization of Mandelbrot-like families, to get results similar to those in chapter V in the complex case also.

Chapter VI is a study of Figure 4.

This work owes a great deal to the ideas of D. Sullivan, who initiated the use of quasi-conformal mappings in the study of rational functions. It fits in with Sullivan's papers ([S1], [S2]), and the article by Mañe, Sad and Sullivan [M-S-S].

We thank Pierrette Sentenac for her help with the inequalities, particularly those in IV,5 and V,3.

We also thank Homer Smith. He wrote the programs which drew the pictures in the article, and also provided expert photographic assistance.

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CHAPTER I

Polynomial-like mappings

1. Definitions and statement of the straightening theorem. — Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree d and let $U = D_R$ be the disc of radius R. If R is large enough then $U' = P^{-1}(U)$ is relatively compact in U and homeomorphic to a disc, and $P : U' \to U$ is analytic, proper of degree d.

The object above will be our model.

Definition. — *A polynomial-like map of degree d is a triple* (U, U', f) *where* U *and* U′ *are open subsets of* $\mathbb{C}$ *isomorphic to discs, with* U′ *relatively compact in* U, *and* $f : U' \to U$ *a* $\mathbb{C}$*-analytic mapping, proper of degree d.*

We will only be interested in the case $d \geqq 2$.

Examples. — (1) The model above. When we refer to a polynomial f as a polynomial-like mapping, a choice of U and U′ such that $U'=f^{-1}(U)$ and that $f: U' \to U$ is polynomial-like will be understood.

(2) Let $f(z)=\cos(z)-2$ and let $U'=\{z \mid |\mathrm{Re}(z)|<2, |\mathrm{Im}(z)|<3\}$. Then $U=f(U')$ is the region represented in Figure 9, and $f: U' \to U$ is polynomial-like of degree 2.

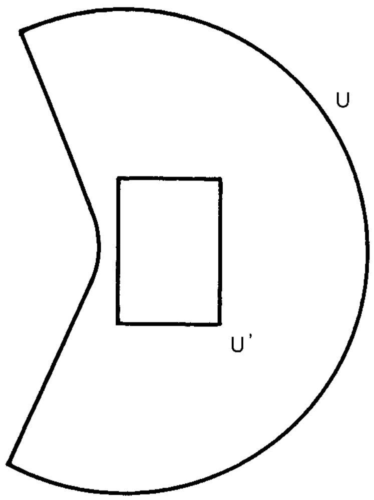

Fig. 9

(3) Let P be a polynomial of degree d and let U be open in $\mathbb{C}$, such that $P^{-1}(U)$ has several connected components $U'_1, U'_2, \ldots, U'_k$ with $k \leq d$. If U'_1 is contained and relatively compact in U, then the restriction of P to U'_1 is polynomial-like of some degree $d_1 < d$.

In this situation, one often gets much better information about the behaviour of P in U'_1 by considering it as a polynomial-like map of degree d_1 than as a polynomial of degree d, especially when $d=128$ and $d_1=2$ as will occur in chapter V.

(4) A small analytic perturbation of a polynomial-like mapping of degree d is still polynomial-like of degree d. More precisely, let $f: U' \to U$ be polynomial-like of degree d. Then f has $d-1$ critical points $\omega_1, \ldots, \omega_{d-1}$, counting with multiplicities. Choose $\varepsilon>0$, $\varepsilon<d(U', \mathbb{C}-U)$ and let U_1 be the component of $\{z \mid d(z, \mathbb{C}-U)>\varepsilon\}$ containing U′. Suppose that ε is so small that U_1 contains the $f(\omega_i)$. Then if $g: U' \to \mathbb{C}$ is an analytic function such that $|g(z)-f(z)|<\varepsilon$ for all $z \in U'$, the set $U'_1=g^{-1}(U_1)$ is homeomorphic to a disc and $g: U'_1 \to U_1$ is polynomial-like of degree d.

If $f: U' \to U$ is a polynomial-like mapping of degree d, we will note

$$K_f = \bigcap_{n \geq 0} f^{-n}(U'),$$

the set of $z \in U'$ such that $f^n(z)$ is defined and belongs to U′ for all $n \in \mathbb{N}$. The set K_f is a compact subset of U′, which we will call the *filled-in Julia set* of f. The *Julia set* J_f of

f is the boundary of K_f. As a dynamical system, f is mainly interesting near K_f; we will neglect what occurs near the boundary of U'.

The following statements are standard for polynomials; they are also valid for polynomial-like mappings. The proofs are simply copies of the classical proofs.

PROPOSITION 1. — *Every attractive cycle has at least one critical point in its immediate basin.*

PROPOSITION 2. — *The set K_f is connected if and only if all the critical points of f belong to K_f. If none of the critical points belong to K_f then K_f is a Cantor set.*

PROPOSITION 3 AND DEFINITION. — *The following conditions are equivalent:*

(i) *Every critical point of f belonging to K_f is attracted by an attractive cycle.*

(ii) *There exists a continuous Riemannian metric on a neigborhood of J_f and $\lambda > 1$ such that for any $x \in J_f$ and any tangent vector t at x we have*

$$\| d_x f(t) \|_{f(x)} \geqq \lambda \| t \|_x.$$

If the above conditions are satisfied, f is said to be hyperbolic.

Sometimes proofs in the setting of polynomial-like mappings give slightly better results.

For instance, by Proposition 1 a polynomial-like mapping of degree d has at most $d-1$ attractive cycles. One can deduce from this that if a polynomial has n attractive periodic cycles and m indifferent ones, then $n+m/2 \leqq d-1$, by perturbing P so as to make half the indifferent points attractive. That is as well as one can do if the perturbation is constrained to be among polynomials of degree d.

However, if one is allowed to perturb among polynomial-like maps of degree d, it is easy to make all the indifferent points attractive, and to see that $n+m \leqq d-1$.

Let $f: U' \to U$ and $g: V' \to V$ be polynomial-like mappings. We will say that f and g are *topologically equivalent* (denoted $f \sim_{\text{top}} g$) if there is a homeomorphism φ from a neighborhood of K_f onto a neighborhood of K_g such that $\varphi \circ f = g \circ \varphi$ near K_f. If φ is quasi-conformal (resp. holomorphic) we will say that f and g are *quasi-conformally* (resp. *holomorphically*) equivalent (denoted $f \sim_{qc} g$ and $f \sim_{\text{hol}} g$). We will say that f and g are *hybrid* equivalent (noted $f \sim_{hb} g$) if they are quasi-conformally equivalent, and φ can be chosen so that $\bar{\partial}\varphi = 0$ on K_f.

We see that

$$f \sim_{\text{hol}} g \quad \Rightarrow \quad f \sim_{hb} g \quad \Rightarrow \quad f \sim_{qc} g \quad \Rightarrow \quad f \sim_{\text{top}} g.$$

If J_f is of measure 0 (no example is known for which this does not hold) the condition $\bar{\partial}\varphi = 0$ on K_f just means that φ is analytic on the interior of K_f.

THEOREM 1 (the Straightening Theorem). — (*a*) *Every polynomial-like mapping $f: U' \to U$ of degree d is hybrid equivalent to a polynomial* P *of degree d.*

(*b*) *If K_f is connected,* P *is unique up to conjugation by an affine map.*

Part (*a*) follows from Propositions 4 and 5 below and part (*b*) is Corollary 2 of Proposition 6.

2. EXTERNAL EQUIVALENCE. — Let $f: U' \to U$ and $g: V' \to V$ be two polynomial-like mappings, with K_f and K_g connected. Then f and g are *externally equivalent* (noted $f \sim_{\text{ext}} g$) if there exist connected open sets U_1, U'_1, V_1, V'_1 such that

$$K_f \subset U'_1 \subset U_1 \subset U,$$
$$K_g \subset V'_1 \subset V_1 \subset V,$$
$$f^{-1}(U_1) = U'_1, \qquad g^{-1}(V_1) = V'_1$$

and a complex-analytic isomorphism

$$\varphi: \quad U_1 - K_f \to V_1 - K_g,$$

such that $\varphi \circ f = g \circ \varphi$.

We will associate to any polynomial-like map $f: U' \to U$ of degree d a real analytic expanding map $h_f : S^1 \to S^1$ also of degree d, unique up to conjugation by a rotation. We will call h_f the *external map* of f.

The construction is simpler if K_f is connected; we will do it in that case first.

Let α be an isomorphism of $U - K_f$ onto $W_+ = \{ z \mid 1 < |z| < R \}$ ($\log R$ is the modulus of $U - K_f$) such that $|\alpha(z)| \to 1$ when $d(z, K_f) \to 0$. Set $W'_+ = \alpha(U' - K_f)$ and $h_+ = \alpha \circ f \circ \alpha^{-1} : W'_+ \to W_+$. Let $\tau : z \mapsto 1/\bar{z}$ be the reflection with respect to the unit circle, and set $W_- = \tau(W_+)$, $W'_- = \tau(W'_+)$, $W = W_+ \cup W_- \cup S^1$ and $W' = W'_+ \cup W'_- \cup S^1$.

By the Schwarz reflection principle [R], h_+ extends to an analytic map $h: W' \to W$; the restriction of h to S^1 is h_f. The mapping is strongly expanding. Indeed, $\tilde{h}: \tilde{W}' \to \tilde{W}$ is an isomorphism, and $\tilde{h}^{-1} : \tilde{W} \to \tilde{W}' \subset \tilde{W}$ is strongly contracting for the Poincaré metric on $\tilde{W}$. This metric restricts to a metric of the form $a|dz|$ with a constant on S^1.

In the general case (K_f not necessarily connected), we begin by constructing a Riemann surface T, an open subset T′ of T and an analytic map $F: T' \to T$ as follows.

Let $L \subset U'$ be a compact connected subset containing $f^{-1}(\overline{U'})$ and the critical points of f, and such that $X_0 = U - L$ is connected. Let X_n be a covering space of X_0 of degree d^n, $\rho_n : X_{n+1} \to X_n$ and $\pi_n : X_n \to X_0$ be the projections and let X be the disjoint union of the X_n. For each n choose a lifting

$$\tilde{f}_n: \quad \pi_n^{-1}(U' - L) \to X_{n+1},$$

of f. Then T is the quotient of X by the equivalence relation identifying x to $\tilde{f}_n(x)$ for all $x \in (U' - L)$ and all $n = 0, 1, \ldots$ The open set T′ is the union of the images of the X_n, $n = 1, 2, \ldots$, and $F: T' \to T$ is induced by the ρ_n.

If K_f is connected, we have just painfuly reconstructed $U - K_f$ and $f: U' - K_f \to U - K_f$. In all cases, T is a Riemann surface isomorphic to an annulus of finite modulus, say $\log R$. We can choose an isomorphism $\alpha: T \to W = \{ z \mid 1 < |z| < R \}$ and continue the construction as above, to construct an expanding map $h_f: S^1 \to S^1$.

If L is chosen differently, neither T, T′ or F are changed.

Let $f_1 : U'_1 \to U_1$ be a polynomial-like restriction of f such that $U'_1 = f^{-1}(U_1)$. The surface T_1 constructed from f_1 is an open subset of T and if h_1 is an external map of f, constructed from an isomorphism α_1 of T_1 onto an annulus, then $\beta = \alpha \circ \alpha_1^{-1}$ gives, using the Schwarz reflection principle again, a real-analytic automorphism of S^1 conjugating h_1 to h.

Let $f : U' \to U$ and $g : V' \to V$ be two polynomial-like mappings, and h_f and h_g be their external maps. If K_f and K_g are connected, the same argument as above shows that f and g are externally equivalent if and only if h_f and h_g are real-analytically conjugate.

If K_f or K_g is not connected, we take real-analytic conjugation of their external maps as the definition of external equivalence.

Remarks. — (1) If there exist open sets U_1, U'_1, V_1, V'_1 and compact sets L and M such that

$$f^{-1}(U') \subset L \subset U'_1 \subset U_1,$$

$$g^{-1}(V') \subset M \subset V'_1 \subset V_1,$$

$$U'_1 = f^{-1}(U_1), \qquad V'_1 = g^{-1}(V_1),$$

L (resp. M) containing the critical points of f (resp. g) and an isomorphism $\varphi : U_1 - L \to V_1 - M$ such that $\varphi \circ f = g \circ \varphi$ on $U'_1 - L$, one can easily construct using φ an isomorphism of $T_{f,1}$ onto $T_{g,1}$ and therefore f and g are externally equivalent.

Unfortunately, there exist pairs f, g of externally equivalent polynomial-like maps, whose equivalence cannot be realized by such a φ. However, any external equivalence can be obtained from a chain of three such realizable equivalences.

(2) Let $\tilde{h} : \mathbb{R} \to \mathbb{R}$ be a lifting of h to the universal covering space. Then h has a unique fixed point a with $h'(a) = \delta > 1$ and there exists a diffeomorphism $\gamma : \mathbb{R} \to \mathbb{R}$ such that $\gamma \circ \tilde{h} \circ \gamma^{-1}$ is $t \mapsto \delta . t$. The map $S : t \mapsto \gamma(\gamma^{-1}(t) + 1)$ satisfies the functional equation $S^d(t) = \delta S(t/\delta)$, which is formally identical to the Cvitanovic-Feigenbaum equation. The trivial map $z \mapsto z^d$ leads to $h(z) = z^d$, $\delta = d$ and $S(t) = t + 1$. Up to conjugation by an affine map, the class of S uniquely determines the class of h. We do not know how to express the property that h is expanding in terms of S.

(3) Let h_1 and h_2 be two expanding maps $S^1 \to S^1$ of degree d. If h_1 and h_2 are real-analytically conjugate, then the eigenvalues along corresponding cycles are equal. It is an open question whether the converse holds. Recently, P. Collet [C] found a partial positive answer. (*)

PROPOSITION 4. — *Let f be a polynomial-like mapping of degree d. Then f is holomorphically equivalent to a polynomial if and only if f is externally equivalent to $z \mapsto z^d$.*

Proof. — Let P be a polynomial. Then P is analytically conjugate to $z \mapsto z^d$ in a neighborhood of infinity. Therefore if U is chosen sufficiently large, and L is chosen sufficiently large in $U' = P^{-1}(U)$, the Riemann surface T constructed will be isomorphic to the one for $z \mapsto z^d$, by an isomorphism which conjugates P to $z \mapsto z^d$. This proves "only if".

(*) Added in proof: The answer is yes (Sullivan, unpublished).

Now let $f: U' \to U$ be a polynomial-like mapping, and $F: T' \to T$ be constructed from f as above. If f is externally equivalent to $P_0: z \mapsto z^d$, there exists an open subset T_1 of T containing X_n for n sufficiently large, and an isomorphism φ of T_1 onto $W_1 = V_1 - \bar{D}$, where V_1 is an open neigborhood of $\bar{D}$ in $\mathbb{C}$, such that $\varphi \circ F = P \circ \varphi$ on $T_1' = F^{-1}(T_1)$.

LEMMA. — *The mapping* φ *extends to an analytic embedding of* T *into* $\mathbb{C} - \bar{D}$, *satisfying* $\varphi \circ F = P_0 \circ \varphi$ *on* T′.

Proof of lemma. — The map $F: T' \to T$ is a covering map of degree d. Let $\sigma: T' \to T'$ be the automorphism induced by the canonical generator of $\pi_1(T) = \mathbb{Z}$, and let σ_0 be the automorphism $z \mapsto e^{2\pi i/d} z$ of $\mathbb{C} - \bar{D}$. Then $\varphi \circ \sigma = \sigma_0 \circ \varphi$ on X_n for n sufficiently large, hence on any connected subset of X to which φ extends analytically. Therefore, if φ is defined on X_n for some $n > 0$, it can be extended to X_{n-1} by the formula $\varphi(x) = P_0(\varphi(F^{-1}(x)))$.

End of proof of the Proposition. — Consider the Riemann surface S obtained by gluing U and $\bar{\mathbb{C}} - {}^{\varphi}(L)$ using $\varphi|_{X_0}$, and the map $g: S \to S$ given by f on U′ and P_0 on $\bar{\mathbb{C}} - {}^{\varphi}(L)$. The surface S is homeomorphic to the Riemann sphere, and is therefore analytically isomorphic to it by the uniformization Theorem. Let $\Phi: S \to \bar{\mathbb{C}}$ be an isomorphism such that $\Phi(\infty) = \infty$, and let $P = \Phi \circ g \circ \Phi^{-1}$. The map $P: \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is analytic, therefore a rational function. Since $P^{-1}(\infty) = \infty$, P is a polynomial. The map $\Phi\varphi$ defines an analytic equivalence between f and P.

Q.E.D.

3. THE AHLFORS-BERS THEOREM. — We will use the "measurable Riemann mapping Theorem", which can be stated as follows:

If $\mu = u(z)\, d\bar{z}/dz$ *is a Beltrami form on* $\mathbb{C}$ *with* $\sup |u(z)| < 1$, *then there exists a quasi-conformal homeomorphism* $\varphi: \mathbb{C} \to \mathbb{C}$ *such that* $\bar{\partial}\varphi/\partial\varphi = \mu$.

This Theorem can also be stated as follows:

Any measurable almost complex structure σ *on a Riemann surface* X *which has bounded dilatation ratio with respect to the initial complex structure* σ_0 *is integrable.*

As the two statements above show, Beltrami forms and almost complex structures are two ways of speaking of the same objects. In this paper we use both, according to convenience, with a preference for the language of almost complex structures, which is more geometric. A dictionary is as follows.

To provide an oriented surface X with an *almost complex structure* σ means to choose, for each $x \in X$ (sometimes almost every $x \in X$), a multiplication by i in $T_x X$ which makes $T_x X$ into a complex vector space, in a way compatible with the orientation.

An $\mathbb{R}$-differentiable map $f: X \to \mathbb{C}$ is *holomorphic at* x *for* σ if $T_x f: T_x X \to \mathbb{C}$ is $\mathbb{C}$-linear for the complex structure σ_x on $T_x X$ and the standard structure on $\mathbb{C}$.

An almost complex structure σ defined almost everywhere on X is called *integrable* if, for any $x \in X$, there is an open neighborhood U of x in X and a homeomorphism $\varphi: U \to V$ with V open in $\mathbb{C}$, which is in the Sobolev space $H^1(U)$ [i. e. such that the distributional derivatives are in $L^2(U)$], and holomorphic for σ at almost every point of U. Such maps can then be used as charts for an atlas which makes X into a $\mathbb{C}$-analytic manifold.

An almost complex structure is best visualized as a field of infinitesimal ellipses: to each $x \in X$, assign an ellipse $u^{-1}(S^1) \subset T_x X$ where $u: T_x X \to \mathbb{C}$ is $\mathbb{C}$-linear for the structure σ_x.

Any field of infinitesimal ellipses defines an almost complex structure; two fields $(E_x)_{x \in X}$ and $(E'_X)_{x \in X}$ define the same almost complex structure if and only if E'_x is $\mathbb{R}$-homothetic to E_x for almost all x.

If X and X′ are oriented surfaces and $\varphi: X' \to X$ is C^1, an almost complex structure σ on X can be pulled back into an almost complex structure $\varphi^* \sigma$ on the open set $W' \subset X'$ on which $\mathrm{Jac}(\varphi) > 0$, as follows. If σ is defined by a field of infinitesimal ellipses (E_x), then $\varphi^* \sigma$ is defined by $(E'_{x'})_{x' \in W'}$, where $E'_{x'} = (T_{x'} \varphi)^{-1} E_{\varphi(x')}$.

If f is holomorphic at $x = \varphi(x')$ for σ, then $f \circ \varphi$ is holomorphic at x' for $\varphi^* \sigma$.

Let U be an open set in $\mathbb{C}$ and denote σ_0 its initial complex structure. A new almost complex structure on σ on U can be defined by its Beltrami form $\mu = u\, d\bar{z}/dz$. For each x, one has

$$u(x) = (\partial f / \partial \bar{z}\,(x)) / (\partial f / \partial z\,(x)),$$

where f is any function holomorphic at x for σ. The correspondance between Beltrami forms and ellipses is as follows: the argument of $u(x)$ is twice the argument of the major axis of E_x, and $|u(x)| = (K-1)/(K+1)$, where $K \geqslant 1$ is the ratio of the lengths of the axes. This ratio K is the *dilatation ratio* of σ at x with respect to the standard structure σ_0.

Let U and U′ be two open sets in $\mathbb{C}$ and $\varphi: U' \to U$ be a C^1-diffeomorphism. If σ is the almost complex structure defined on U by a Beltrami form $\mu = u\, d\bar{z}/dz$, the structure $\sigma' = \varphi^* \sigma$ is defined by $\varphi^! \mu = \mu' = v\, d\bar{w}/dw$, where $v = \lambda (u + a)/(1 + \bar{a} u)$, with

$$\lambda = \overline{(\partial \varphi / \partial w)} / (\partial \varphi / \partial w) \qquad \text{and} \qquad a = (\partial \varphi / \partial \bar{w}) / \overline{(\partial \varphi / \partial w)}.$$

If φ is holomorphic for the original structure, then σ and σ' have the same dilatation ratio.

Concerning the dependence on parameters, we shall use the two following results: Let U be a bounded open set in $\mathbb{C}$.

(*a*) Let $(\mu_n) = (u_n\, d\bar{z}/dz)$ be a sequence of Beltrami forms on U and $\mu = u\, d\bar{z}/dz$ be another Beltrami form. Suppose that there is a $m < 1$ such that $\|u\|_\infty \leqq m$ and $\|u_n\|_\infty \leqq m$ for each n, and that $u_n \to u$ in the L^1 norm. Let $\varphi: U \to D$ be a quasi-conformal homeomorphism such that $\bar{\partial}\varphi / \partial \varphi = \mu$. Then there exists a sequence (φ_n) of quasi-conformal homeomorphisms $U \to D$ such that $\bar{\partial}\varphi_n / \partial \varphi_n$ for each n and that $\varphi_n \to \varphi$ uniformly on U([A], [L−V]).

(*b*) Let Λ be an open set in $\mathbb{C}^n$ and $(\mu_\lambda = u_\lambda\, d\bar{z}/dz)_{\lambda \in \Lambda}$ be a family of Beltrami forms. Suppose $\lambda \mapsto u_\lambda(z)$ is holomorphic for almost each $z \in U$, and that there is a $m < 1$ such that $\|u_\lambda\|_\infty \leqq m$ for each $\lambda \in \Lambda$. For each λ, extend μ_λ to $\mathbb{C}$ by $\mu_\lambda = 0$ on $\mathbb{C} - U$, and let $\varphi_\lambda: \mathbb{C} \to \mathbb{C}$ be the unique quasi-conformal homeomorphism such that $\bar{\partial}\varphi_\lambda / \partial \varphi_\lambda = \mu_\lambda$ and $\varphi(z) - z \to 0$ when $|z| \to \infty$. Then $(\lambda, z) \mapsto (\lambda, \varphi_\lambda(z))$ is a homeomorphism of $\Lambda \times \mathbb{C}$ onto itself, and for each $z \in \mathbb{C}$ the map $\lambda \mapsto \varphi_\lambda(z)$ is $\mathbb{C}$-analytic.

4. MATING A HYBRID CLASS WITH AN EXTERNAL CLASS.

PROPOSITION 5. — *Let $f: U' \to U$ be a polynomial-like mapping, and $h: S^1 \to S^1$ be an $\mathbb{R}$-analytic expanding map of the same degree d. There then exists a polynomial-like mapping $g: V' \to V$ which is hybrid equivalent to f and whose exterior class is h.*

Proof. — Let $A \subset U$ be a compact manifold with C^1-boundary, homeomorphic to $\bar{D}$, and containing K_f in its interior. Moreover, we require that $A' = f^{-1}(A)$ be homeomorphic to $\bar{D}$ and that $A' \subset \mathring{A}$. Set $Q_f = A - \mathring{A}'$.

The mapping h extends to an analytic map $V' \to V$ where V' and V are neighborhoods of S^1 in $\mathbb{C}$. If V' is chosen sufficiently small, there exists $R > 1$ such that $B = \{ z \mid 1 < |z| \leqq R \}$ is contained in V and $B' = h^{-1}(B)$ is homeomorphic to B and contained in $\mathring{B}$. Set $Q_h = B - \mathring{B}'$.

Let ψ_0 be an orientation-preserving C^1-diffeomorphism of ∂A onto ∂B. Since $\partial A'$ and $\partial B'$ are d-fold covers of ∂A and ∂B respectively, there exists a diffeomorphism $\psi_1: A' \to B'$ such that $\psi_0 \circ f = h \circ \psi_1$. Let $\psi: Q_f \to Q_h$ be a diffeomorphism inducing ψ_0 on ∂A and ψ_1 on $\partial A'$.

Call σ_0 the standard complex structure of $\mathbb{C}$ and σ_1 the complex structure $\psi^* \sigma_0$ on Q_f. Define the complex structure σ on A by taking $(f^n)^*(\sigma_1)$ on $f^{-n}(Q_f - \partial A')$ and σ_0 on K_f. This complex structure is not defined at inverse images of the critical points ω_i (if $\omega_i \varepsilon / K_f$); it is discontinuous along the curves $f^{-n}(\partial A')$ although these discontinuities could be avoided by a more careful choice of ψ. However, the much nastier discontinuities along ∂K_f cannot be avoided.

Since f is holomorphic, the dilatation of σ is equal to that of σ_1, hence bounded since ψ was chosen of class C^1. By the measurable Riemann mapping Theorem, σ defines a complex structure on A, and there exists a homeomorphism $\varphi: \mathring{A} \to D$ which is holomorphic for σ on $\mathring{A}$ and σ_0 on D.

Set $g = \varphi \circ f \circ \varphi^{-1}: \varphi(\mathring{A}') \to D$. Clearly g is holomorphic, hence polynomial-like of degree d. The mapping φ is a hybrid equivalence of f with g and $\psi \circ \varphi^{-1}$ is an exterior equivalence of h and g.

Q.E.D.

5. UNIQUESS OF THE MATING. — Let $f: U' \to U$ and $g: V' \to V$ be two polynomial-like mappings of degree $d > 1$, with K_f and K_g connected. Let $\varphi: U_1 \to V_1$ be a hybrid equivalence and $\psi: U_2 - K_f \to V_2 - K_g$ an external equivalence. Define $\Phi: U_2 \to V_2$ by $\Phi = \psi$ on $U_2 - K_f$ and $\Phi = \varphi$ on K_f.

We will give a topological condition making Φ continuous, and show that if this condition is satisfied, then Φ is in fact holomorphic.

One way to state the condition is to require that φ and ψ operate the same way on the $d-1$ prime ends of K_f that are fixed by f. We shall restate the condition so as to avoid the theory of prime ends.

Note that the condition is void if $d = 2$.

Let Q be a space having the homotopy type of an oriented circle; let $f: Q \to Q$ and $\alpha: Q \to Q$ have degree d and 1 respectively and satisfy $\alpha \circ f = f \circ \alpha$. Define

$[\alpha;\ f] \in \mathbb{Z}/(d-1)$ as follows. Let $\tilde{f}: \tilde{Q} \to \tilde{Q}$ and $\tilde{\alpha}: \tilde{Q} \to \tilde{Q}$ be liftings to the universal covering space, and $\tau: \tilde{Q} \to \tilde{Q}$ be the automorphism giving the generator of $\pi_1(Q)$ specified by the orientation.

Clearly, $\tilde{f} \circ \tilde{\alpha} = \tau^i \circ \tilde{\alpha} \circ \tilde{f}$ for some i; if we replace $\tilde{f}$ by $\tau \circ \tilde{f}$ then i does not change, whereas replacing $\tilde{\alpha}$ by $\tau \circ \tilde{\alpha}$ changes i to $i+d-1$.

Therefore the class $[\alpha, f]$ of i in $\mathbb{Z}/(d-1)$ is independant of the choices.

We will require the above invariant in a slightly more general context. Let Q_1 and Q_2 be two spaces having the homotopy type of an oriented circle, and Q'_1, Q'_2 subsets such the inclusions are homotopy equivalences.

Let $f: Q'_1 \to Q_1$ and $g: Q'_2 \to Q_2$ be maps of degree d and $\varphi, \psi: Q_1 \to Q_2$ be maps of degree 1, mapping Q'_1 into Q'_2, and satisfying $g \circ \varphi = \varphi \circ f$ and $g \circ \psi = \psi \circ f$.

Define $[\varphi, \psi;\ f, g] \in \mathbb{Z}/(d-1)$ as follows: construct $\tilde{\varphi}, \tilde{\psi}, \tilde{f}, \tilde{g}$ by lifting to the universal covering spaces. Then

$$\tilde{\varphi} \circ f = \tau^i \circ \tilde{g} \circ \tilde{\varphi}$$

and

$$\tilde{\psi} \circ f = \tau^j \circ \tilde{g} \circ \tilde{\psi}$$

for some i and j. The class of $j-i$ in $\mathbb{Z}/(d-1)$ is independant of the choices and will be noted $[\varphi, \psi;\ f, g]$.

If $Q'_1 = Q_1$ and φ is a homeomorphism, then $[\varphi, \psi;\ f, g] = [\varphi^{-1}\psi;\ f]$.

Proposition 6. — *Let $f: U' \to U$ and $g: V' \to V$ be two polynomial-like mappings of degree $d>1$, with K_f and K_g connected. Let $\varphi: U_1 \to V_1$ be a hybrid equivalence and $\psi: U_2 - K_f \to V_2 - K_g$ be an external equivalence. If $[\varphi, \psi;\ f, g] = 0$ then the map Φ which agrees with φ on K_f and with ψ on $U_2 - K_f$ is a holomorphic equivalence between f and g.*

Lemma 1. — *The map Φ is a homeomorphism.*

Proof. — Let $\alpha = \psi^{-1} \circ \varphi$; we must show that α is close to the identify near K_f.

Let $\rho(z)\,|dz|$ be the Poincaré metric of $U - K_f$ and let d_P be the associated distance; d will denote the Euclidian distance. There is a constant M such that $\rho(z) \geqq M/d(z, K_f)$.

The mapping $\tilde{f}: U' - K_f \to U - K_f$ on covering spaces is bijective; let h be its inverse. Then h is contracting for d_P.

Choose a compact set $C \subset U - K_f$ such that the union

$$\bigcup_{i>0} \bigcup_j h^i(\tau^j(C))$$

is the inverse image in $U - K_f$ of a neighborhood of K_f; let $m = \sup_{x \in C} d_P(\tilde{\alpha}(x), x)$.

The map h commutes with $\tilde{\alpha}$ because of the hypothesis $[\varphi, \psi; f, g]=0$; and τ is an isometry. It follows that

$$d_P(\tilde{\alpha}(h^i(\tau^j(x)), h^i(\tau^j(x))) = d_P(h^i(\tilde{\alpha}(\tau^j(x))), h^i(\tau^j(x)))$$
$$< d_P(\tilde{\alpha}(\tau^j(x)), \tau^j(x)) = d_P(\tau^j(\tilde{\alpha}(x)), \tau^j(x)) < m$$

for $x \in C$.

Therefore $d(\tilde{\alpha}(x), x) \to 0$ as $x \to K_f$. This shows that Φ is continuous; to show that Φ^{-1} is continuous, you repeat the proof above exchanging φ and ψ.

Q.E.D.

LEMMA 2 (S. Rickman [Ri]). — *Let* $U \subset \mathbb{C}$ *be open,* $K \subset U$ *be compact,* φ *and* Φ *be two mappings* $U \to \mathbb{C}$ *which are homeomorphisms onto their images. Suppose that* φ *is quasi-conformal, that* Φ *is quasi-conformal on* $U-K$ *and that* $\varphi=\Phi$ *on* K. *Then* Φ *is quasi-conformal, and* $D\Phi=D\varphi$ *almost everywhere on* K.

Proof. — We may assume that $\Phi(U)$ and $\varphi(U)$ are bounded. We need to show that Φ is in the Sobolev space $H^1(U)$, and find bounds for the excentricity of its derivative.

Since φ is in $H^1(U)$ it is enough to show that $u=\mathrm{Re}(\Phi-\varphi)\in H^1(U)$ [and similarly for $\mathrm{Im}(\Phi-\varphi)$].

Let $\eta_n: \mathbb{R} \to \mathbb{R}$ be a C^1-function such that

$$\begin{aligned} \eta_n(x) &= x - 1/n && \text{for } x > 2/n, \\ \eta_n(x) &= x + 1/n && \text{for } x < -2/n, \\ \eta_n(x) &= 0 && \text{for } -1/2n < x < 1/2n, \\ \eta_n'(x) &< 1 && \text{for } x \in \mathbb{R}. \end{aligned}$$

The sequence $u_n=\eta_n \circ u$ is a Cauchy sequence in H^1, with limit u. Since $u_n=0$ on a neighborhood of K for all n, $Du=0$ almost everywhere on K.

Q.E.D.

Proposition 6 is now immediate: we have to show that Φ satisfies the Cauchy-Riemann equations almost everywhere. On K_f, $D\Phi=D\varphi$ almost everywhere, and on U_2-K_f, Φ is holomorphic.

Q.E.D.

COROLLARY 1. — *Let* f *and* g *be two polynomial-like mappings of degree* 2 *with* K_f *and* K_g *connected. If* f *and* g *are hybrid equivalent and externally equivalent, then they are holomorphically equivalent.*

COROLLARY 2. — *Let* P *and* Q *be two polynomials with* K_f *and* K_g *connected. If* P *and* Q *are hybrid equivalent, then they are conjugate by an affine map.*

Proof. — If P is a polynomial of degree d, there is an external equivalence $\psi_P: \mathbb{C}-K_P \to \mathbb{C}-D$ between P and $z \mapsto z^d$ defined on $\mathbb{C}-K_P$. Then $\psi_{P,Q}=\psi_Q^{-1}\circ\psi_P$ is an isomorphism of $\mathbb{C}-K_P$ onto $\mathbb{C}-K_Q$, and the map Φ constructed in Proposition 4 is an isomorphism of $\mathbb{C}$ onto itself, therefore affine.

COROLLARY 3. — *Let c_1 and c_2 be two points of* M. *If the polynomials $z \mapsto z^2 + c_1$ and $z \mapsto z^2 + c_2$ are hybrid equivalent, then $c_1 = c_2$.*

6. QUASI-CONFORMAL EQUIVALENCE IN DEGREE 2.

PROPOSITION 7. — *Suppose c_1 and c_2 are in $\mathbb{C}$, with c_1 in ∂M. If the polynomials $P_1 : z \mapsto z^2 + c_1$ and $P_2 : z \mapsto z^2 + c_2$ are quasi-conformally equivalent, then $c_1 = c_2$.*

Proof. — Let $\varphi : U \to V$ be a quasi-conformal equivalence of P_1 with P_2. If K_{P_1} is of measure zero, then φ is a hybrid equivalence, and the result follows from Corollary 3 to Proposition 6.

In the general case, consider the Beltrami form $\mu = \bar{\partial}\varphi / \partial\varphi$ and let μ_0 be the form which agrees with μ on K_{P_1} and with 0 on $\mathbb{C} - K_{p_1}$. Set $k = \| \mu_0 \|_\theta$; certainly $k < 1$. For any $t \in D_{1/k}$ there exists a unique quasi-conformal homeomorphism $\Phi_t : \mathbb{C} \to \mathbb{C}$ such that

$$\bar{\partial}\Phi_t / \partial\Phi_t = t\,\mu_0,$$

$\Phi_t(0) = 0$ and $\Phi_t(z)/z \to 1$ as $z \to \theta$.

Then $\Phi_t \circ P_1 \circ \Phi_t^{-1}$ is a polynomial of the form $z \mapsto z^2 + u(t)$, where $u : D_{1/k} \to \mathbb{C}$ is holomorphic. Since $u(0) = c_1 \in \partial$M and $u(t)$ is in M for all t, the function u is constant, in particular $u(1) = c_1$. Then $\varphi \circ \Phi_1^{-1}$ is a hybrid equivalence of P_1 and P_2, and the Proposition follows from Corollary 3 of Proposition 6.

Q.E.D.

CHAPTER II

Analytic families of polynomial-like mappings

1. DEFINITIONS.

DEFINITION. — *Let Λ be a complex analytic manifold and $\mathbf{f} = (f_\lambda : U'_\lambda \to U_\lambda)_{\lambda \in \Lambda}$ be a family of polynomial-like mappings. Set $\mathcal{U} = \{ (\lambda, z) \mid z \in U_\lambda \}$, $\mathcal{U}' = \{ (\lambda, z) \mid z \in U'_\lambda \}$* and *$f(\lambda, z) = (\lambda, f_\lambda(z))$. Then* **f** *is an analytic family if the following conditions are satisfied:*

(1) *$\mathcal{U}$ and $\mathcal{U}'$ are homeomorphic over Λ to $\Lambda \times D$.*

(2) *The projection from the closure of $\mathcal{U}'$ in $\mathcal{U}$ to Λ is proper.*

(3) *The mapping $f : \mathcal{U}' \to \mathcal{U}$ is complex-analytic and proper.*

For such a family the degree of f_λ is independant of λ if Λ is connected; we will call it the degree of **f**. Set $K_\lambda = K_{f_\lambda}$, $J_\lambda = J_{f_\lambda}$ and $\mathcal{K}_\mathbf{f} = \{ (\lambda, z) \mid z \in K_\lambda \}$. The set $\mathcal{K}_\mathbf{f}$ is closed in $\mathcal{U}$ and the projection of $\mathcal{K}_\mathbf{f}$ onto Λ is proper, since $\mathcal{K}_\mathbf{f} = \bigcap_n f^{-n}(\mathcal{U}')$. Let $M_\mathbf{f}$ denote the set of λ for which K_λ is connected.

By the straightening Theorem, if **f** is an analytic family of polynomial-like mappings of degree d, we can find for each λ a polynomial P_λ of degree d and a homeomorphism $\varphi_\lambda : V_\lambda \to W_\lambda$, where V_λ and W_λ are neighborhoods of K_λ and K_{P_λ} respectively, which defines a hybrid equivalence between f_λ and P_λ. In this chapter, we will investigate whether P_λ and φ_λ can be chosen continuous, or even analytic in λ.

2. TUBINGS. — Choose $R>1$; set $R'=R^{1/d}$ and $Q_R=\{ z \mid R'\leqq |z| \leqq R \}$.

PROPOSITION 8 AND DEFINITION. — *Let* $\mathbf{f}=(f_\lambda: U'_\lambda \to U_\lambda)$ *be an analytic family of polynomial-like mappings of degree* d, *with* Λ *contractible. Then there exists a* C^1-*embedding* $T: (\lambda, x)\mapsto(\lambda, T_\lambda(x))$ *of* $\Lambda\times Q_R$ *into* $\mathcal{U}$ *such that* $T(\Lambda\times Q_R)$ *is of the form* $\mathcal{A}-\mathring{\mathcal{A}}'$ *with* $\mathcal{A}$ *and* $\mathcal{A}'$ *homeomorphic over* Λ *to* $\Lambda\times D$, $\mathcal{K}_{\mathbf{f}}\subset\mathcal{A}'\subset\mathring{\mathcal{A}}$ *and* $T_\lambda(x^d)=f_\lambda(T_\lambda(x))$ *for* $|x|=R'$.

Such a mapping will be called a tubing *for* $\mathbf{f}$.

Proof. — Let Ξ' be the set of $(\lambda, z)\in\mathcal{U}'$ for which z is a critical point of f_λ and let $\Xi=f(\Xi')$. The projections of Ξ and Ξ' onto Λ are proper. Let $\varphi: (\lambda, x)\mapsto(\lambda, \varphi_\lambda(x))$ be a diffeomorphism of $\Lambda\times D$ onto $\mathcal{U}$ and $\eta: \Lambda\to]0, 1[$ a C^1-function such that $\varphi_\lambda(D_{\eta(\lambda)})$ contains $\overline{U}'_\lambda$ and the critical values of f_λ for all $\lambda\in\Lambda$.

Define the embedding $T^0: (\lambda, x)\mapsto(\lambda, T^0_\lambda(x))$ of $\Lambda\times S^1_R$ into $\mathcal{U}$ by

$$T^0_\lambda(x)=\varphi_\lambda\left(\frac{\eta(\lambda)}{r}.x\right).$$

For each λ, the curve $\Gamma_\lambda=T^0_\lambda(S^1_R))$ encloses the critical values of f, and therefore $\Gamma'_\lambda=f^{-1}_\lambda(\Gamma_\lambda)$ is homeomorphic to a circle. Then there exists a diffeomorphism $T^1_\lambda: S^1_{R'}\to\Gamma'_\lambda$ such that $f_\lambda(T^1_\lambda(x))=T^0_\lambda(x^d)$ for $|x|=R'$; the region bounded by Γ_λ and Γ'_λ is an annulus and there exists an embedding T_λ of Q_R into U_λ inducing T^0_λ and T^1_λ.

In order to find T^1_λ and T^0_λ depending continuously or differentiably on λ, one must find a continuous or a C^1-section of a principal fibration over Λ, with fiber principal under the group $\mathcal{D}$ of diffeomorphisms of Q_R inducing the identity on S^1_R and a rotation $x\mapsto e^{2\pi ip/d}x$ on S^1_R. Since Λ is assumed contractible, there is such a section.

Q.E.D.

Remark. — According to a Theorem of Cerf, $\mathcal{D}$ can be written $\mathbb{Z}\times\mathcal{D}_0$, with $\mathcal{D}_0$ contractible. Therefore the hypothesis "Λ contractible" can be replaced by "Λ simply connected". One can even do better. If instead of

$$T^0_\lambda(x)=\varphi_\lambda\left(\frac{\eta(\lambda)}{r}.x\right)$$

we require only that T be of the form

$$\varphi_\lambda\left(u(\lambda).\frac{\eta(\lambda)}{r}.x\right)$$

with $u: \Lambda\to S^1$ of class C^1, then the space of possible $(T^0_\lambda, T^1_\lambda, T_\lambda)$ is homeomorphic to the space $\mathcal{D}'$ of diffeomorphisms of Q_R inducing

$$x\mapsto u, x \quad \text{on } S^1_R,$$
$$x\mapsto v.x \quad \text{on } S^1_{R'},$$

with $v \in S^1$ and $u = v^d$. The space $\mathscr{D}'$ can be written $\mathbb{Z}/(d-1) \times \mathscr{D}'_0$ with $\mathscr{D}'_0$ contractible. Therefore the only obstruction to finding a section to the corresponding fiber bundle is in $H^1(\Lambda; \mathbb{Z}/(d-1))$.

The hypothesis "Λ contractible" can be replaced by "$H^1(\Lambda; \mathbb{Z}/(d-1)) = 0$", and in particular be omitted altogether if $d = 2$.

We will be primarily interested in cases where Λ is homeomorphic to D.

A tubing T of $\mathbf{f}$ will be called *horizontally analytic* if the map $\lambda \mapsto T_\lambda(x)$ is complex-analytic for all $x \in Q_{\mathbf{R}}$.

Proposition 9. — *For any $\lambda_0 \in \Lambda$, there exists a neighborhood Λ' of λ_0 in Λ and a horizontally analytic tubing of $(f_\lambda)_{\lambda \in \Lambda'}$.*

Proof. — Let τ^0 be a C^1-embedding of a neighborhood V_1 of $S^1_{\mathbf{R}}$ into $U_{\lambda_0} - \overline{U}'_{\lambda_0}$, enclosing the critical values of f_{λ_0}. There exists an embedding τ^1 from a neighborhood V'_1 of $S^1_{\mathbf{R}'}$ into U'_{λ_0} such that $f_{\lambda_0}(\tau^1(x)) = \tau^0(x^d)$ for $x \in V_1$, and a C^1-embedding τ of a neighborhood V of $Q_{\mathbf{R}}$ into U_{λ_0} that agrees with τ^0 near $S^1_{\mathbf{R}}$ and with τ^1 near $S^1_{\mathbf{R}'}$.

Define $\tilde{\tau}: \Lambda'' \times V_1 \to \mathscr{U}'$ by $\tilde{\tau}_\lambda(x) \in f_\lambda^{-1}(\tau(x^d))$, the choice in this finite set being imposed by continuity, the requirement that $\tilde{\tau}_{\lambda_0} = \tau|_{V'}$, and that Λ'' be a neighborhood of λ_0 in Λ.

Let $h: V \to \mathbb{R}_+$ be a C^1-function, with support in V_1 and equal to 1 on a neighborhood of $S^1_{\mathbf{R}'}$. Set $T(\lambda, x) = (\lambda, (1 - h(x)) . \tau(x) + h(x) . \tilde{\tau}_\lambda(x))$. Then T is a C^1-mapping above Λ'', inducing τ over λ_0 and therefore an embedding of $\Lambda' \times Q_{\mathbf{R}}$ for some neighborhood Λ' of λ_0. The mapping T is horizontally analytic and satisfies $T(\lambda, x^d) = f(T(\lambda, x))$ for all $(\lambda, x) \in \Lambda' \times S^1_{\mathbf{R}'}$. So T is a horizontally analytic tubing.

Q.E.D.

Remark. — Let R and R_1 be any two numbers > 1 and $T: \Lambda \times Q_{\mathbf{R}} \to \mathscr{U}$ a tubing of f. Define $\alpha: Q_{\mathbf{R}_1} \to Q_{\mathbf{R}}$ by $\alpha(re^{it}) = r^{\mathscr{L}} e^{it}$, where $l = \log R / \log R_1$. Then $T_1 = T \circ (1_\Lambda \times \alpha)$ is a tubing of f using $Q_{\mathbf{R}_1}$. If T is horizontally analytic, so is T_1. Therefore the choice of R is unimportant.

3. Review and statements. — In the remainder of this chapter we will assume we have an analytic family $\mathbf{f} = (f_\lambda: U'_\lambda \to U_\lambda)_{\lambda \in \Lambda}$ of polynomial-like mappings of degree d. We will suppose Λ contractible, and that a tubing T of $\mathbf{f}$ has been chosen.

Define $\mathscr{A} = T(\Lambda \times D_{\mathbf{R}})$, $\mathscr{A}' = T(\Lambda \times D_{\mathbf{R}'})$ and $Q_{\mathbf{f}} = \mathscr{A} - \mathring{\mathscr{A}}' = T(\Lambda \times Q_{\mathbf{R}})$; set $\Psi = T^{-1}$. By repeating the constructions which prove Propositions 4 and 5, we find for each λ a polynomial P_λ of degree d, monic and without a term of degree $d-1$, and a hybrid equivalence φ_λ of f_λ with P_λ, defined on $\mathring{A}_\lambda$.

If $d = 2$, the polynomial P_λ is of the form $z \mapsto z^2 + \chi(\lambda)$, and this defines a mapping $\chi: \Lambda \to \mathbb{C}$.

Let us go through the steps of the construction:

(1) Consider the Beltrami form

$$\mu_{\lambda, 0} = \bar{\partial}\psi_\lambda / \partial\psi_\lambda,$$

defined on Q_λ.

(2) Let

$$\mu_{\lambda, n}=(f_\lambda^n)^* \mu_{\lambda, 0},$$

be the corresponding Beltrami form on

$$Q_{\lambda, n}=f_\lambda^{-n}(Q_\lambda).$$

(3) Define the Beltrami form μ_λ on $\mathring{A}_\lambda$ by setting

$$\mu_\lambda(z)=\mu_{\lambda, n}(z) \qquad \text{if} \quad z \in Q_{\lambda, n},$$

and

$$\mu_\lambda(z)=0 \qquad \text{if} \quad z \in K_\lambda.$$

(4) Using the measurable Riemann mapping Theorem, the form μ_λ defines a complex structure σ_λ on A_λ; the mapping $\psi_\lambda: \mathring{Q}_\lambda \to \mathring{Q}_R$ is holomorphic for σ_λ in the domain and σ_0 in the range.

(5) Glue $\mathring{A}_\lambda$ with the structure σ_λ to $\bar{\mathbb{C}}-\bar{D}_R$ using ψ_λ; you get a Riemann surface Σ_λ homeomorphic to S^2. The maps f_λ and $P_0: z \mapsto z^d$ patch together to give an analytic map $g_\lambda: \Sigma_\lambda \to \Sigma_\lambda$.

(6) Using the uniformization Theorem, there exists a unique isomorphism $\Phi_\lambda: \Sigma_\lambda \to \bar{\mathbb{C}}$ tangent to the identity at infinity and such that the polynomial

$$P_\lambda=\Phi_\lambda \circ g_\lambda \circ \Phi_\lambda^{-1},$$

has no term of degree $d-1$; φ_λ is the restriction of Φ_λ to $\mathring{A}_\lambda$.

The step which causes problems is step 3. The mapping $\lambda \mapsto \mu_\lambda$ is not necessarily continuous, even for the weak topology on L^∞. In general, the map is discontinuous at points where f_λ has a rationally indifferent non-persistent cycle. This is closely related to the fact that the mapping $\lambda \mapsto K_\lambda$ is not continuous at such points for the Hausdorff topology. We will analyse a counter-example in chapter III.

For people who dislike cut and paste techniques, steps 4, 5 and 6 can be restated as follows. Extends ψ_λ to a diffeomorphism ψ_λ of $\mathring{A}_\lambda$ onto D_R. Let θ_λ be the complex structure on D_R gotten by transporting μ_λ by Ψ_λ. Let ν_λ be the Beltrami form on $\bar{\mathbb{C}}$ defining θ_λ on D_R, and extended by 0. There exists a unique quasi-conformal homeomorphism $\Phi_\lambda: \mathbb{C} \to \mathbb{C}$ such that

$$\bar{\partial}\Phi_\lambda/\partial\varphi_\lambda=\nu_\lambda$$

and

$$\Phi_\lambda(z)-z \to z \to 0 \quad \text{as} \quad |z| \to \theta.$$

Define $g_\lambda: \mathbb{C} \to \mathbb{C}$ by

$$g_\lambda=\Psi_\lambda \circ f_\lambda \circ \Psi_\lambda^{-1} \quad \text{on } D_{R'}$$

and

$$g_\lambda(z)=z^d \quad \text{on } \mathbb{C}-D_R.$$

Then

$$\varphi_\lambda = \Phi_\lambda \circ \Psi_\lambda$$

and

$$P_\lambda = \Phi_\lambda \circ g_\lambda \circ \Phi_\lambda^{-1},$$

up to conjugation by a translation.

In this chapter we will prove the following Theorems.

THEOREM 1. — *Let $\mathscr{R}$ be the open subset of the first Mañe-Sad-Sullivan decomposition of* Λ (cf. *II*, 4). *Then* φ_λ *and* P_λ *depend continuously on* λ *for* $\lambda \in \mathscr{R}$, *and* P_λ *depends analytically on* λ *for* $\lambda \in \mathscr{R} \cap \mathring{M}_{\mathbf{f}}$.

The first part is Proposition 12 and the second is Corollary 1 to Proposition 13.

THEOREM 2. — *If* $d=2$, *the mapping* $\chi: \Lambda \to \mathbb{C}$ *is continuous on* Λ, *and analytic on* $\mathring{M}_{\mathbf{f}}$.

Continuity is Proposition 14 and analyticity is Corollary 1 of Proposition 13.

4. THE FIRST MAÑE-SAD-SULLIVAN DECOMPOSITION. — Let $(P_\lambda)_{\lambda \in \Lambda}$ be an analytic family of polynomials (or rational functions). In [M-S-S], Mañe, Sad and Sullivan describe two decomposition of Λ into a dense open set and a closed complement. For the first decomposition, the open set is the set of λ for which P_λ is structurally stable on a neighborhood of the Julia set; for the second decomposition structural stability of P_λ is required on the whole Riemann sphere.

In this paragraph we will review the first decomposition in the framework of polynomial-like mappings.

Let $\mathbf{f} = (f_\lambda : U'_\lambda \to U_\lambda)_{\lambda \in \Lambda}$ be an analytic family of polynomial-like mappings.

An indifferent periodic point z_0 of f_{λ_0} is called *persistent* if for each neighborhood V et z_0 there is a neighborhood W of λ_0 such that, for each $\lambda \in W$, the function f_λ has in V an indifferent periodic point of the same period.

Let $\mathscr{I}$ be the set of λ for which f_λ has a non-persistent indifferent periodic point, $\mathscr{F} = \bar{\mathscr{I}}$, and $\mathscr{R} = \Lambda - \mathscr{F}$ the complementary open set.

PROPOSITION 10. — (*a*) *The open set* $\mathscr{R}$ *is dense in* Λ.

(*b*) *For any* λ_0, *there exists a neighborhood* W *of* λ_0 *in* $\mathscr{R}$, *a neighborhood* V *of* J_{λ_0} *in* U_{λ_0} *and an embedding* $\tau: (\lambda, z) \mapsto (\lambda, \tau_\lambda(z))$ *of* $W \times V$ *into* $\mathscr{U}$ *such that:*

(i) $\tau(\lambda, z)$ *is holomorphic in* λ *and quasi-conformal in* z, *with dilatation ratio bounded by a constant independant of* λ.

(ii) *The image of* τ *is a neighborhood of* $\mathscr{I}_W = \{ (\lambda, z) \mid \lambda \in W, z \in J_\lambda \}$, *which is closed in* $\mathscr{U} \cap (W \times \mathbb{C})$.

(iii) *The map* τ_{λ_0} *is the identity of* V, *and for all* $\lambda \in W$ *we have* $f_\lambda \circ \tau_\lambda = \tau_\lambda \circ f_{\lambda_0}$.

The proofs of [M-S-S], completed by [S-T] (*see* also [B-R]) for the case of persistent Siegel discs, can simply be copied in the setting of polynomial-like mappings.

COROLLARY. — *Let* W *be a connected component of* $\mathscr{R}$. *If* $\lambda_1, \lambda_2 \in W$, *then* K_{λ_1} *and* K_{λ_2} *are homeomorphic. In particular, either* $W \subset M_{\mathbf{f}}$ *or* $W \cap M_{\mathbf{f}} = \varnothing$.

PROPOSITION 11. — (*a*) $\mathring{\mathrm{M}}_{\mathbf{f}} \subset \mathscr{R}$.

(*b*) *If* $d=2$, *then* $\mathscr{R}=\Lambda-\partial \mathrm{M}_{\mathbf{f}}$.

Proof. — (*a*) Choose $\lambda_0 \in \mathring{\mathrm{M}}_{\mathbf{f}}$. By contradiction, suppose f_{λ_0} has a non-persistent indifferent periodic point α_0 of period k. There then exists a neighborhood V of 0 in $\mathbb{C}$, an analytic map $t \mapsto \lambda(t)$ of V into Λ and analytic maps

$$\alpha, \quad \omega_1, \ldots, \omega_{d-1},$$

of V into $\mathbb{C}$ with $\lambda(0)=\lambda_0$, $\alpha(0)=\alpha_0$, such that $\alpha(t)$ is a periodic point of $f_{\lambda(t)}$ of period k and eigenvalue $\rho(t)$, where $\rho: \mathrm{V} \to \mathbb{C}^*$ is a non-constant holomorphic function, and such that

$$\omega_1(t), \ldots, \omega_{d-1}(t),$$

are the $(d-1)$ critical points of $f_{\lambda(t)}$ for $t \in \mathrm{V}$.

Let (t_n) be a sequence in V converging to 0, such that $|\rho(t_n)|<1$ for all n. For each n, $\alpha(t_n)$ is an attractive periodic point, so there exist i, j such that

$$f_{\lambda(t_n)}^{kp+i}(\omega_j(t_n))$$

converges to $\alpha(t_n)$ without reaching it as $p \to \infty$. By choosing a subsequence we can assume i and j independant of n.

Since $\lambda(t) \in \mathrm{M}_{\mathbf{f}}$ for all $t \in \mathrm{V}$, we can define a sequence (u_p) of analytic functions on V by

$$u_p(t)=f_{\lambda(t)}^{kp+i}(\omega_j(t)).$$

This sequence is bounded on any compact subset of V since $u_p(t) \in \mathrm{U}'_{\lambda(t)}$. If a subsequence converges to a function $h: \mathrm{V} \to \mathbb{C}$, then $h(t_0)=\alpha(t_n)$ for all n, so $h=\alpha$. Therefore $u_p(t) \to \alpha(t)$ for all $t \in \mathrm{V}$.

But in V there are points t^* such that $|\rho(t^*)|>1$ and $u_p(t^*) \neq \alpha(t^*)$ for all p. The point $\alpha(t^*)$ is a repulsive periodic point and cannot attract the sequence $u_p(t^*)$.

(*b*) By the Corollary to Proposition 10, $\mathscr{R} \subset \Lambda-\partial \mathrm{M}_{\mathbf{f}}$, and $\mathring{\mathrm{M}}_{\mathbf{f}} \subset \mathscr{R}$ by (*a*). We will use the hypothesis $d=2$ to prove $\Lambda-\mathrm{M}_{\mathbf{f}} \subset \mathscr{R}$. For any $\lambda \in \Lambda$, the map f_λ has a unique critical point $\omega(\lambda)$. If $\lambda \in \Lambda-\mathrm{M}_{\mathbf{f}}$, then $\omega(\lambda) \in \mathrm{U}'-\mathrm{K}_\lambda$, so f_λ is hyperbolic and all periodic points are repulsive. Therefore $(\Lambda-\partial \mathrm{M}_{\mathbf{f}}) \cap \mathscr{I}$ is empty, and since $\Lambda-\mathrm{M}_{\mathbf{f}}$ is open, $\Lambda-\mathrm{M}_{\mathbf{f}} \subset \mathscr{R}$.

Q.E.D.

5. CONTINUITY ON $\mathscr{R}$.

PROPOSITION 12. — *On the open set* $\mathscr{R}$ *both* φ_λ *and* P_λ *depend continuously on* λ.

Proof. — Using the notations of 3, clearly $\|\mu_\lambda\|_\infty=\|\mu_{\lambda,0}\|_\infty$ is bounded on any compact subset of Λ by a constant <1. According to I,3, we need to show that $\mu_\lambda \to \mu_{\lambda_0}$ as $\lambda \to \lambda_0$ in the L^1-norm.

Since $\mu_\lambda = \lim \hat{\mu}_{\lambda, n}$ pointwise, where

$$\begin{aligned} \hat{\mu}_{\lambda, n} &= \mu_{\lambda, i} \quad \text{on} \quad f^{-i}(Q_\lambda) \quad \text{for } i \leqq n \\ &= 0 \quad \text{on} \quad A_{\lambda, n} = f_\lambda^{-n-1}(A_\lambda), \end{aligned}$$

and since for each n the form $\hat{\mu}_{\lambda, n}$ depends continuously on λ for the L^1-norm, it is enough to show that $\| \hat{\mu}_{\lambda, n} - \mu_\lambda \|_1$ tends to 0 uniformly on any compact subset of $\mathscr{R}$.

This will follow from the statement that the area of $A_{\lambda, n} - K_\lambda$ tends to zero uniformly on every compact subset of $\mathscr{R}$.

Note that this is false on any open subset of Λ which meets $\mathscr{F}$. Indeed, for every n the area of $A_{\lambda, n}$ depends continuously on λ, but the area of K_λ is discontinuous for every value of λ for which f_λ has a non-persistent rationally indifferent fixed point.

Choose $\lambda_0 \in \mathscr{R}$, and using the notations of Proposition 10 set $\tau_\lambda(z) = \tau(\lambda, z)$, $V_\lambda = \tau_\lambda(V)$, $B_\lambda = V_\lambda \cup K_\lambda$ and $B_{\lambda, n} = f_\lambda^{-n}(B_\lambda)$. There exists a neighborhood W' of λ_0 with compact closure in W and $p \in \mathbb{N}$ such that $A_{\lambda, p} \subset B_\lambda$ for all $\lambda \in W'$. Then $A_{\lambda, n+p} \subset B_{\lambda, n}$, so it is enough to show that the area $\mathbf{m}_n(\lambda)$ of $B_{\lambda, n} - K_\lambda$ tends to 0 uniformly on any compact subset of W'.

Clearly

$$\mathbf{m}_n(\lambda) = \int_{B_{\lambda_0, n} - K_{\lambda_0}} \operatorname{Jac}(\tau_\lambda).$$

Set

$$\mathbf{n}_n(\lambda) = \int_{B_{\lambda_0, n} - K_{\lambda_0}} \| D\tau \|^2.$$

The functions $\mathbf{n}_n$ form a decreasing sequence of plurisubharmonic functions. There exist numbers a, b, with $0 < a < b < \infty$ such that on W'

$$a\, \mathbf{n}_n(\lambda) < \mathbf{m}_n(\lambda) < b\, \mathbf{n}_n(\lambda),$$

since the τ_λ are quasi-conformal with bounded dilatation ratio.

Since $\mathbf{m}_n(\lambda) \to 0$ pointwise, $\mathbf{n}_n(\lambda) \to 0$ pointwise, hence $\mathbf{n}_n(\lambda) \to 0$ uniformly on any compact subset of W', so $\mathbf{m}_n(\lambda) \to 0$ uniformly on any compact subset of W'.

Q.E.D.

6. The locus of hybrid equivalence.

PROPOSITION 13. — *Let* $\mathbf{f} = (f_\lambda : U'_\lambda \to U_\lambda)$ and $\mathbf{q} = (g_\lambda : V'_\lambda \to V_\lambda)$ *be two analytic families of polynomial-like mappings of degree d parametrized by the same manifold* Λ. *Let* $\mathscr{R}$ *be the open subset of* Λ *given by the first Mañe-Sad-Sullivan decomposition for* $\mathbf{f}$ *and let* W *be a connected component of* $\mathscr{R}$ *contained in* $M_{\mathbf{f}}$. *Then the set* $\Gamma \subset W$ *of those* λ *for which* f_λ *and* g_λ *are hybrid equivalent is a complex-analytic subset of* W.

Addendum if $d > 2$. — Let $T_{\mathbf{f}}$ and $T_{\mathbf{g}}$ be tubings of $\mathbf{f}$ and $\mathbf{g}$ respectively, defined on $W \times Q_R$, and with images $Q_{\mathbf{f}} = \mathscr{A} - \mathring{\mathscr{A}}'$ and $Q_{\mathbf{g}} = \mathscr{B} - \mathring{\mathscr{B}}'$ respectively. Since K_{f_λ} is connected for all $\lambda \in W$, $T_{\mathbf{f}}$ can be extended to a homeomorphism $\hat{T}_{\mathbf{f}}: W \times (\bar{D}_R - \bar{D}) \to \mathscr{A} - \mathscr{K}_{\mathbf{f}}$. Choose $\lambda \in \Gamma$ and let α be a hybrid equivalence of f_λ and g_λ. Since this requires that K_{g_λ} be connected, T_{g_λ} can be extended to $\hat{T}_{g_\lambda}: \bar{D}_R - \bar{D} \to B_\lambda - K_{g_\lambda}$.

Set $\mathbf{i}(\alpha) = [\hat{T}_{g_\lambda} \circ \hat{T}_f^{-1}, \alpha; f_\lambda, g_\lambda] \in \mathbb{Z}/(d-1)$, (*cf.* I,4).

Then Γ is the union of subsets Γ_i for $i \in \mathbb{Z}/(d-1)$, where Γ_i is the set of $\lambda \in W$ for which there exists a hybrid equivalence α of f_λ and g_λ with $\mathbf{i}(\alpha) = i$.

We shall show that for each $i \in \mathbb{Z}/(d-1)$ the set Γ_i is an analytic subset of W. By changing T_f or T_g, $\mathbf{i}(\alpha)$ is can be changed to any element of $\mathbb{Z}/(d-1)$, so we only need to show that Γ_0 is analytic.

Proof. – Choose $\lambda_0 \in W$ and let Λ' be a neighborhood of λ_0 in W. Choose $T_g: \Lambda' \times Q_R \to \mathscr{V}$ a horizontally analytic tubing of $\mathbf{g}$ and $\tau: \Lambda' \times \check{U} \to \mathscr{U}$ an M-S-S trivialisation of $\mathbf{f}$ (*cf.* Proposition 10).

Let $T_{f_{\lambda_0}}: Q_R \to U_{\lambda_0}$ be a tubing of f_{λ_0}, whose image is contained in $\check{U}$ (since K_{f_λ} is connected, there are no critical values of f_{λ_0} in $U_{\lambda_0} - K_{\lambda_0}$). Now define a horizontally analytic tubing $T_{\mathbf{f}}$ of $\mathbf{f}$ over Λ' by $T_{\mathbf{f}, \lambda} = \tau_\lambda \circ T_{f_{\lambda_0}}$. The images $Q_{\mathbf{f}}$ and $Q_{\mathbf{g}}$ of $T_{\mathbf{f}}$ and $T_{\mathbf{g}}$ are respectively of the form $\mathscr{A} - \mathring{\mathscr{A}}'$ and $\mathscr{B} - \mathring{\mathscr{B}}'$, where $\mathscr{A}$ and $\mathscr{B}$ are homeomorphic to $\Lambda' \times D$.

For any $\lambda \in \Lambda'$, set

$$\gamma_\lambda = T_{g, \lambda} \circ T_{f, \lambda}^{-1}: Q_{\mathbf{f}, \lambda} \to Q_{\mathbf{g}, \lambda}$$

and

$$\tilde{\gamma}_\lambda = \tau_\lambda \circ \gamma_\lambda: Q_{\mathbf{f}, \lambda_0} \to Q_{\mathbf{g}, \lambda},$$

and define the Beltrami forms

$$\nu_{\lambda, 0} = \bar{\partial}\gamma_\lambda / \partial\gamma_\lambda \quad \text{on} \quad Q_{\mathbf{f}, \lambda}$$

and

$$\tilde{\nu}_{\lambda, 0} = \bar{\partial}\tilde{\gamma}_\lambda / \partial\tilde{\gamma}_\lambda \quad \text{on} \quad Q_{f, \lambda_0}.$$

Further define ν_λ on A_λ and $\tilde{\nu}_\lambda$ on A_{λ_0} by

$$\nu_\lambda = (f^n)^*_{\nu_{\lambda, 0}} \quad \text{on } Q_{f_\lambda, n}, \qquad \nu = 0 \quad \textit{on } K_{f_\lambda}.$$
$$\tilde{\nu}_\lambda = (f^n)^*_{\nu_{\lambda, 0}} \quad \text{on } Q_{f_{\lambda_0}, n}, \qquad \tilde{\nu} = 0 \quad \text{on } K_{f_{\lambda_0}}.$$

Call θ_λ and $\tilde{\theta}_\lambda$ the complex structures on A_λ and A_{λ_0} defined by ν_λ and $\tilde{\nu}_\lambda$ respectively. The map $\lambda \mapsto \tilde{\nu}_\lambda$ is a complex analytic map of Λ' into $L^\infty(A_{\lambda_0})$, so there exists a unique complex structure $\tilde{\theta}$ on $\Lambda' \times A_0$ inducing $\tilde{\theta}$ on $\{\lambda\} \times A_{\lambda_0}$ for each $\lambda \in \Lambda'$ and such that $\Lambda' \times \{x\}$ is an analytic submanifold for each $x \in A_{\lambda_0}$ (*cf.* I,3).

Let $\theta = \tau_*(\tilde{\theta})$ be the corresponding complex structure on $\check{\mathscr{U}} = \tau(\Lambda' \times \check{U})$; it induces θ_λ on $\tau_\lambda(\check{U})$ for each λ and the $\tau(\Lambda' \times \{x\})$ are complex analytic submanifolds of $\check{\mathscr{U}}$.

Since $\theta=\sigma_0$ on $\mathscr{K}\cap\check{\mathscr{U}}$, it can be extended to $\mathscr{K}\cup\check{\mathscr{U}}$ by setting $\theta=\sigma_0$ on $\mathscr{K}$. Proposition 13 now follows from the following two Lemmas.

LEMMA 1. — *For any $\lambda\in\Lambda'$, the following conditions are equivalent:*

(i) $\lambda\in\Gamma_0$.

(ii) *There exists an isomorphism*

$$\alpha:\quad (\mathring{A}_\lambda,\ \theta_\lambda)\to(\mathring{B}_\lambda,\ \sigma_0)$$

extending γ_λ, and such that $\alpha\circ f_\lambda=g_\lambda\circ\alpha$ on A'_λ.

(iii) *There exists a map $\alpha:\ \mathring{A}_\lambda\to\mathbb{C}$, holomorphic with respect to θ_λ and extending γ_λ.*

LEMMA 2. — *Let Ω be a complex analytic manifold, $\pi:\ \Omega\to\Lambda$ be a submersion, and Ω' be open in Ω. Suppose both Ω and Ω' are homeomorphic over Λ to $\Lambda\times D$, and that $\mathring{Q}=\Omega-\overline{\Omega}'$ is homeomorphic over Λ to $\Lambda\times\mathring{Q}_R$. Let $h:\ Q\to\mathbb{C}$ be a holomorphic function, and call h_λ the restriction of h to $\mathring{Q}_\lambda=Q\cap\pi^{-1}(\lambda)$. Then the set of λ for which h_λ extends to a holomorphic function on $\Omega_\lambda=\pi^{-1}(\lambda)$ is a closed analytic subset of Λ.*

Proof of Lemma 1. — Clearly (ii) implies both (i) and (iii). To see that (i) implies (ii), let β be a hybrid equivalence of f_λ and g_λ such that

$$\mathbf{i}(\beta)=[\check{T}_{g_\lambda}\circ\check{T}_{f_\lambda}^{-1},\ \beta;f_\lambda,g_\lambda]=0.$$

Define $\alpha:\ A_\lambda\to B_\lambda$ by $\alpha=\check{T}_{g_\lambda}\circ\check{T}_{f_\lambda}^{-1}$ on $A_\lambda-K_{f_\lambda}$ and $\alpha=\beta$ on K_{f_λ}. By Proposition 6 (I,4), α is an isomorphism of $\mathring{A}_\lambda$ with the complex structure θ_λ onto $\mathring{B}_\lambda$ with the complex structure σ_0.

Next we show that (iii) implies (ii). For any $z\in B'_\lambda$, the cardinal of $\alpha^{-1}(z)$, counted with multiplicity, is the winding number of a loop $\alpha|_\Gamma$ around z, where Γ is a loop in $\mathring{A}_\lambda$ near $\partial A'_\lambda$; it is 1. If we had chosen z in $\mathbb{C}-B_\lambda$, we would have found 0. Therefore α is bijective from $\mathring{A}_\lambda$ onto $\mathring{B}_\lambda$, and hence an isomorphism of $(\mathring{A}_\lambda,\ \theta_\lambda)$ onto $(\mathring{B}_\lambda,\ \sigma_0)$.

The function $\alpha\circ f_\lambda-g_\lambda\circ\alpha$ is continuous on A_λ, analytic on $\mathring{A}_\lambda$ for the structure θ_λ and vanishes on the boundary, so it vanishes on A_λ.

Q.E.D. Lemma 1.

Proof of Lemma 2. — Let λ_0 be a point of Λ, ζ_0 be an isomorphism of $\Omega_{\lambda_0}=\pi^{-1}(\lambda_0)$ onto D and let A be a closed disc embedded in Ω_{λ_0} containing $\overline{\Omega}'_{\lambda_0}$. According to a Theorem of Grauert, there exists a Stein neighborhood U of A in Ω and a holomorphic function $\zeta:\ U\to\mathbb{C}$ extending ζ_0.

We can find neighborhoods Λ' of λ_0 in Λ and W of A in U, and an open subset $V\subset\mathbb{C}$ such that

$$\varphi:\ x\mapsto(\pi(x),\ \zeta(x))$$

is an isomorphism of W onto $\Lambda'\times V$. Set $\Gamma=\zeta_0(\partial A)$ and $\Gamma_\lambda=\varphi^{-1}(\{\lambda\}\times\Gamma)$.

By restriction W if necessary, we may assume that $\Gamma_\lambda \subset \mathring{Q}_\lambda$ for all $\lambda \in W$. The function h_λ extends to an analytic function on Q_λ if and only if the integrals

$$\int_{\Gamma_\lambda} h_\lambda \zeta_\lambda^n d\zeta_\lambda$$

vanish for all $n \geqq 0$. Since these integrals are analytic functions of λ, the set defined by these conditions is analytic.

Q.E.D.

COROLLARY 1. — *Using the notations of* II,3, P_λ *depends analytically on* λ *for* λ *in* $\mathring{M}_{\mathbf{f}}$.

Proof. — By Proposition 11 (*a*) and Proposition 12, the map $\lambda \mapsto P_\lambda$ from $M_{\mathbf{f}}$ into the affine space E of monic polynomials of degree d with no term of degree $d-1$ is continuous. According to Proposition 6, its graph is the set of $(\lambda, P) \in \Lambda \times E$ for which there is a hybrid equivalence α of f_λ with P_λ such that $i(\lambda)=0$. This is an analytic subspace.

COROLLARY 2. — *Suppose* $d=2$ *and let* c *be a point of the standard Mandelbrot set* M. *The set* $\chi^{-1}(c)$ *is analytic.*

Proof. — Let $\mathbf{p}$ be the constant family $p_\lambda = P_c : z \mapsto z^2 + c$. The set $\mathscr{R}$ for $\mathbf{p}$ is all of Λ. Corollary 2 follows by applying Proposition 13 to $\mathbf{p}$ and $\mathbf{f}$.

7. CONTINUITY OF χ IN DEGREE 2.

LEMMA (valid for any degree). — *Choose* $\lambda_0 \in \Lambda$ *and let* (λ_n) *be a sequence in* Λ *converging to* λ_0. *Then there exists a subsequence* $(\lambda_k^*) = (\lambda_{n_k})$ *such that the* $P_{\lambda_k^*}$ *converge to a polynomial* $\tilde{P}$ *and such that the* $\varphi_{\lambda_k^*}$ *converge uniformly on every compact subset of* A_{λ_0} *to a quasi-conformal equivalence* $\tilde{\varphi}$ *of* f_{λ_0} *with* $\tilde{P}$.

Remarks. — (1) If $\lambda_0 \in \mathscr{R}$, then the Lemma follows from Proposition 12, with $\tilde{P} = P_{\lambda_0}$.

(2) If $\lambda_0 \in \mathscr{F}$, then $\bar{\partial}\tilde{\varphi}$ may fail to vanish on K_{λ_0}, even if $d=2$. If $d \geqq 3$, the polynomial $\tilde{P}$ is not necessarily hybrid equivalent to f_{λ_0}, and may depend on the choice of the subsequence. We will show examples of all these pathologies in chapter III.

Proof of Lemma. — Since all the φ_{λ_n} are quasi-conformal with the same dilatation ratio, they form an equicontinuous family. Moreover, any compact subset of A_{λ_0} is contained in all but finitely many of the A_{λ_n}. The Lemma follows by Ascoli's Theorem.

PROPOSITION 14. — *If* $d=2$, *the map* $\chi : \Lambda \to \mathbb{C}$ *is continuous.*

Proof. — By Proposition 12, χ is continuous on $\mathscr{R}$. It is therefore enough to show that for any sequence (λ_n) in Λ converging to a point $\lambda_0 \in \mathscr{F}$ you can choose a subsequence $\lambda_k^* = \lambda_{n_k}$ such that the $\chi(\lambda_k^*)$ converge to $\chi(\lambda_0)$.

First let us show that $c_0 = \chi(\lambda_0)$ belongs to ∂M. Let (μ_n) be a sequence in $\mathscr{I}$ converging to λ_0. By the preceding Lemma, you can choose a subsequence μ_k^* such that $\chi(\mu_k^*)$ converge to a point $\tilde{c}'$ for which $\tilde{P}' : z \to z^2 + \tilde{c}'$ is quasi-conformally equivalent to $P_0 : z \to z^2 + c_0$.

For any n, the point $c'_n=\chi(\mu_n)$ belongs to ∂M since P'_n: $z\to z^2+c'_n$ has an indifferent periodic point, like f_{μ_n}. Therefore $\tilde{c}'=\lim c'_n$ belongs to ∂M, and $c_0=\tilde{c}'$ by Proposition 7 (I,5).

Now let (λ_n) be an arbitrary sequence of points of Λ converging to λ_0. According to the Lemma, a subsequence (λ_k^*) can be chosen so that $\chi(\lambda_k^*)$ converges to a point $\tilde{c}$ such that $\tilde{P}$: $z\mapsto z^2+\tilde{c}$ is quasi-conformally equivalent to P_0. By Proposition 7, $\tilde{c}=c_0$, so $c_0=\lim\chi(\lambda_k^*)$.

Q.E.D.

CHAPTER III

Negative results

1. NON-ANALYTICITY OF χ. — Let $f=(f_\lambda)_{\lambda\in\Lambda}$ be an analytic family of polynomial-like mappings of degree 2, and $\chi:\Lambda\to\mathbb{C}$ be the map defined in section II,3. The following example shows that χ is not generally analytic on a neighborhood of $\partial M_{\mathbf{f}}$.

Let f: $(f_\lambda)_{\lambda\in\Lambda}$ be an analytic family of polynomial-like mappings of degree 2. Let ω_λ be the critical point of f_λ; we will assume that Λ is a neighborhood of 0 in $\mathbb{C}$, and that there is an analytic map $\lambda\mapsto\alpha_\lambda$ such that α_λ is a repulsive fixed point of f_λ for all $\lambda\in\Lambda$, and that $f_0^k(\omega_0)=\alpha_0$. Set $\rho(\lambda)=f'_\lambda(\alpha_\lambda)$.

Let g_λ be a branch of f_λ^{-1} defined near α_λ, and such that $g_\lambda(\alpha_\lambda)=\alpha_\lambda$. Choose a sequence of analytic functions $\lambda\mapsto\alpha_{\lambda,n}$, restricting Λ if necessary, so that:

(i) $\alpha_{\lambda,0}=\alpha_\lambda$ and $\alpha_{\lambda,1}\neq\alpha_\lambda$.

(ii) $f(\alpha_{\lambda,n+1})=\alpha_{\lambda,n}$.

(iii) $g(\alpha_{\lambda,n})=\alpha_{\lambda,n+1}$ for n sufficiently large.

If $\lambda\mapsto f_\lambda^k(\omega_\lambda)-\alpha_\lambda$ has a simple zero at $\lambda=0$, there exists a sequence $(\lambda_n)_{n\geqq n_0}$ converging to 0, such that $f_{\lambda_n}^k(\omega_{\lambda_n})=\alpha_{\lambda_n,n}$; and the ratios λ_{n+1}/λ_n converge to $1/\rho(0)$.

Using the notations of II,4, set $c_0=\chi(0)$, $a_{c_0}=\varphi_0(\alpha_0)$ and $a_{c_0,n}=\varphi_0(\alpha_{0,n})$. For c near c_0, the polynomial P_c: $z\mapsto z^2+c$ has a repulsive fixed point a_c near a_{c_0}, and we may define analytic functions $a_{c,n}$ of c such that $a_{c,0}=a_c$ and $a_{c,n+1}=P_c^{-1}(a_{c,n})$.

If the zero of $c\mapsto P_c^k(0)-a_c$ at c_0 is simple (which in fact it always is), there exists a sequence $(c_n)_{n\geqq n_0}$ converging to c_0 such that $P_{c_n}^k(0)=a_{c_n,n}$; the ratios

$$\frac{c_{n+1}-c_0}{c_n-c_0},$$

converge to $1/P'_{c_0}(a_{c_0})$.

There exists a neighborhood V of c_0 such that for $n\geqq n_1$, c_n is the only point of V such that $P_{c_n}^k(0)=a_{c_n,n}$, therefore $\chi(\lambda_n)=c_n$ for all n sufficiently large. If $\rho(0)=f'_0(\alpha_0)\neq P'_{c_0}(a_{c_0})$, the ratios λ_{n+1}/λ_n and

$$\frac{c_{n+1}-c_0}{c_n-c_0}$$

will have different limits. Therefore χ is not analytic near 0.

Remark. — The reader who examines Figures 3 and 6 carefully will observe that the "trunk of the elephant" is more open in Figure 6. This can be understood as an example of the above computation. In both pictures, the tip of the trunk corresponds to a mapping for which the critical point lands on the 12'th move on a repulsive cycle of length 2. Moreover, the tree which describes the combinatorics of the mappings [D-H] is the same. However, the eigenvalues of the cycles in the two drawings are different.

2. NON-CONTINUITY OF $\lambda \mapsto \varphi_\lambda$ IN DEGREE 2. — Let $(f_\lambda)_{\lambda\in\Lambda}$ be an analytic family of polynomial-like mappings of degree 2. We will take Λ a neighborhood of 0 in $\mathbb{C}$, U'_λ and U_λ symmetrical with respect to $\mathbb{R}$ and we will assume that $f_\lambda(z)=f_\lambda(z)$ for all $\Lambda\in\mathbb{R}$. Choose for each λ a hybrid equivalence $\varphi_\lambda: V_\lambda \to \mathbb{C}$ of f_λ with a polynomial $P_\lambda: z \to z^2+\chi(\lambda)$, with the φ_λ all quasi-conformal, of dilatation ratio bounded by a constant independant of λ. Suppose that all V_λ contain a fixed neighborhood of K_0, are symmetric with respect to $\mathbb{R}$ and that $\varphi_\lambda(\bar{z})=\overline{\varphi_\lambda(z)}$ for $\lambda\in\mathbb{R}$. (We are not assuming that the φ_λ are obtained from a tubing.)

If (λ_n) is a sequence converging to 0, there is a subsequence (λ_{n_k}) such that the $(\varphi_{\lambda_{n_k}})$ converge to a limit $\tilde{\varphi}$, which is a quasi-conformal equivalence of f_0 with a polynomial $\tilde{P}: z \to z^2+\tilde{c}$..

Suppose that f_0 has a fixed point α_0 with $f'(\alpha_0)=1$, and that for $\lambda>0$ the map f has two conjugate fixed points α_λ and $\bar{\alpha}_\lambda$ which are then necessarily repulsive. Then $\chi(0)=1/4$, and $\chi(\lambda)>1/4$ for $\lambda>0$.

PROPOSITION 15. — *If $\lambda_n>0$ for all n, then $\tilde{\varphi}=\varphi_0$ only if f_0 is holomorphically equivalent to $z\to z^2+1/4$.*

Proof. — If $G \subset U$ is open, G/f_λ will be the quotient of G by the equivalence relation generated by setting x equivalent to $f_\lambda(x)$ whenever both x and $f_\lambda(x)$ are in G. Pick $u>0$ and for $\lambda\geqq 0$ let G_λ (resp. H_λ) denote the disc centered at $\operatorname{Re}(\alpha_\lambda)-u$ [resp. $\operatorname{Re}(\alpha_\lambda)+u$], the edge of which contains α_λ and $\bar{\alpha}_\lambda$. Set $X_\lambda=G_\lambda/f_\lambda$ and $Y_\lambda=H_\lambda/f_\lambda$. If u and λ are sufficiently small, then both X_λ and Y_λ are isomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$. For $\lambda>0$ both X_λ and Y_λ can be identified to $(G_\lambda \cup H_\lambda)/f_\lambda$, and this induces an isomorphism $E_\lambda: X_\lambda \to Y_\lambda$. There is no mapping E_0 but if (λ_n) is a sequence of positive numbers converging to zero, then a subsequence (λ_{n_k}) can be chosen so that the $E_{\lambda_{n_k}}$ converge to an isomorphism $\tilde{E}: X_0 \to Y_0$.

The Beltrami form $\mu_\lambda=\bar{\partial}\varphi_\lambda/\partial\varphi_\lambda$ is f-invariant for all λ, so for all $\lambda>0$ we find Beltrami forms μ_λ^X and μ_λ^Y on X_λ and Y_λ respectively. For $\lambda>0$ we have $\mu_\lambda^X=E_\lambda^*\mu_\lambda^Y$. Therefore if (λ_n) is a sequence for which the (φ_{λ_n}) and the $E_{\lambda_n})$ have limits $\tilde{\varphi}$ and $\tilde{E}$, set $\tilde{\mu}=\bar{\partial}\tilde{\varphi}/\partial\tilde{\varphi}$, and we get $\tilde{\mu}^X=\tilde{E}^*\mu_0^Y$. If $\tilde{\varphi}=\varphi_0$, we get $\mu_0^X=\tilde{E}^*\mu_0^Y$, but $\mu_0^X=0$, at least if $\omega_0<\alpha_0$ and if u is sufficiently small, for in that case $G_0 \subset \mathring{K}_0$. So $\mu_0^Y=0$, and $\mu_0=0$ on H_0.

Since $K_0 \cup H_0$ is a neighborhood of α_0 which is in the Julia set, and since μ_0 is f_0-invariant, we find that μ_0 vanishes on a neighborhood of K_0. This is just another way of saying that φ_0 is a holomorphic equivalence of f_0 and P_0 (*Fig.* 10).

Q.E.D.

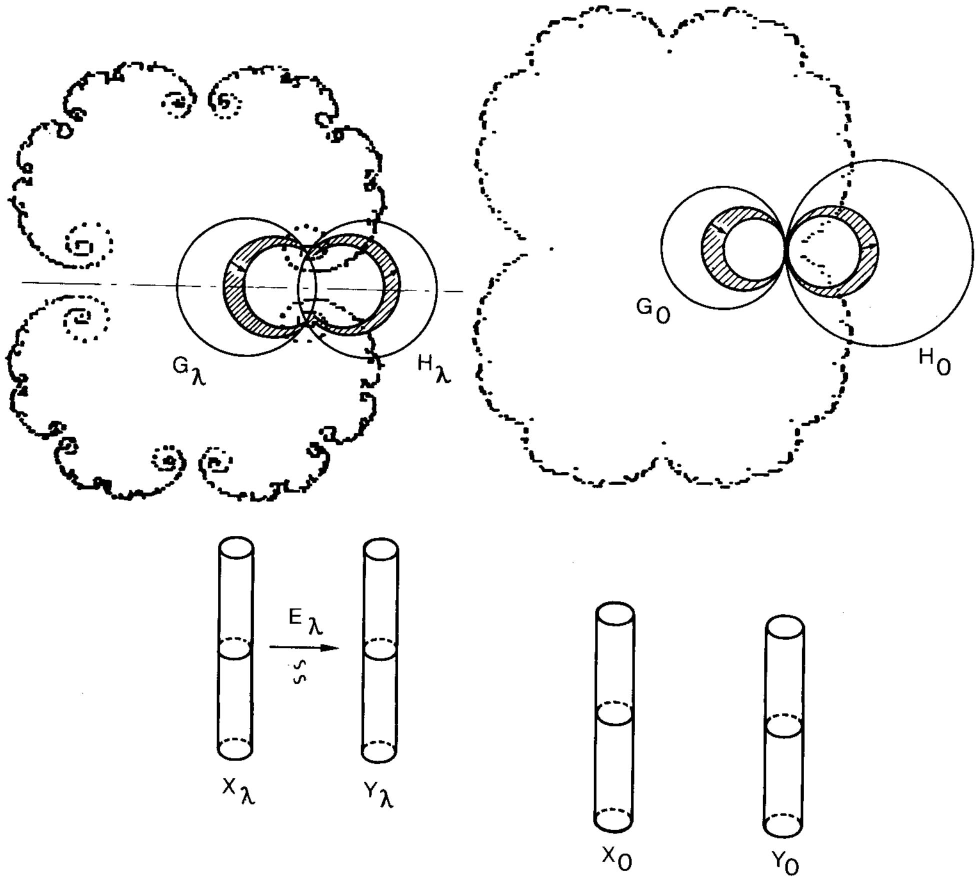

Fig. 10

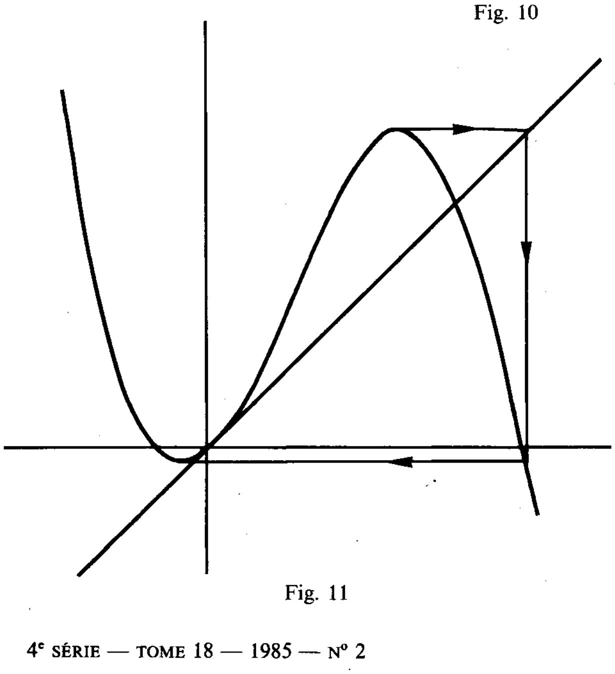

Fig. 11

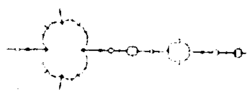

Fig. 12.

3. A FAMILY OF POLYNOMIALS. — In this paragraph we will study a family of polynomials of degree 3, the properties of which will be used in section 4 to construct a third counter-example.

For any $a \in \mathbb{R}_+$, define $Q = Q_a$ by

$$Q_a(z) = z + az^2 - z^3.$$

This polynomial has two real critical points ω and ω' with $\omega < 0 < \omega'$, and the point $\alpha = 0$ is an indifferent fixed point with $Q'(\alpha) = 1$. Set $\omega_i = Q^i(\omega)$ and $\omega'_i = Q^i(\omega')$; the ω_i are for each a an increasing sequence tending to 0, and each ω_i is an increasing function of a.

There is a value a_∞ of a for which $\omega'_2 = 0$; for $a \geqq a_\infty$ the point ω'_2 is a decreasing function of a. Thus there is a decreasing sequence (a_n) tending to a_∞ such that $\omega'_2 = \omega_n$ for $a = a_n$. Set $I = [a_0, a_\infty[$ and $I_n = [a_n, a_{n+1}]$.

For $a \in I$, the graph of $Q|_{\mathbb{R}}$ and the Julia set of Q look as Figures 11-12.

There exists a neighborhood of 0 on which Q is polynomial-like of degree 2, hybrid equivalent to $z \to z^2 + 1/4$. If a and b are both in $I_n =]a_n, a_{n+1}[$, the polynomials Q_a and Q_b are quasi-conformally conjugate; if a and b are simply in I, by [M-S-S] and Proposition 4 there exists a quasi-conformal homeomorphism $\psi_{a,b}: \mathbb{C} \to \mathbb{C}$, holomorphic on $\mathbb{C} - K_a$, conjugating Q_a to Q_b on a neighborhood of J_a and on $\mathbb{C} - K_a$. *The* $\psi_{a,b}$ can be chosen to depend continuously on a, b and with dilatation ratio bounded on compact subsets of $I \times I$; let κ_n be a bound for $(a, b) \in I_n \times I_n$.

We shall now describe an invariant of the Q_a as a varies in I_n.

A polynomial-like mapping $f: U' \to U$ will be *of real type* if U and U′ are symmetrical with respect to the real axis, and $f(\bar{z}) = \overline{f(z)}$. Let $f: U' \to U$ be a polynomial-like mapping of real type, quasi-conformally equivalent to some Q_a with $a \in I$. We will define an invariant $\theta(f) \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$.

The map f has an indifferent fixed point α with $f'(\alpha) = 1$, and two critical points $\omega = \omega_f$ and $\omega' = \omega'_f$. By a change of variables we may assume $\alpha = 0$, $f''(\alpha) > 0$ and $\omega < 0 < \omega'$.

Let G and H be open discs of radius a small number u, centered at $-u$ and u respectively. Set $X = G/f$ and $Y = H/f$, and let ω^X and ω'^X be the points of X given by the orbits of ω_f and ω'_f respectively. More generally, if F is an f-invariant subset of U we will call F^X (resp. F^Y) the image of $F \cap G$ in X (resp. the image of $F \cap H$ in Y).

The manifolds X and Y are isomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$. If x and y are two points of X, let $x - y$ be the point $h(x) - h(y)$ of $\mathbb{C}/\mathbb{Z}$, where h is an isomorphism of X onto $\mathbb{C}/\mathbb{Z}$ respecting the orientation of the equator. Clearly $x - y$ does not depend on the choice of h. If x and y are in Y, we may similarly define $x - y$. More generally, if μ is an f-invariant Beltrami form on U and $x, y \in X$, we set

$$[x - y]_\mu = h_\mu(x) - h_\mu(y),$$

where $h: X \to \mathbb{C}/\mathbb{Z}$ is a quasi-conformal homeomorphism such that $\bar{\partial} h / \partial h = \mu^X$.

Set $\theta(f)=\omega'^{X}-\omega^{X}$. If $f\sim_{hb} g$, then $\theta(f)=\theta(g)$. Indeed, $G\subset\mathring{K}_f$, and therefore a hybrid equivalence of f with g gives an analytic isomorphism of X_f onto X_g.

If μ is an f-invariant Beltrami form on U (or even on K_f), set $\theta_\mu(f)=[\omega'^{X}-\omega^{X}]_\mu$.

4. NON-CONTINUITY OF $\lambda\mapsto P_\lambda$ IN DEGREE 3.

PROPOSITION 16. — *There exists an analytic family $(f_\lambda)_{\lambda\in\Lambda}$ of polynomial-like mappings of degree* 3, *and a sequence (λ_n) of points of $M_{\mathbf{f}}\subset\Lambda$, converging to a point λ_0 such that the polynomials P_{λ_n} constructed in* II,3 *have a limit $\tilde{P}$ which is not affine conjugate to P_{λ_0}.*

Our construction will give polynomials $\tilde{P}$ and P_{λ_0} which will be affine conjugate to elements of the family $(Q_a)_{a\in I}$ studied in paragraph 2. (In order to have actual elements of that family, we would have to change the convention that they be monic.) We will see that P_{λ_0} and $\tilde{P}$ are not affine conjugate by showing that $\theta(P_{\lambda_0})\neq\theta(\tilde{P})$.

We will use the counter-example of section 2. Our Λ will be a neighborhood of 0 in $\mathbb{C}$, $\lambda_0=0$, and all constructions will be equivariant under $z\mapsto\bar{z}$.

We will start from a polynomial $Q=Q_a$ with $a\in I$, an isomorphism $E: X_Q\to T_Q$ and a Beltrami form ν defined on a neighborhood U_Q of K_Q. We will assume that U_Q is homeomorphic to D, that $U'_Q=Q^{-1}(U_Q)$ (necessarily homeomorphic to D) is relatively compact in U_Q, and that $Q^*\nu=\nu|_{U'}$ and $E^*\nu^Y=\nu^X$ (we will say that ν is invariant by Q and E).

We will require more of Q and E. Set $J_Q=\partial K_Q$; in this case $J_Q^Y\cap\mathbb{R}$ is infinite, in fact a Cantor set.

LEMMA 1. — Q *and* E *can be chosen so that ω^X and ω'^X are different points of* $E^{-1}(J_Q^Y)$.

Proof of Lemma. — When a varies in $\mathring{I}$, the set J_Q remains homeomorphic to itself, as do J_Q^Y and $J_Q^Y\cap\mathbb{R}^Y$; the J_{Q_a} (and the $J_{Q_a}^Y$ and the $J_{Q_a}^Y\cap\mathbb{R}^Y$) are the fibers of a trivial fibration over I.

Thus there exist two continuous maps $a\mapsto x(a)$ and $a\mapsto y(a)$ such that $x(a)$ and $y(a)$ are distinct points of $J_{Q_a}^Y\cap\mathbb{R}^Y$. For any $a\in I$, choose E_a so that $E_a(\omega_a^X)=x(a)$. Then for any $n\in\mathbb{N}$, there exists $a_n\in I$ such that $Q^2(\omega')=Q^n(\omega)$ when $a=a_n$. As a varies from a_n to a_{n+1}, the angle $\omega_a'^X-\omega_a^X$ makes one complete revolution. Therefore there exists $a\in\,]a_n, a_{n+1}[$ for which $E_a(\omega_a'^X)=y(a)$.

Q.E.D.

Proof of the Proposition. — We will assume that Q and E have been chosen so as to satisfy Lemma 1. Choose a neighborhood U_Q of K_Q as above, and a Beltrami form ν on U_Q, invariant by Q and E.

We shall take f_0 to be the mapping $Q: U'_Q\to U_Q$, with U_Q and U'_Q carrying the complex structure defined by ν. More precisely, let ψ be a quasi-conformal diffeomorphism of U_Q onto an open subset $U_0\subset\mathbb{C}$, such that $\bar{\partial}\psi/\partial\psi=\nu$ and $\psi(0)=0$. Set $f_0=\psi\circ Q\circ\psi^{-1}: U'_0\to U_0$, where $U'_0=\psi(U'_Q)$. We may normalize ψ so that $f_0(z)=z+z^2+O(|z|^3)$ near 0.

For any $\lambda\in\mathbb{C}$ set $f_\lambda(z)=f_0(z)+\lambda$. In a neighborhood Λ of 0 the mapping f_λ is polynomial-like of degree 3 from U'_0 onto $U=U_0+\lambda$. For $\lambda\in\mathbb{R}_+\cap\Lambda$, the mapping f_λ

has two conjugate repulsive fixed points α and $\bar{\alpha}$ close to 0. The interval $[\omega_\lambda, f_\lambda(\omega')]$ is sent into itself, so ω_λ and ω'_λ both belong to K_λ, hence K_λ is connected and the polynomial P_λ obtained by straightening f_λ does not depend on the choice of a tubing.

Clearly $\theta(P_0)=\theta(f_0)=\theta_\nu(Q)$. We will show that for appropriate choices of ν and of the sequence (λ_n) in $\Lambda \cap \mathbb{R}_+^*$, the P_{λ_n} have a limit $\tilde{P}$ such that $\theta(\tilde{P}) \neq \theta_\nu(Q)$.

As in section III,3, define for each $\lambda \in \Lambda \cap \mathbb{R}_+^*$ cylinders X_λ and Y_λ, and an isomorphism $E_\lambda : X_\lambda \to Y_\lambda$. As before there is no E_0, but $E^\psi = \psi^Y \circ E \circ (\psi^X)^{-1}$ is an isomorphism of X_0 onto Y_0.

Lemma 2. — *There exists a sequence (λ_n) in $\Lambda \cap \mathbb{R}_+^*$ converging to* 0, *such that the* E_{λ_n} *converge to* E^ψ.

Proof. — Let u and v be two points of $U'_0 \cap \mathbb{R}$ close to 0, such that $u<0<v$ and $E^\psi(u^X)=v^Y$. For any integer $n \geqq 1$, the mapping $g_n : \lambda \mapsto f_\lambda^n(u)$ is increasing for λ in some interval $[0, \lambda_n^+]$, with $g_n(0)<0$ and $g_n(\lambda_n^+)>v$, therefore there exists $\lambda_n \in]0, \lambda_n^+[$ such that $g_n(\lambda_n)=v$ (*Fig.* 13).

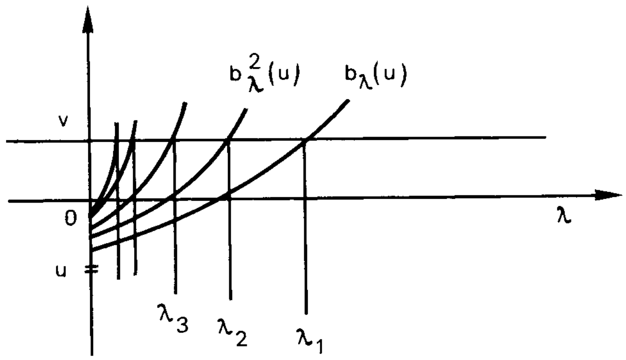

Fig. 13

Then $E_{\lambda_n}(u^X)=v^Y$, and so the E_{λ_n} have the limit E^ψ. For all $\lambda>0$, there exists an n for which $g_n(\lambda)>v$, so the sequence (λ_n) converges to 0.

Q.E.D.

Proposition 16 now follows from Lemmas 3 and 4.

Lemma 3. — *If* (λ_n) *satisfies the conditions of Lemma* 2 *and if a subsequence* (λ_k^*) *is chosen so that the* $P_{\lambda_k^*}$ *have a limit* $\tilde{P}$, *there exists a Beltrami form* δ *on* $\mathbb{C}$, *invariant under* Q *and* E, *vanishing on* $\mathbb{C}-K_Q$, *and such that* $\theta(\tilde{P})=\theta_\delta(Q)$.

Proof. — The mapping ψ is a quasi-conformal equivalence of Q with f_0 and $\nu=\bar{\partial}\psi/\partial\psi$. Let

$$T_Q : \bar{D}_R - D_{R'} \to A_Q - \mathring{A}'_Q,$$

be an analytic tubing of Q (since K_Q is connected, this is possible by Proposition 4), and choose a tubing $T=(T_\lambda)$ of f such that $T_0=\psi \circ T_Q$. Construct (φ_λ) and (P_λ) from T as in II,3. By the Lemma of II,7 there is a subsequence (λ_k^*) such that the $\varphi_{\lambda_k^*}$ converge to a quasi-conformal equivalence $\tilde{\varphi}$ of f_0 with $\tilde{P}_0$.

Then $\tilde{\mu}=\bar{\partial}\tilde{\varphi}/\partial\tilde{\varphi}$ is a Beltrami form on U_0, invariant under f_0 and E, which agrees on $A-\mathring{A}'$ with $\bar{\partial}\psi^{-1}/\partial\psi^{-1}$. The mapping $\Phi=\tilde{\varphi}\circ\psi$ is a quasi-conformal equivalence of Q with $\tilde{P}$, and $\delta=\bar{\partial}\Phi/\partial\Phi$ is a Beltrami form on a neighborhood of K_Q, invariant under Q and E and vanishing on A_Q-K_Q. Set $\mu=0$ on $\mathbb{C}-K_Q$ to get a Beltrami form on $\mathbb{C}$. Then $\theta(\tilde{P})=\theta_\delta(Q)$.

Q.E.D.

LEMMA 4. — *Let* Q *and* E *satisfy the conditions of Lemma* 1.

(*a*) *For any* $\varepsilon>0$, *there exists a Beltrami form* ν *on a neigborhood of* K_Q, *invariant under* Q *and* E, *such that* $0<\theta_\nu(Q)<\varepsilon$.

(*b*) *There exists* $m>0$ *such that any Beltrami form* δ *on* $\mathbb{C}$, *invariant under* Q *and* E *and vanishing on* $\mathbb{C}-K_Q$, *satisfies* $\theta_\nu(Q)\in[m, 1-m]$.

Proof of (*a*). — The polynomial Q is Q_a for a in some I_n. Choose $\varepsilon>0$, and let ε' be such that for any quasi-conformal homeomorphism $h: \mathbb{C}/\mathbb{Z}\to\mathbb{C}/\mathbb{Z}$ of dilatation ratio $\leq\kappa_n$ (κ_n is defined in II.2), $d(h(u), h(v))<\varepsilon$ if $d(u, v)<\varepsilon'$.

Choose a quasi-conformal homeomorphism Ψ of Y_Q onto itself, holomorphic on a neighborhood of K_Q^Y, and such that

$$0<\Psi(E(\omega'))-\Psi(E(\omega))<\varepsilon'.$$

Set $\mu_0=\bar{\partial}\Psi/\partial\Psi$, and let μ be the corresponding Beltrami form on $K_Q\cup H$, vanishing on K_Q and invariant under $Q^{-1}|_H$. Let V_0 be a neighborhood of K_Q with $V_1=Q^{-1}(V_0)$ relatively compact in V_0.

There then exists a unique Beltrami form ν_0, invariant under Q, agreeing with μ on $(V_0-V_1)\cap H$, and vanishing on $(V_0-V_1)-H$ and on K_Q. In general, $\nu_0^Y\neq\mu_0$, but if ν_n is the Beltrami form on $V_n=Q^{-n}(V_0)$ defined in the same way, then $\|\nu_n^Y\|_\infty<\|\mu_0\|_\infty$ for all n, and $\nu_n^Y\to 0$ *in* L^1. So $0<[E(\omega')-E(\omega)]_{\nu_n}<\varepsilon'$ for sufficiently large n.

Let ν be the Beltrami form on V_n, invariant under Q and E and agreeing with ν_n on V_n-K_Q. There is a quasi-conformal equivalence Φ of $Q=Q_a$ with a polynomial Q_b, $b\in I_n$, such that

$$\bar{\partial}\Phi/\partial\Phi=\nu \quad \text{on} \quad K_Q$$

and

$$=0 \quad \text{on} \quad \mathbb{C}-K_Q$$

and Φ agrees with $\psi_{a,b}$ on J_Q.

Let ν^* be the form which agrees with ν (and hence with ν_n) on V_n-K_Q and with $\bar{\partial}\psi_{a,b}/\partial\psi_{a,b}$ on K_Q. Then

$$\theta_\nu(Q)=[E(\omega')-E(\omega)]_\nu=[E(\omega')-E(\omega)]_\nu^*.$$

However, the identity of Y, taken with the structure defined by ν_n in the domain and that defined by ν^* in the range, is quasi-conformal with dilatation ratio bounded by κ_n. Considering how ε' was chosen, we see that $\theta_\nu(Q)<\varepsilon$.

Proof of part (*b*). — The polynomial Q is Q_a for some $a \in I_n$, and some $n \in \mathbb{N}$. Let Φ be a quasi-conformal homeomorphism of $\mathbb{C}$ such that $\bar{\partial}\Phi/\partial\Phi = \delta$, $\Phi(z)/z \to 1$ as $z \to \infty$ and $\Phi(0) = 0$.

Then $\Phi \circ Q \circ \Phi^{-1}$ is Q_b for some $b \in I_n$, and there exists $\Psi_{a,b}$, quasi-conformal of dilatation ratio $\leqq \kappa_n$, holomorphic on $\mathbb{C} - K_a$ and conjugating Q_a to Q_b there. On $\mathbb{C} - K_a$, Φ and $\psi_{a,b}$ agree, since they are both holomorphic and conjugate Q_a to Q_b. By continuity, they also agree on J_a.

Using Φ we define $\Phi^X : X_a \to X_b$ and $\Phi^Y : Y_a \to Y_b$; then $E_b = \Phi^Y \circ E \circ (\Phi^X)^{-1}$ is an isomorphism of X_b onto Y_b since δ is E-invariant.

Let ω and ω' be the critical points of $Q = Q_a$; then

$$\theta_\delta(Q) = \Phi^X(\omega'^X) - \Phi^X(\omega^X) = E_b(\Phi^X(\omega'^X)) - E_b(\Phi^X(\omega^X)) = \Phi^Y(E(\omega'^X)) - \Phi^Y(E(\omega^X)).$$

The mapping $\Psi_{a,b}$ conjugates Q_a to Q_b on J_a, so we can consider

$$\Psi_b^Y = \psi_{a,b}|_J^Y : J_a^Y \to J_b^Y.$$

Similarly we can define

$$\psi_{b'}^Y = \psi_{a,b'}|_J^Y$$

for any $b' \in I_n$, and these mappings form an equicontinuous family. Therefore if u and v are two distinct points of J_a^Y, we have

$$m_{u,v} = \inf_{b' \in I_n} |\psi_{b'}^Y(u) - \psi_{b'}^Y(v)| > 0.$$

But we had assumed that ω and ω' satisfied the conditions of Lemma 1, so $E(\omega^X)$ and $E(\omega'^X)$ are in J_a^Y, and so Φ^Y and Ψ_b^Y agree on these points. Therefore

$$\theta_\delta(Q) = |\Phi^Y(E(\omega'^X)) - \Phi^Y(E(\omega^X))| > m,$$

where $m = m_{E(\omega^X), E(\omega'^X)}$ depends only on $Q = Q_a$ and E.

Q.E.D.

This ends the proof of Lemma 4 and of Proposition 16.

CHAPTER IV

One parameter families of maps of degree 2

1. TOPOLOGICAL HOLOMORPHY. — In this chapter, we will consider an analytic family $f = (f_\lambda)_{\lambda \in \Lambda}$ of polynomial-like mappings of degree 2, where $\dim(\Lambda) = 1$, i. e. Λ is a Riemann surface. We will particularly study the straightening map $\chi : \Lambda \to \mathbb{C}$ (defined using some tubing) and the set $M_{\mathbf{f}} = \chi^{-1}(M)$.

Recall that $M_{\mathbf{f}}$ and the restriction of χ to $M_{\mathbf{f}}$ do not depend on the tubing; we have also shown that χ is not in general analytic.

We will show in this chapter that χ has the same topological properties it would have if it were analytic, and we will give conditions for χ to induce a homeomorphism, sometimes even quasi-conformal, of M_f onto M.

The point of these conditions is that they are preserved by small perturbations.

The first two sections are devoted to generalities about "topologically holomorphic" mappings.

Reminder. — Let X and Y be oriented topological surfaces and $\varphi: Y \to X$ a continuous map. If $y \in Y$ is isolated in its fiber, the local degree $i_y(\varphi)$ of φ at y is defined as follows: set $x = \varphi(y)$ and choose neighborhoods U and V of x and y respectively, homeomorphic to D and such that $\varphi(V) \subset U$ and $\{y\} = V \cap \varphi^{-1}(x)$. If γ is a loop in $V - \{y\}$ with winding number 1 around y then $i_y(\varphi)$ is the winding number of $\varphi \circ \gamma$ around x.

If φ is proper and X and Y are connected, then φ has a degree. Indeed, the cohomology groups with compact support $H_c^2(X)$ and $H_c^2(Y)$ are canonically isomorphic to $\mathbb{Z}$ and $\varphi^*: H_c^2(Y) \to H_c^2(X)$ is multiplication by the integer deg φ.

If φ is proper, X and Y are connected and $y \in Y$ has $\varphi^{-1}(y)$ discrete hence finite, then

$$\deg \varphi = \sum_{x \in \varphi^{-1}(y)} i_x(\varphi).$$

DEFINITION. — *Let* X *and* Y *be oriented surfaces as above, and* $\varphi: Y \to X$ *be a continuous map. Let* M *be closed in* X *and* $P = \varphi^{-1}(M)$. *We will say that* φ *is* topologically holomorphic *over* M *if for all* $y \in P$, y *is isolated in its fiber and* $i_y(\varphi) > 0$.

PROPOSITION 17. — *Suppose that* $\varphi: Y \to X$ *is topologically holomorphic over* M *and let* $P = \varphi^{-1}(M)$.

(*a*) *for all* $p \in P$ *there exist open connected neigborhoods* U *and* V *of* $m = \varphi(p)$ *and* p *respectively, with compact closure, such that* φ *induces a proper map* $V \to U$ *of degree* $d = i_p \varphi$;

(*b*) *if* $d = 1$ *then* φ *induces a homeomorphism of* $P \cap V$ *onto* $M \cap U$. *More generally,* $\varphi|_V$ *can be written* $\pi \circ \tilde{f}$, *here* $\pi: \tilde{U} \to U$ *is the projection of the d-fold cover of* U *ramified m, and* $\tilde{f}: V \to \tilde{U}$ *is a proper mapping of degree* 1. *The mapping* $\tilde{f}$ *induces a homeomorphism of* $P \cap V$ *onto* $\pi^{-1}(M \cap U)$.

Proof. — (*a*) Let B be a compact neighborhood of p, containing no other point of $\varphi^{-1}(m)$. Since $m \notin \varphi(\partial B)$, there is an open neighborhood U of m homeomorphic to D and such that $U \cap \varphi(\partial B) = \varnothing$. Let V be the connected component of $\varphi^{-1}(U)$ containing p; since $V \subset \mathring{B}$, clearly $\varphi: V \to U$ is proper. By the reminder above, $\deg(\varphi|_V) = i_p \varphi$.

(*b*) Just apply the lifting criterion of covering space theory. More specifically, we need to know that $\varphi_*(\pi_1(V - p))$ is contained in (in fact equal to) the subgroup of $\pi_1(U - m)$ generated by d, which is the very definition of local degree.

Q.E.D.

COROLLARY. — *With the hypotheses of the above Proposition, the points of* P *where* $i_p(\varphi) > 1$ *form a closed discrete subset of* P.

Indeed, if $q \in V \cap P$ and $q \neq p$ then $i_q(\varphi) = i_q(\tilde{\varphi})$, and since $\deg(\tilde{\varphi}) = 1$ the Corollary follows.

The next statement can be viewed as a maximum principle for topologically holomorphic mappings.

PROPOSITION 18. — *Let X and Y be oriented surfaces, with Y connected and non-compact, $\varphi: Y \to X$ a continuous mapping and $M \subset X$ a closed subset, such that φ is topologically holomorphic over M. Set $P = \varphi^{-1}(M)$. If W is a relatively compact component of $Y - P$ then $\varphi(W)$ is a relatively compact component of $X - M$ and φ induces a proper mapping $W \to \varphi(W)$ of degree > 0.*

LEMMA. — *Let $p \in Y$ satisfy $i_p(\varphi) = 1$; let $m = \varphi(p)$ and U, V be as in Proposition 17. Then φ induces a bijection of the set $\pi_0(V - P)$ of connected components of $V - P$ onto $\pi_0(U = M)$, and for each component V′ of $V - P$, φ induces a proper mapping $V' \to \varphi(V')$ of degree 1.*

Proof. — Consider the diagram

$$\begin{array}{ccccccc} & & H_c^1(P \cap V) & \to & H_c^2(V - P) & \to & H_c^2(V) \\ & & \uparrow \varphi_M^* & & \uparrow \varphi_{U-M}^* & & \uparrow \varphi_U^* \\ H_c^1(U) & \to & H_c^1(M \cap U) & \to & H_c^2(U - M) & \to & H_c^1(U) \end{array}.$$

Since φ induces a homeomorphism of $P \cap V$ onto $M \cap U$ (Prop. 17, *b*), φ_M^* is an isomorphism. Since U and V are connected oriented surfaces

$$H_c^2(U) = H_c^2(V) = \mathbb{Z}$$

and φ_U^* is multiplication by $i_p(\varphi) = 1$, so it is also an isomorphism. Finally, $H_c^1(U) = 0$ since U is homeomorphic to D.

Therefore φ_{U-M}^* is surjective. But

$$H_c^2(U - M) = \mathbb{Z}^{\pi_0(U-M)} \quad \text{and} \quad H_c^2(V - P) = \mathbb{Z}^{\pi_0(V-P)} = \bigoplus_{i \in \pi_0(U-M)} Z_i,$$

where $Z_i = \mathbb{Z}^{\pi_0(V \cap \varphi^{-1}(U_i))}$ and U_i ranges over the connected components of $U - M$.

For all $i \in \pi_0(U - M)$ the mapping $\varphi_i^*: \mathbb{Z} = H_c^2(U_i) \to Z_i$ is therefore surjective. So $\pi^{-1}(U_i)$ has at most one component, and the commutativity of the righthand square of the diagram shows that there is precisely one, and that the degree is 1.

Q.E.D.

Proof of Proposition 18. — Let W_1 be the component of $X - M$ containing $\varphi(W)$. Since W is relatively compact in Y, φ induces a proper mapping $W \to W_1$, which we must show to have degree > 0.

Clearly ∂W is compact and infinite; since φ has only finitely many ramification points in ∂W, there exists $m \in \varphi(\partial W)$ with $i_p \varphi = 1$ for all $p \in \varphi^{-1}(m) \cap \partial W$. Let $p_1, \ldots, p_k$ be these points, and let U, $V_1, \ldots, V_k$ be as in Proposition 17.

Since $\bar{W}$ is compact, there exists a neighborhood U^* of m in U such that

$$\varphi^{-1}(U^*) \cap \bar{W} \subset V_1 \cup \ldots \cup V_k.$$

Let V'_1 be a connected component of $W \cap V_1$ which intersects $\varphi^{-1}(U^*) \cap V_1$ (there is one since $p_1 \in \partial W$).

The open set V'_1 is a component of $V_1 - P$, and $U' = \varphi(V'_1)$ is a connected component of $U - M$.

By the Lemma there is a unique component V'_i of $V_i - P$ such that $\varphi(V'_i) = U'$ for each $i = 1, \ldots, k$. Let $i_1 = 1, \ldots, i_I$ be the values of i for which $V'_i \subset W$. There may be other components of $\varphi^{-1}(U') \cap W$, but if V'' is such a component, then $\varphi(V'') \neq U'$ since $\varphi(V'') \cap U^* = \varnothing$, so the degree of $\varphi: V'' \to U'$ is zero. The degree of $\varphi: W \to W_1$ is equal to the degree of $\varphi: \varphi^{-1}(U') \to U'$, which is $I \geqq 1$.

Q.E.D.

2. The Riemann-Hurwitz formula. — The results of this paragraph will not be used in an essential way in the remainder of the paper. We have included them to illustrate the strength of topological holomorphy.

For any topological space T we will note $h^i(T)$ the $\mathbb{Z}$-rank of the Čech cohomology group $H^i(T; \mathbb{Z})$, which will be an element of $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

For the spaces which we will consider, all subsets of orientable surfaces, $h^i(T)$ is also the dimension of the vector space $H^i(T; k)$ for any field k.

Proposition 19. — *Let* X *and* Y *be oriented connected non-compact surfaces,* M *a closed connected subset of* X, $\varphi: Y \to X$ *a continuous mapping, topologically holomorphic over* M *and* $P = \varphi^{-1}(M)$. *Suppose that* φ *induces a proper map* $P \to M$. *Then:*

(*a*) *For any* $m \in M$ *set* $d_m = \sum_{p \in \varphi^{-1}(m)} i_p(\varphi)$. *The number* $d = d_m$ *is independant of the choice of* m *in* M.

(*b*) *There exist neighborhoods* U *of* M *in* X *and* V *of* P *in* Y *such that* φ *restricts to a proper mapping* $V \to U$ *of degree* d.

(*c*) *The inequalities* $h^0(P) \leqq d$ *and* $h^1(P) < dh^1(M) + h^1(Y)$ *are satisfied.*

(*d*) *If* $h^1(M)$ *and* $h^1(Y)$ *are finite, then the set* C *of ramification points of* φ *in* P *is finite, and*

$$h^1(P) - h^0(P) = d(h^1(M) - 1) + \sum_{p \in C} (i_p(\varphi) - 1).$$

Under the hypotheses of this Proposition, the mapping $\varphi: P \to M$ deserves to be called a ramified covering space; part (*d*) is the Riemann-Hurwitz formula in this setting.

Proof. — (*b*) There exists a closed neighborhood N of P in Y such that φ induces a proper mapping of N into X. Indeed, if $h: Y \to \mathbb{R}_+$ is a proper continuous function, then the function $h_1: x \mapsto \sup_{y \in \varphi^{-1}(x)} h(y)$ is locally bounded on M, and so there is a continuous function $g: X \to \mathbb{R}_+$ such that $g(x) > h_1(x)$ for all $x \in M$. Then $N = \{ y \in Y \mid h(y) \leqq g(\varphi(y)) \}$ satisfies the requirement.

The set $\varphi(\partial N)$ is closed in X, and does not intersect M. Let U be the connected component of $X-\varphi(\partial N)$ containing M and set $V=\varphi^{-1}(U)\cap N=\varphi^{-1}(U)\cap \mathring{N}$. The sets U and V are open and φ restricts to a proper mapping $V\to U$.

(*a*) Let d be the degree of $\varphi: V\to U$. For any $m\in M$, the sum $\sum_{p\in\varphi^{-1}(m)} i_p\varphi$ is equal to d since $\varphi^{-1}(m)\subset P\subset V$.

(*c*) Since $\varphi: P\to M$ is open and closed, we have that $\varphi(W)=M$ for each connected component W of P. Since a fiber of φ has at most d points, there are at most d components, i. e. $h^0(P)\leqq d$.

Consider the diagram

$$\begin{array}{ccccccc} H^1(Y) & \to & H^1(P) & \to & H^2(Y, P) & & \\ \uparrow & & \uparrow & & \uparrow & & \\ H^1(X) & \to & H^1(M) & \to & H^2(X, M) & \to & 0. \end{array}$$

Clearly $h^1(P)\leqq h^1(Y)+h^2(Y, P)$. But $h^2(Y, P)$ is equal to the number of connected relatively compact components of $Y-p$. By Proposition 18, $h^2(Y, P)\leqq dh^2(X, M)$. Since $h^2(X, M)\leqq h^1(M)$, we have that $h^1(P)\leqq h^1(Y)+dh^1(M)$.

(*d*) We do not known that φ is topologically holomorphic over a neighborhood of M. The following Lemma deals with this difficulty.

LEMMA. — *There exists a neighborhood* U *of* M *and a closed surface* $Z\subset Y\times U$ *such that* $Z\cap(Y\times M)$ *is the graph of* $\varphi|_P$, *and that the projection* $\mathrm{pr}_2: Y\times U\to U$ *restricts to* $q: Z\to U$ *which makes* Z *a ramified covering space of* U *of degree* d. *Moreover* $i_{(p, \varphi(p))}q=i_p\varphi$ *for all* $p\in P$.

Proof of Lemma. — For any open subset U′ of X, let Z(U′) be the set of closed surfaces $Z'\subset Y\times U'$ such that $Z'\cap Y\times M$ is the graph of $\varphi|_{P\cap\varphi^{-1}(U')}$ and that $q_{Z'}: Z'\to U'$ (induced by pr_2) is a ramified covering space, and moreover that $i_{(p, \varphi(p))}q_{Z'}=i_p(\varphi)$ for $p\in P\cap\varphi^{-1}(U')$.

This defines a sheaf of sets Z on X which induces a sheaf Z_M on M. The stalk of Z (or of Z_M) at $m\in M$ is $Z_m=\varinjlim_{U'\ni m} Z(U')$ and for any open set $A\subset M$ we have $Z_M(A)=\varinjlim Z(U')$, with the direct limit taken over open subsets U′ of X with $U'\cap M=A$ [G]. The problem is to construct an element of $Z_M(M)$.

First we will show how to construct an element of Z_m for $m\in M$. Let U_m be a neighborhood of m homeomorphic to D and for each $p\in\varphi^{-1}(m)$ let V_p be a neighborhood of p satisfying the conditions of Proposition 17. For each p choose a continuous mapping $\psi_p: \tilde{U}_p\to Y$, where $\tilde{U}_p$ is the covering of U_m of degree $d_p=i_p\varphi$, and where ψ_p is chosen to extend the inverse of the homeomorphism $P\cap V\to\pi_p^{-1}(M\cap U_m)$ induced by φ. This is possible by the Tietze extension Theorem, if V_p was chosen contained in an open subset homeomorphic to a disc.

Let $\tilde{Z}_p$ be the graph of ψ_p and Z_p its image in $Y\times U_m$ by $\sigma\circ(\pi_p\times 1_Y)$, where $\sigma(x, y)=(y, x)$.

The projection $\mathrm{pr}_1 : \tilde{\mathrm{U}}_p \times \mathrm{Y} \to \tilde{\mathrm{U}}_p$ induces a homeomorphism of $\tilde{\mathrm{Z}}_p$ onto $\tilde{\mathrm{U}}_p$, therefore pr_2 makes Z_p into a ramified covering space of U_m of degree d_p, and $\mathrm{Z}_m = \bigcup_{p \in \varphi^{-1}(m)} \mathrm{Z}_p$ defines an element of Z_m.

The key point is that the sheaf Z_M is soft on $\mathrm{M}^* = \mathrm{M} - \varphi(\mathrm{C})$. This means that if $\mathrm{F} \subset \mathrm{A} \subset \mathrm{B} \subset \mathrm{M}^*$ with A and B open in M^* and F closed, for any $\mathrm{Z}' \in \mathrm{Z}_\mathrm{M}(\mathrm{A})$ there exists $\mathrm{Z}'' \in \mathrm{Z}_\mathrm{M}(\mathrm{B})$ and a neighborhood A' of F in A such that $\mathrm{Z}''|_{\mathrm{A}'} = \mathrm{Z}'|_{\mathrm{A}'}$. Indeed, this is a local property, and Z_M is locally on M^* isomorphic to $\mathscr{I}^d_\mathrm{M}$, where $\mathscr{I}$ is the sheaf on X defined by

$$\mathscr{I}(\mathrm{U}') = \{ f \in \mathrm{C}(\mathrm{U}'; \mathbb{R}) \mid f|_{\mathrm{U}' \cap \mathrm{M}} = 0 \}$$

and $\mathscr{I}_\mathrm{M}$ is the sheaf induced by $\mathscr{I}$ on M. It is clear, and classical, that sections of $\mathscr{I}$ and $\mathscr{I}_\mathrm{M}$ can be spliced using partitions of unity, and so those sheaves are soft.

Since $\varphi(\mathrm{C})$ is closed and discreet, there exists an open neighborhood A of $\varphi(\mathrm{C})$ in M and a section Z_0 of Z_M over A. Let F be a closed neighborhood of $\varphi(\mathrm{C})$ in M contained in A and Z_1 be an element of $\mathrm{Z}_\mathrm{M}(\mathrm{M}^*)$ agreeing with Z_0 on $\mathrm{A}' - \varphi(\mathrm{C})$, where $\mathrm{F} \subset \mathrm{A}' \subset \mathrm{A}$. We get a section Z of $\mathrm{Z}_\mathrm{M}(\mathrm{M})$ by splicing $\mathrm{Z}_0|_{\mathrm{A}'}$ and Z_1.

Q.E.D.

Proof of Proposition 19 (*d*). — If $h^1(\mathrm{M})$ is finite, there is a basis (U_α) of open neighborhoods of M in X such that the maps $\mathrm{H}^1(\mathrm{U}_\alpha) \to \mathrm{H}^1(\mathrm{M})$ are isomorphisms.

Let Z satisfy the conditions of the Lemma, and set $\mathrm{V}_\alpha = q^{-1}(\mathrm{U}_\alpha)$. The V_α form a basis of neighborhoods of P in Z, so for some α_0, V_{α_0} contains no ramification points of q except those in P; we will only consider $\alpha > \alpha_0$.

Then, according to the Riemann-Hurwitz formula which is classical for ramified covering maps of surfaces,

$$h^1(\mathrm{V}_\alpha) - h^0(\mathrm{V}_\alpha) = d\,(h^1(\mathrm{U}_\alpha) - 1) + \sum_{p \in \mathrm{P}} (i_p \varphi - 1).$$

It is easy to show that $\mathrm{H}^1(\mathrm{V}_\alpha) \to \mathrm{H}^1(\mathrm{V}_\beta)$ is an isomorphism for $\beta \geqq \alpha$. Since $\mathrm{H}^1(\mathrm{P}) = \varinjlim \mathrm{H}^1(\mathrm{V}_\alpha)$, we see that $h^1(\mathrm{P}) = h^1(\mathrm{V}_\alpha)$, so

$$h^1(\mathrm{P}) - h^0(\mathrm{P}) = d\,(h^1(\mathrm{M}) - 1) + \sum_{p \in \mathrm{P}} (i_p \varphi - 1).$$

Since $h^1(\mathrm{P})$ is finite, as was shown in (*c*), we see that $\sum_{p \in \mathrm{P}} (i_p \varphi - 1)$ is finite, i. e. C is finite.

Q.E.D.

3. Topological holomorphy of χ.

Theorem 4. — *Let* $\mathbf{f} = (f_\lambda)_{\lambda \in \Lambda}$ *be an analytic family of polynomial-like mappings of degree* 2, *with* Λ *connected of complex dimension* 1. *Then if the mapping* $\chi : \Lambda \to \mathbb{C}$ *defined using some tubing* T *of* $\mathbf{f}$ *is not constant, it is topologically holomorphic over* M.

Proof. — If χ is not constant, then $\chi^{-1}(c)$ is discreet for all $c \in \mathrm{M}$ by Corollary 2 of Proposition 13. So we need to know that $i_\lambda(\chi) > 0$ for all $\lambda \in \mathrm{M}_\mathbf{f}$.

Let $\mathscr{R}$ and $\mathscr{F}$ be the complementary open and closed subsets of Λ given by the M-S-S decomposition. We will distinguish three cases.

(*a*) $\lambda \in \mathscr{R}$, $c \in \mathring{\mathrm{M}}$. Then χ is holomorphic on a neighborhood of λ by Theorem 2.

(b) $\lambda \in \mathscr{R}$, $c \in \partial \mathrm{M}$. According to Proposition 7, f_λ, is quasi-conformally equivalent to f_λ for λ' sufficiently close to λ. But by Proposition 4, $f_{\lambda'}$ is therefore hybrid equivalent to f_λ, and so χ is constant on a neighborhood of λ, and so constant.

(*c*) $\lambda \in \mathscr{F}$, $c \in \partial \mathrm{M}$. Let Δ be a disc in Λ containing λ and no other point of $\chi^{-1}(c)$; set $\gamma = \partial\Delta$ and $i = i_\lambda(\chi)$. By the definition of $\mathscr{F}$ there is $\lambda \in \mathring{\Delta}$ such that $\chi(\gamma)$ has winding number i around $c' = \chi(\lambda')$, and that $f_{\lambda'}$ has a non-persistent indifferent periodic point.

There then exists $\lambda'' \in \mathring{\Delta}$ so that $\chi(\gamma)$ has winding number i around $c'' = \chi(\lambda'')$ *and* λ'' has an attractive periodic point, so $c'' \in \mathring{\mathrm{M}}$.

Now $i = \sum_{m \in \Delta \cap \chi^{-1}(c'')} i_\mu \chi$; but each term of the sum is > 0 since χ is holomorphic at such a μ (Thm. 3) and there is at least one term in the sum, for $\mu = \lambda''$.

Q.E.D.

Proposition 20. — *Under the hypotheses of Theorem* 4, *if the tubing* T *is horizontally analytic, then* χ *is quasi-conformal on* $\Lambda' - \mathrm{M}_\mathbf{f}$ *for any relatively compact open subset* Λ' *of* Λ.

Remark. — Here the definition of quasi-conformal is not quite the ordinary one, since χ may not be injective. Saying that χ is quasi-conformal on $\mathrm{W} \subset \Lambda - \mathrm{M}_\mathbf{f}$ means that $\chi|_\mathrm{W}$ is in the Sobolev space $\mathrm{H}^1_{\mathrm{loc}}(\mathrm{W})$ and that there exists a Beltrami form μ on W with L^∞-norm $= k < 1$, such that $\bar{\partial}\varphi = \mu\,\partial\varphi$. In other words, χ is analytic for the structure defined by μ.

Lemma. — *Let* V *and* W *be open subsets of* $\mathbb{C}$, $k \in]0, 1[$, $(\mathrm{T}_\lambda)_{\lambda \in \mathrm{W}}$ *a familly of quasi-conformal embeddings of* V *into* $\mathbb{C}$, *all with dilatation bounded by* k, *and such that* $\lambda \mapsto \mathrm{T}_\lambda(x)$ *is analytic for all* $x \in \mathrm{V}$.

Let $\sigma: \mathrm{W} \to \mathbb{C}$ *be an analytic mapping, and let* $\mathrm{W}' \subset \mathrm{W}$ *be the open subset of all* λ *such that* $\sigma(\lambda) \in \mathrm{T}_\lambda(\mathrm{W})$. *Then the mapping* $h: \lambda \mapsto \mathrm{T}_\lambda^{-1}(\sigma(\lambda))$ *of* W' *into* $\mathbb{C}$ *is quasi-conformal with dilatation* $\leqq k$.

Proof. — Suppose first that $(\lambda, x) \mapsto \mathrm{T}_\lambda(x)$ is of class C^1. Then h is also of class C^1, and

$$\frac{\partial h}{\partial \lambda}(\lambda) = \frac{\partial \mathrm{T}}{\partial x}(\lambda, h(\lambda)) \cdot \left(\sigma'(\lambda) - \frac{\partial \mathrm{T}}{\partial \lambda}(\lambda, h(\lambda))\right),$$
$$\frac{\partial h}{\partial \bar{\lambda}}(\lambda) = \frac{\partial \mathrm{T}}{\partial \bar{x}}(\lambda, h(\lambda)) \cdot \overline{\left(\sigma'(\lambda) - \frac{\partial \mathrm{T}}{\partial \lambda}(\lambda, h(\lambda))\right)}.$$

Therefore h is quasi-conformal with dilatation bounded by k.

In the general case, the family (T_λ) can be uniformly approximated by families $T_{n,\lambda}$ of class C^1, still analytic in λ and all k-quasi-conformal. Then the corresponding h_n are k-quasi-conformal, and converge uniformly to h. So h is k-quasi-conformal.

Q.E.D.

Proof of Proposition 20. — The tubing T is an embedding of $\Lambda \times (\bar{D}_R - D_{R'})$ into $\mathcal{U} \subset \Lambda \times \mathbb{C}$, with $R' = R^{1/2}$. It can be extended to an embedding $\hat{T}$ of $\bar{\Lambda}(\bar{D}_R - D_{R'-\varepsilon})$ with $\hat{T}_\lambda(x) \in f_\lambda^{-1}(x^2)$ for $R' - \varepsilon \leqq |x| \leqq R'$.

Then $\hat{T}$ is holomorphic in λ, and k'-quasi-conformal over Λ' for some $k' < 1$.

Call ω_λ the critical point of f_λ, and let $W_n \subset \Lambda$ be the open set of those λ such that $f_\lambda^n(\omega_\lambda)$ is defined and belongs to $\hat{T}(\Lambda \times (D_R - \bar{D}_{R'-\varepsilon}))$. On $W_n \cap \Lambda'$, the mapping $h_n: \lambda \mapsto \hat{T}^{-1}(f_\lambda^n(\omega_\lambda))$ is k'-quasi-conformal according to the Lemma, but $h_n(\lambda) = \Phi(\chi(\lambda))^{2^n}$, where $\Phi: \mathbb{C} - M \overset{\approx}{\to} \mathbb{C} - \bar{D}$ is the mapping described in [D-H] and [D]. Therefore χ is k'-quasi-conformal on $W_n \cap \Lambda'$.

For all $\lambda \in \Lambda - M_f$, the point $f_\lambda(\omega_\lambda)$ belongs to the open set $\mathring{A}_\lambda$ bounded by the curve $T_\lambda(\partial D_R)$ since the inverse image of this curve is connected.

For the last n for which $f_\lambda^n(\omega_\lambda)$ is defined and belongs to $\mathring{A}_\lambda$, $\lambda \in W_n$, so $\bigcup W_n = \Lambda - M_f$. So χ is k'-quasi-conformal on $\Lambda' - M_f$.

Q.E.D.

4. The case M_f compact. — Suppose that the hypotheses of Theorem 4 are satisfied, and that M_f is compact. Then $\chi: M_f \to M$ is a ramified covering space, and since M is connected [D-H], χ has a degree δ, which we will call the *parametric degree* of the family **f**, as opposed to the degree of the family, which is 2 in this case.

If $\delta = 0$ then M_f is empty.

If $\delta = 1$ then χ restricts to a homeomorphism $M_f \to M$; in that case we say that **f** is *Mandelbrot-like*.

Call ω_λ the critical point of f_λ. A sufficient condition for M_f to be compact is that there exist $A \subset \Lambda$ such that $f_\lambda(\omega_\lambda) \in U_\lambda - U'_\lambda$ for $\lambda \in \Lambda - A$.

Proposition 21. — *Suppose λ homeomorphic to* D *and* M_f *compact. Let* $A \subset \Lambda$ *a subset homeomorphic to* $\bar{D}$ *such that* $M_f \subset \mathring{A}$. *The parametric degree* δ *of* **f** *is equal to the number of times* $f_\lambda(\omega_\lambda) - \omega_\lambda$ *turns around* 0 *as* λ *describes* ∂A.

Proof. — Let $\lambda_0 \in \Lambda$ be a point such that $f_{\lambda_0}(\omega_{\lambda_0}) = \omega_{\lambda_0}$. Then $\chi(\lambda_0) = 0$, and χ is holomorphic on a neighborhood of λ_0. Moreover, the multiplicity $i_{\lambda_0}(\chi)$ of λ_0 as a zero of χ is equal to its multiplicity as a zero of $\lambda \mapsto f_\lambda(\omega_\lambda) - \omega_\lambda$.

Indeed, the hybrid equivalence φ_λ of f_λ with $z \mapsto z^2 + \chi(\lambda)$ is an analytic function of (λ, z), and

$$\chi(\lambda) = \varphi_\lambda(f_\lambda(\omega_\lambda))$$

with $\varphi_\lambda(\omega_\lambda) = 0$ and $d\varphi_\lambda/dz(\omega_\lambda) \neq 0$.

Therefore, $\delta = \sum_{\lambda \in \chi^{-1}(0)} i_\lambda(\chi)$ is the number of zeroes of $\lambda \mapsto f_\lambda(\omega_\lambda) - \omega_\lambda$, counted with multiplicities.

Q.E.D.

5. FURTHER RESSEMBLANCE OF $\mathrm{M}_{\mathbf{f}}$ AND M. — Set $\mathrm{I} = [0,1]$, let Λ be a Riemann surface and

$$\mathbf{f} = (f_{s,\lambda} : \mathrm{U}'_{s,\lambda} \to \mathrm{U}_{s,\lambda})_{(s,\lambda) \in \mathrm{I} \times \Lambda},$$

be a family of polynomial-like mappings of degree 2. We will assume that the conditions (1) and (2) of Definition II,1,1 are satisfied. We will also assume that $f : \mathcal{U}' \to \mathcal{U}$ is continuous, holomorphic in (λ, z), and proper.

Suppose that for each $s \in \mathrm{I}$, the analytic family $\mathbf{f}_s = (f_{s,\lambda})_{\lambda \in \Lambda}$ is Mandelbrot-like, and that the $\mathrm{M}_{\mathbf{f}_s}$ are all contained in a common compact set $\mathrm{A} \subset \Lambda$.

We will then say that $\mathbf{f}_0$ and $\mathbf{f}_1$ are connected by a continuous path of Mandelbrot-like families.

PROPOSITION 22. — *Let $\mathbf{f} = (f_\lambda)_{\lambda \in \Lambda}$ and $\mathbf{g} = (g_\lambda)_{\lambda \in \Lambda}$ be two Mandelbrot-like families parametrized by the same Riemann surface Λ. If $\mathbf{f}$ and $\mathbf{g}$ can be connected by a continuous path of Mandelbrot-like families, then the homeomorphism $\chi_{f,g} = \chi_{\mathbf{g}}^{-1} \circ \chi_{\mathbf{f}} : \mathrm{M}_{\mathbf{f}} \to \mathrm{M}_{\mathbf{g}}$ is quasi-conformal in the sense of Mañe-Sad-Sullivan.*

Proof. — It is enough to prove the property for $\mathbf{g}$ sufficiently close to $\mathbf{f}$. Set $\eta_\lambda = g_\lambda - f_\lambda$. If $\mathbf{g}$ is sufficiently close to $\mathbf{f}$, there exists $\mathrm{R} > 1$ and Λ' relatively compact in Λ, containing $\mathrm{M}_{\mathbf{f}}$, such that for any $t \in \mathbb{C}$ with $|t| < \mathrm{R}$, the family $\mathbf{f}_t = (f_\lambda + t.\eta_\lambda)_{\lambda \in \Lambda'}$, appropriately restricted, is Mandelbrot-like.

Then $(\chi_{\mathbf{f}_t}^{-1})_{t \in \mathrm{D}_r}$ is a complex analytic family of topological embeddings of M into Λ.

The Proposition follows from the λ-Lemma [M-S-S] (note that in our case, the parameter is s and the variable is λ).

Q.E.D.

Example 1. — Let Λ and V be open in $\mathbb{C}$, both containing the disc $\bar{\mathrm{D}}_4$ of radius 4, and

$$(\lambda, z) \mapsto f_\lambda(z) = z^2 + \lambda + \eta_\lambda(z)$$

be a complex analytic mapping of $\Lambda \times \mathrm{V}$ into $\mathbb{C}$ with $\eta'_\lambda(0) = 0$ and $|\eta_\lambda(z)| \leqq 1$ for all $(\lambda, z) \in \Lambda \times \mathrm{V}$.

Set $\mathrm{U}_\lambda = \mathrm{D}_{10}$ for all λ, and $\mathrm{U}'_\lambda = f_\lambda^{-1}(\mathrm{U}_\lambda)$. Then $f_\lambda : \mathrm{U}'_\lambda \to \mathrm{U}_\lambda$ is a polynomial-like mapping of degree 2 for $|\lambda| \leqq 4$, and $\mathbf{f} = (f_\lambda : \mathrm{U}'_\lambda \to \mathrm{U}_\lambda)_{\lambda \in \mathrm{D}_4}$ is a Mandelbrot-like family.

Indeed, if $|\lambda| \leqq 4$ and $|z| = 4$ then

$$|f_\lambda(z)| \geqq 16 - 4 - 1 > 10,$$

so the equation $f_\lambda(z) = w$ has two solutions $z_1, z_2 \in \mathrm{D}_4$ if $|w| < 10$, and $f_\lambda : \mathrm{U}'_\lambda \to \mathrm{U}_\lambda$ is proper of degree 2.

By the maximum principle, U'_λ is topologically a union of discs, and since there is at least one critical point, namely 0, the Riemann-Hurwitz formula implies that $U'_\lambda \approx D$ and $U'_\lambda \subset \bar{D}_4 \subset D_{10}$.

Moreover, if $|\lambda|=4$ then $|f_\lambda(0)| \geqq 4-1=3$ and $|f_\lambda^2(0)| \geqq 9-4-1=4$, so $f_\lambda^2(0) \notin U'_\lambda$ and $\lambda \notin M_f$.

Finally, the parametric degree is 1 by Proposition 21.

Remark. — For such a family, the mapping $\chi: M_f \to M$ is quasi-conformal in the sense of M-S-S according to Proposition 22.

PROPOSITION 23. — *Choose* $\varepsilon < 1$, *and suppose that the conditions of example* 1 *above are satisfied. Suppose that* $|\eta_\lambda(z)| \leqq \varepsilon$ *for* $|\lambda| < 4$, $|z| < 4$. *Then:*

(*a*) $|\chi(\lambda)-\lambda| \leqq 6\varepsilon$ *for all* $\lambda \in M_f$.

(*b*) *If* $\lambda_1, \lambda_2 \in M_f$, *then*

$$\frac{1}{k_-(\varepsilon)}|\lambda_1-\lambda_2|^{1/\beta(\varepsilon)} \leqq |\chi(\lambda_1)-\chi(\lambda_2)| \leqq k_+(\varepsilon)|\lambda_1-\lambda_2|^{\beta(\varepsilon)},$$

with

$$k_-(\varepsilon)=8^{2\varepsilon/(1+\varepsilon)}, \qquad k_-(\varepsilon)=8^{2\varepsilon/(1-\varepsilon)}$$

and

$$\beta(\varepsilon)=(1-\varepsilon)/(1+\varepsilon).$$

Proof. — Set $\eta_\lambda=\varepsilon h_\lambda$, so that $|h_\lambda(z)| \leqq 1$ for $|\lambda| < 4$, $|z| < 4$. Set $F_{s,\lambda}(z)=z^2+\lambda+sh_\lambda(z)$. For each $s \in D$, $F_s=(F_{s,\lambda})_{\lambda \in D_4}$ is a Mandelbrot-like family, and if χ_s is defined using F_s, the mapping χ_s^{-1} is an embedding of M into D_4. For every $c \in M$, the mapping $\gamma_c: s \mapsto \chi_s^{-1}(c)$ of D into D_4 is analytic, since its graph is an analytic subset of $D \times D_4$ by Proposition 13, Corollary 2.

(*a*) If $s \in D$, then $|\gamma_c(s)| \leqq 4$ and $|c| \leqq 2$, so $|\gamma_c(s)-c| \leqq 6$. Since $\gamma_c(0)=c$, we see that $|\gamma_c(s)-c| \leqq 6|s|$ by Schwarz's Lemma. If we take $s=\varepsilon$, we find

$$|\chi^{-1}(c)-c| \leqq 6\varepsilon.$$

(*b*) If $c_1 \neq c_2$, then the mappings γ_{c_1} and γ_{c_2} have disjoint graphs. The inequality in the Proposition follows from the following Lemma.

LEMMA. — *Let* $u, v: D \to D_R$ *be two holomorphic functions with disjoint graphs. Then*

$$\left|\frac{u(0)-v(0)}{2R}\right|^{1/\beta(z)} \leqq \frac{|u(z)-v(z)|}{2R} \leqq \left|\frac{u(0)-v(0)}{2R}\right|^{\beta(z)},$$

with

$$\beta(z)=\frac{1-|z|}{1+|z|}.$$

Proof. — Set $w=\log(u-v)/2\mathrm{R}$, where some branch of the logarithm is chosen, which is possible since $u-v$ does not vanish and D is simply connected. Let $w_0=w(0)$. The function w takes its values in the left half-plane $\mathrm{Re}(w)<0$, and the mapping

$$w \mapsto \frac{w-w_0}{w+w_0},$$

is an isomorphism of that half-plane onto D. Then

$$\left|\frac{\mathrm{Re}(w-w_0)}{\mathrm{Re}(w+w_0)}\right| \leqq \left|\frac{w-w_0}{w+w_0}\right| \leqq |z|,$$

where the first inequality is true because the level curves of

$$w \mapsto \left|\frac{w-w_0}{w+w_0}\right|$$

are circles, and the second follows from Schwarz's Lemma.

So we find that

$$\beta(z) \leqq \frac{\mathrm{Re}(w)}{\mathrm{Re}(w_0)} \leqq \frac{1}{\beta(z)}.$$

Q.E.D.

This finishes the proof of Proposition 23.

Example 2. — Let $(f_\lambda: \mathrm{U}'_\lambda \to \mathrm{U}_\lambda)_{\lambda\in\Lambda}$ be an analytic family of polynomial-like mappings of degree 2, parametrized by an open subset of $\mathbb{C}$ homeomorphic to D. We will assume that U_λ is convex for every λ, and that there is a compact subset $\mathrm{A} \subset \Lambda$ such that $f_\lambda(\omega_\lambda)$ is not in the convex hull of U'_λ for $\lambda \in \Lambda - \mathrm{A}$. This implies that $\mathrm{M}_\mathbf{f}$ is compact. We suppose that the parametric degree $\delta=1$.

Proposition 24. — *Under these conditions, the homeomorphism* $\chi: \mathrm{M}_\mathbf{f} \to \mathrm{M}$ *is quasi-conformal in the sense of M-S-S.*

Proof. — By an affine change of coordinates depending on λ we may assume that $\omega_\lambda=0$ and $f''(\omega_\lambda)=2$ for all λ, i. e. that

$$f_\lambda(z)=z^2+\lambda+O(z^3).$$

The function $c: \Lambda \to \mathbb{C}$ has a simple zero at some $\lambda_0 \in \Lambda$, and we can identify Λ with an open subset of $\mathbb{C}$ so that $\lambda_0=0$ and $c'(0)=1$. Then

$$f_\lambda(z)=z^2+\lambda+O(|z|^3+|\lambda|^2),$$

and shrinking Λ slightly if necessary, we may assume that the distance of U′ to the complement of U is bounded below by a number $m>0$ independant of λ.

Set $\Lambda_s=s^{-2}\Lambda$; and for $\lambda\in\Lambda_s$ set $\mathrm{U}'_{s,\lambda}=s^{-1}\mathrm{U}'_{s^2\lambda}$ and $\mathrm{U}_{s,\lambda}=s^{-2}\mathrm{U}_{s^2\lambda}$. Define $f_{s,\lambda}: \mathrm{U}'_{s,\lambda} \to \mathrm{U}_{s,\lambda}$ by $f_{s,\lambda}=s^{-2} f_{s^2\lambda}(sz)$, and define $f_{0,\lambda}(z)=z^2+\lambda$.

The mapping $(s, \lambda, z) \mapsto f_{s,\lambda}(z)$ gives a continuous path of Mandelbrot-like families connecting (f_λ) to $z \mapsto z^2+\lambda$.

We conclude by Proposition 22.

Q.E.D.

CONJECTURE. — *For any complex analytic family* $\mathbf{f}=(f_\lambda)_{\lambda\in\Lambda}$ *of polynomial-like mappings of degree* 2 *with* $\Lambda \approx \mathrm{D}$, *there exists a quasi-conformal mapping* $\varphi_1 : \Lambda \to \mathrm{D}$ *and a holomorphic map* $\varphi_2 : \mathrm{D} \to \mathbb{C}$ *such that* $\chi_{\mathbf{f}} = \varphi_2 \circ \varphi_1$ *on* $\mathrm{M}_{\mathbf{f}}$.

CHAPTER V

Small copies of M in M

1. TUNABLE POINTS OF M. — Let $\mathrm{P}_c : z \mapsto z^2+c$ and let $c \in \mathrm{M}$ be a point for which 0 is periodic of period k for P_c. We will call c *tunable* if there is a neighborhood Λ of c in $\mathbb{C}$ and a Mandelbrot-like family $\mathbf{f}=(f_\lambda : \mathrm{U}'_\lambda \to \mathrm{U}_\lambda)_{\lambda\in\Lambda}$ such that for every $\lambda \in \Lambda$, the mapping f_λ is the restriction of P_λ^k to U'_λ, and $0 \in \mathrm{U}'_\lambda$.

Call M_c the set $\mathrm{M}_{\mathbf{f}}$, and $x \mapsto c \perp x$ the inverse of the homeomorphism $\chi : \mathrm{M}_c \to \mathrm{M}$. Then $\mathrm{M}_c \subset \mathrm{M}$ and $\partial \mathrm{M}_c \subset \partial \mathrm{M}$. Indeed, for any $\lambda \in \Lambda$,

$$\mathrm{K}_{f_\lambda} \subset \mathrm{K}_{\mathrm{P}_\lambda^k} = \mathrm{K}_{\mathrm{P}_\lambda}$$

and if K_{f_λ} is connected, the connected components of $\mathrm{K}_{\mathrm{P}_\lambda}$ are not points. Any point in $\partial \mathrm{M}_c$ can be approximated by λ for which f_λ has an indifferent periodic point; such a point is also an indifferent periodic point of P_λ.

We will say that c is *semi-tunable* if there exists an analytic family $\mathbf{f}=(f_\lambda : \mathrm{U}'_\lambda \to \mathrm{U}_\lambda)_{\lambda\in\Lambda}$ of polynomial-like mappings of degree 2, such that as above Λ is a neighborhood of c in $\mathbb{C}$ and each f_λ is a restriction of P_λ^k, and that χ restricts to a homeomorphism of $\mathrm{M}_{\mathbf{f}}$ onto $\mathrm{M} - \{1/4\}$ whose inverse extends to a continuous map $\mathrm{M} \to \mathbb{C}$.

In fact, every $c \in \mathrm{M}$ for which 0 is periodic under P_c is semi-tunable, and is tunable if and only if the component of $\mathring{\mathrm{M}}$ of which it is the center is primitive. The proof depends on detailed knowledge of external radii of M, and will be part of another publication. Here we will show by more general arguments that there are infinitely many tunable points in M.

2. CONSTRUCTION OF A SEQUENCE (c_n). — Let c_0 be a point of M such that 0 is preperiodic. In this paragraph we will approximate c_0 by a sequence (c_n) such that for each c_n, 0 is periodic. It is easy, using Picard's Theorem, to show that such a sequence exists, but we will require more specific information about it. Our construction is analogous to that of III,1.

Let l be the smallest integer for which $\zeta_0 = \mathrm{P}_{c_0}^l(0)$ is a repulsive periodic point; let k be its period. There exists a neighborhood Λ of c_0 and an analytic function $\zeta : \Lambda \to \mathbb{C}$ such that $\zeta(c_0) = \zeta_0$ and that $\zeta(\lambda)$ is a repusive periodic point of period k for all P_λ with $\lambda \in \Lambda$.

Set

$$\rho(\lambda)=(P_\lambda^k)'(\zeta(\lambda)),$$

$$\rho_0=\rho(c_0) \qquad \text{and} \qquad \mu=\inf_{\lambda\in\Lambda}|\rho(\lambda)|.$$

We may assume that $\mu>1$.

LEMMA 1. — *The function* $\lambda\mapsto P_\lambda^l(0)-\zeta(\lambda)$ *has a simple zero at* c_0.

There are several ways of proving this Lemma; we will give an arithmetic one, using an idea of A. Gleason's.

Proof. — It is equivalent to show that

$$\lambda\mapsto P_\lambda^{l+k}(0)-P_\lambda^l(0)$$

has a simple zero. Set $F_n(c)=P_c^n(0)$; then F_n is monic of degree 2^{n-1}, with integer coefficients, and

$$F'_n=2F_{n-1}F'_{n-1}+1.$$

Let $\bar{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$, and let c be a solution of $F_{l+k}(c)=F_l(c)$ in $\bar{\mathbb{Q}}$. If the smallest l for which c is a solution is >0, then

$$F_{k+l-1}(c)+F_{l-1}(c)=0,$$

and $l\geqq 2$. Set

$$G=F_{k+l-1}+F_{l-1},$$

so that G is monic of degree 2^{k+l-2}, and

$$G'/2=F_{k+l-2}F'_{k+l-2}+F_{l-2}F'_{l-2}+1$$

has integer coefficients.

Extend the 2-adic valuation of $\mathbb{Q}$ to a valuation v on $\bar{\mathbb{Q}}$, and set

$$A=\{z\in\bar{\mathbb{Q}}\,|\,v(z)\geqq 0\} \qquad \text{and} \qquad \mathbf{m}=\{z\in\bar{\mathbb{Q}}\,|\,v(z)>0\}.$$

Since c is an algebraic integer, we see that $c\in A$. Moreover,

$$(F_{k+l-2}(c))^2+(F_{l-2}(c))^2=-2c\in\mathbf{m},$$

so

$$F_{k+l-2}(c)\equiv F_{l-2}(c)\equiv -F_{l-2}(c) \bmod \mathbf{m}.$$

The computation of F' gives

$$F'_{k+l-2}(c)\equiv F'_{l-2}\equiv 1 \bmod 2A.$$

so that

$$G'(c)/2\equiv 1 \bmod \mathbf{m}, \qquad G'(c)\neq 0,$$

and

$$(F_{l+k}-F_l)'(c)=2\,F_l(c)\,G'(c)\neq 0.$$

Q.E.D.

By restricting Λ if necessary, we may assume that c_0 is the only zero of ζ in Λ, and we can find a neighborhood $\mathscr{V}$ of (c_0, ζ_0) in $\mathbb{C}^2$, an $R>0$ and an isomorphism $(\lambda, z)\mapsto(\lambda, v_\lambda(z))$ of $\mathscr{V}$ onto $\Lambda\times D_R$ such that

$$v_\lambda(P_\lambda^k(z))=\rho(\lambda).\,v_\lambda(z)$$

if $v_\lambda(z)<R/\rho(\lambda)$. Set $V_\lambda=\{z\,|\,(\lambda, z)\in\mathscr{V}\}$ and $g(\lambda)=v_\lambda(P_\lambda^l(0))$, so $g'(c_0)\neq 0$.

All repulsive periodic points are in the Julia set, so $\zeta_0\in J_{c_0}$; also the inverse images of 0 are dense in J_{c_0}. There is therefore a point $\alpha_0\in V_0$ such that $P_{c_0}^p(\alpha_0)=0$; then $(P_{c_0}^p)'(\alpha_0)\neq 0$.

Shrinking Λ again if necessary, there exists an analytic mapping $\lambda\mapsto\alpha_0(\lambda)$ such that $\alpha_0(c_0)=\alpha_0$ and that $P_\lambda^p(\alpha(\lambda))=0$ for all $\lambda\in\Lambda$. Now shrinking $\mathscr{V}$ and modifying R and the v_λ appropriately, we may assume that $v_\lambda(\alpha(\lambda))=1$ for $\lambda\in\Lambda$.

We define $\alpha_n\colon\Lambda\to\mathscr{V}$ by $v_n(\alpha_n(\lambda))=\rho(\lambda)^{-n}$, so that $P_\lambda^{nk}(\alpha_n(\lambda))=\alpha_0(\lambda)$.

If n is large enough, there exists a unique $\lambda\in\Lambda$ such that $g(\lambda)=\rho(\lambda)^{-n}$, i. e. that $P_\lambda^l(0)=\alpha_n(\lambda)$. Let c_n denote that value of λ. Then for P_{c_n}, the critical point 0 is periodic of period $l+nk+p$.

Proposition 25. — *The sequence $\rho_0^n.(c_n-c_0)$ has a finite limit $C_1\neq 0$.*

Proof. — Since $|g(c_n)|\leq\mu^{-n}$, the sequence (c_n) converges to c_0 (at least exponentially). Since $f(c_n)=\rho(c_n)^{-n}$, we have that

$$\rho_0^n(c_n)=\left(\frac{\rho_0}{\rho(c_n)}\right)^n\frac{c_n-c_0}{g(c_n)-g(c_0)}.$$

The first factor tends to 1; indeed, $\rho(c_n)\to\rho_0$ at least exponentially, so $n\,\mathrm{Log}(\rho(c_n)/\rho_0)\to 0$.

The second factor goes to $1/g'(c_0)$, and the Proposition follows with $C_1=1/g'(c_0)$.

Q.E.D.

3. Tunability of the c_n.

Theorem 5. — (*a*) *For n sufficiently large, c_n is tunable.*

(*b*) *There exists a constant C_2 such that the sequence of mappings*

$$\varphi_n\colon\quad x\mapsto C_2\,\rho_0^{2n}(c_n\perp x-c_n)$$

converge uniformly on M *to the canonical injection* $M\to\mathbb{C}$. *The mappings φ_n are bihölder, with both exponent and coefficient tending to* 1.

Proof. — Call $\mathscr{P}$ the mapping $(\lambda, z) \mapsto (\lambda, P_\lambda(z))$. By shrinking Λ, we may find $R' < R-1$ such that $\mathscr{P}^p$ induces an isomorphism of

$$\mathscr{V}' = \{ (\lambda, z) \in \mathscr{V} \mid |v_\lambda(z) - 1| < R \}$$

onto a neighborhood $\mathscr{U}$ of $\Lambda \times \{0\}$ in $\Lambda \times \mathbb{C}$.

Set $U_\lambda = \{ z \mid (\lambda, z) \in \mathscr{U} \}$ and define $u_\lambda : U_\lambda \to D_{R'}$ by

$$u_\lambda(P_\lambda^p(z)) = v_\lambda(z) - 1.$$

By decreasing R' and further shrinking Λ we may assume that $\mathscr{P}^l(\mathscr{U}) \subset \mathscr{V}$ and that

$$\mathscr{P}^l(\mathscr{U}) \supset \{ (\lambda, z) \mid |v_\lambda(z)| < m_1 \}$$

for some m_1.

Let

$$\mathscr{V}'_n = \{ (\lambda, z) \in \mathscr{V} \mid |v_\lambda(z) - \rho(\lambda)^{-n}| < R' \rho(\lambda)^{-n} \},$$

so that $\mathscr{P}^{nk}$ induces an isomorphism of $\mathscr{V}'_n$ onto $\mathscr{V}$. Let $\mathscr{U}'_n = \mathscr{U} \cap \mathscr{P}^{-l}(\mathscr{V}'_n)$. Choose n_0 such that $R/\mu^{n_0} < m_1$. If $n \geqq n_0$, $\mathscr{P}^l$ restricts to a proper map $\mathscr{U}'_n \to \mathscr{V}'_n$ of degree 2, and $\mathscr{P}^{l+nk+p}$ restricts to a proper mapping $\mathscr{U}'_n \to \mathscr{U}$.

Let $h_\lambda = v_\lambda \circ P_\lambda^l \circ u_\lambda^{-1} : D_{R'} \to D_R$ be the map $P_\lambda^l : U_\lambda \to V_\lambda$ written in the local coordinates u_λ and v_λ. The function $(\lambda, u) \Rightarrow h_\lambda(u)$ can be written

$$h_\lambda(u) = g(\lambda) + a(\lambda) u^2 + \eta_\lambda(u),$$

with $a(\lambda) \neq 0$, holomorphic in λ and with $|\eta_\lambda(u)| \leqq A_1 |u|^3$, where A_1 is a constant independant of $(\lambda, u) \in \Lambda \times D_{R'}$. Set $m_2 = \inf |a(\lambda)|$; we may assume $m_2 > 0$.

Let $f_{n,\lambda}$ be the mapping

$$P_\lambda^{l+nk+p} : U'_{n,\lambda} \to U_\lambda$$

written in the local coordinate u_λ both in the domain and in the range. Then

$$f_{n,\lambda} : u \mapsto \rho(\lambda)^n h_{\lambda(u)-1} = \rho(\lambda)^n a(\lambda) u^2 + \rho(\lambda)^n g(\lambda) - 1 + \rho(\lambda)^n \eta_\lambda(u);$$

$f_{n,\lambda}$ is defined on

$$u_\lambda(U'_{n,\lambda}) = \{ u \mid |\rho(\lambda)^n h_\lambda(u) - 1| < R' \}.$$

Set

$$\tilde{u}_{\lambda,n}(z) = \rho(\lambda)^n a(\lambda) u_\lambda(z),$$

$$c_2 = a(c_0) g'(c_0),$$

$$\tilde{\lambda}_n = c_2 \rho_0^{2n} (\lambda - c_n) \quad \text{and} \quad \Lambda_n = \{ \lambda \mid |\tilde{\lambda}_n| < 4 \}.$$

Let $\tilde{f}_{n,\lambda}$ be the mapping $P_\lambda^{l+nk+p} : U'_{n,\lambda} \to U_\lambda$ written in the local coordinate $\tilde{u}_{\lambda,n}$ and define $\tilde{\eta}_{n,\lambda}$ by

$$(*) \qquad \tilde{f}_{n,\lambda}(\tilde{u}) = \tilde{u}^2 + \tilde{\lambda} + \tilde{\eta}_{n,\lambda}(\tilde{u}).$$

LEMMA 2. — *(a) If n is sufficiently large, then $\Lambda_n \subset \Lambda$ and $\tilde{u}_{\lambda,n}(U'_{n,\lambda}) \supset D_4$.*

(b) For such an n let

$$\varepsilon_n = \sup_{\substack{\lambda \in \Lambda_n \\ \tilde{u} \in D_4}} \tilde{\eta}_{n,\lambda}(\tilde{u}).$$

Then the sequence (ε_n) converges to 0.

Proof. — Since λ_n is the open disc centered at c_n and of radius $4/(c_2 \rho_0^{2n})$, the first part of (a) is true.

If $\lambda \in \Lambda$ and

$$|\tilde{u}| < \frac{4}{|\rho(\lambda)|^n |a(\lambda)|},$$

we can define $\tilde{\eta}_{\lambda,n}(\tilde{u})$ using the formula $(*)$. Then

$$\tilde{\eta}_{\lambda,n}(\tilde{u}) = h - \tilde{\lambda} + \rho(\lambda)^{2n} a(\lambda)\, \eta_\lambda(\tilde{u}/(\rho(\lambda)^n a(\lambda))),$$

where

$$H = \rho(\lambda)^n\, a(\lambda)\, (\rho(\lambda)^n\, g(\lambda) - 1).$$

The modulus of the last term is bounded by

$$\frac{64\, A_1}{|\rho(\lambda)|^n |a(\lambda)|^2}$$

if $|\tilde{u}| < 4$.

Moreover,

$$\frac{H}{\tilde{\lambda}} = \left(\frac{\rho(\lambda)}{\rho_0}\right)^{2n} \frac{a(\lambda)}{a(c_0)}\, \frac{g(\lambda) - \rho(\lambda)^{-n}}{g'(c_0)(\lambda - c_n)}.$$

When $\lambda \in \Lambda_n$ and $n \to \theta$, $|\lambda - c_0|$ converges to 0 at least exponentially, so the first two factors tend to 1.

The last factor can be written

$$\frac{g(\lambda) - g(c_n)}{g'(c_0)(\lambda - c_n)} - \frac{1 - (\rho(\lambda)/\rho(c_n))^n}{g'(c_0)(\lambda - c_n)\,\rho(\lambda)^n}.$$

The first term tends to 1, and the second is equivalent to

$$n \frac{\rho'(c_n)}{\rho(c_n)}\, \frac{1}{g'(c_0)\,\rho(\lambda)^n}$$

which tends to 0.

Finally, $H/\tilde{\lambda}$ converges to 1, and since $|\tilde{\lambda}|<4$, we have that $H-\tilde{\lambda}\to 0$. This gives the bound in (*b*).

Since $\tilde{u}\in\tilde{u}_\lambda(U')$ precisely when $\tilde{f}_{n,\lambda}(\tilde{u})$, as defined by formula (∗), satisfies

$$|\tilde{f}_{n,\lambda}(\tilde{u})| < \rho(\lambda)|^n |a(\lambda)| R',$$

and since whenever $\lambda\in\Lambda_n$, $|\tilde{u}|<4$ and n is sufficiently large, the above inequality is satisfied, the second part of (*b*) is true.

Q.E.D. for the Lemma.

End of proof of Theorem.

We have verified the conditions of Proposition IV,23, which gives the required result.

Q.E.D. for Theorem 5.

CHAPTER VI

Carrots for dessert

1. A DESCRIPTION OF FIGURE 4. — Let

$$P_\lambda(z)=(z-1)(z+1/2-\lambda)(z+1/2+\lambda)$$

and let

$$N_\lambda:\quad z\mapsto z-P_\lambda(z)/P'_\lambda(z)$$

be the associated Newton's method. Since

$$N'_\lambda=P_\lambda P''_\lambda/P'^2_\lambda,$$

the critical points of N_λ are the zeroes 1, $-1/2+\lambda$ and $-1/2-\lambda$ of P_λ, and 0 where P''_λ vanishes.

Color λ in blue, red, or green if $N^n_\lambda(0)\to 1$, $-1/2+\lambda$, $-1/2-\lambda$, and leave it in white if none of the above.

You obtain (Figs. 14 to 16).

Figure 4 is an enlargement of Figure 16. For $\lambda=\lambda_0=-.019\,134+.296\,783\,i$, the center of the copy of the Mandelbrot set appearing in white, 0 is periodic of period 3 for N_λ. Set $F_\lambda=N^3_\lambda$.

Setting $\Lambda=D(\lambda_0, .001)$, $U_\lambda=D(.55)$ and $U'_\lambda=U_\lambda\cap F^{-1}_\lambda(U_\lambda)$, one can verify that, after a change of variables, the hypotheses of Proposition 23 are satisfied. Therefore the family

$$\mathbf{F}=(F_\lambda:\ U'_\lambda\to U_\lambda)_{\lambda\in\Lambda}$$

is a Mandelbrot-like family.

For $\lambda\in M_{\mathbf{F}}$, the point 0 is trapped in U'_λ for F_λ, and therefore does not converge to one of the roots of P_λ under N_λ; so such a λ will appear in white. This explains the white copy of M in the picture.

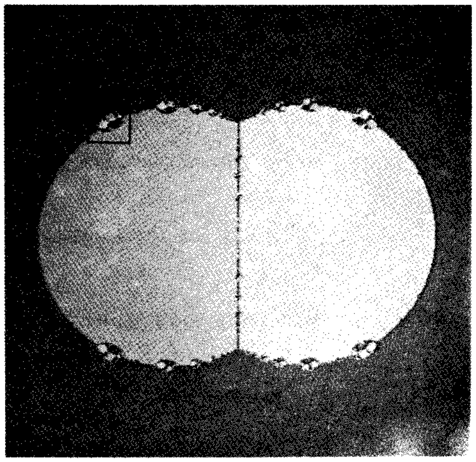

Fig. 14

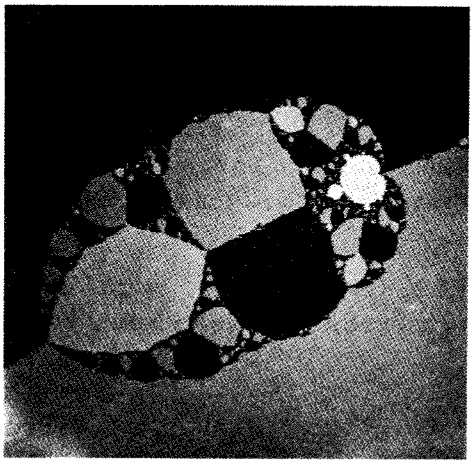

Fig. 15.

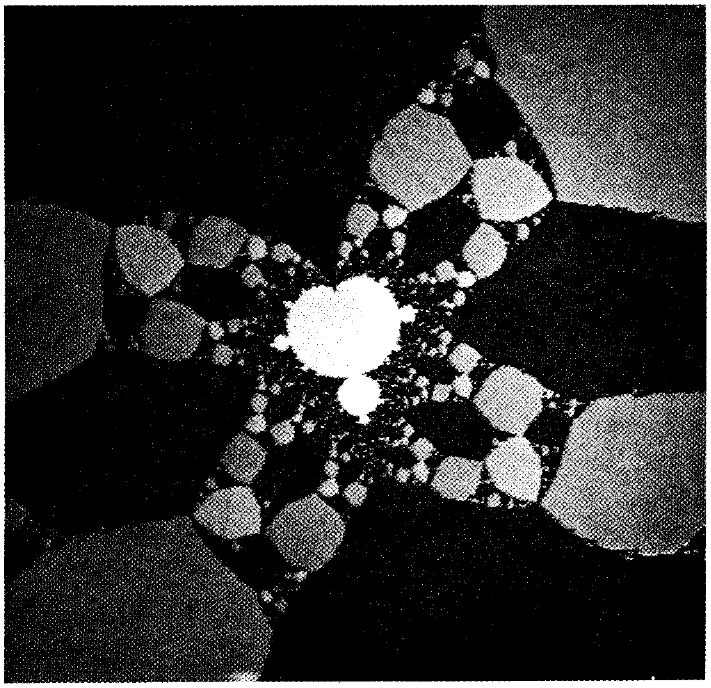

Fig. 16

In [1] we studied the standard Mandelbrot set M, and in particular we described the conformal representation $\Phi: \mathbb{C}-\mathrm{M} \to \mathbb{C}-\bar{\mathrm{D}}$ tangent to the identify at infinity.

Let $\mathscr{R}(\mathrm{M}, \theta)=\{c \mid \arg \Phi(\theta)=\theta\}$ (angles are counted in whole turns, not radians) be the external ray of angle θ. Then each external ray of rational angle θ ends in a point $\gamma_{\mathrm{M}}(\theta)$ of M [D-H]; we say that it is the point of external argument θ.

In Figure 2, we see a collection of blue carrots which point precisely to the points of $\mathrm{M_F}$ which are the inverse images under χ_{F} of the points of M with argument of the form $p/2^k$. If k is large, the corresponding carrot is small.

If we focus on the red and green region, we see a tree-like structure, on which we can read the dyadic expansion of the external arguments of points of $\partial \mathrm{M_F}$.

In this chapter, we give an interpretation of this phenomenon.

2. CARROTS FOR $z \mapsto z^2$. — Fix $\mathrm{R}>1$ and let Q be the closed annulus $\{z \mid \mathrm{R}' \leqq |z| \leqq \mathrm{R}\}$, where $\mathrm{R}'=\mathrm{R}^{1/2}$.

Let I be an open arc in $\mathrm{S}^1_{\mathrm{R}}$, and let I′ be the preimage of I under $f_0: z \mapsto z^2$. Then $\mathrm{I}'=\mathrm{I}'_0 \cup \mathrm{I}'_1$, where I'_0 and I'_1 are arcs in $\mathrm{S}^1_{\mathrm{R}'}$.

Let A be an open set in Q such that

$$\mathrm{A} \cap \partial \mathrm{Q}=\mathrm{I} \cup \mathrm{I}'.$$

Suppose that A has two connected components A_0 and A_1, such that

$$\mathrm{A}_0 \cap \partial \mathrm{Q}=\mathrm{I} \cup \mathrm{I}'_0 \quad \text{and} \quad \mathrm{A}_1 \cap \partial \mathrm{Q}=\mathrm{I}'_1.$$

Suppose moreover that Q−A is connected (*Fig.* 17).

Consider now the semi-closed annulus

$$\Omega=\{z \mid 1<|z| \leqq \mathrm{R}\}.$$

For any $z \in \Omega$, there is an n such that $f_0^n(z) \in \mathrm{Q}$ [unique unless $f_0^n(z) \in \mathrm{S}^1_{\mathrm{R}'}$ and $f_0^{n+1}(z) \in \mathrm{S}^1_{\mathrm{R}}$ for some n]. Define $\mathrm{C} \subset \Omega$ as the set of values of z such that $f_0^n(z) \in \mathrm{A}$ for this n. (If there are two values of n, they give the same condition.)

PROPOSITION 26 AND DEFINITION. — *The open set* C *has an infinite number of connected components. For each* $\tau \in \mathbb{Q}/\mathbb{Z}$ *of the form* $p/2^k$, *there is a unique connected component* C_τ *of* C *such that* $e^{2\pi i \tau} \in \overline{\mathrm{C}}$. *We call it the* carrot *of argument* τ. *For* $\tau=p/2^k$ (*p odd*), *the carrot* C_τ *is contained in* $\mathrm{D}_{\mathrm{R}^{1/2^{k-1}}}$, *and not in* $\mathrm{D}_{\mathrm{R}^{1/2^k}}$.

Proof. — There exists a homeomorphism $\varphi: \Omega \to \Omega$ such that $\varphi(z^2)=(\varphi(z))^2$ and $\varphi(\mathrm{A}_0) \supset \mathrm{Q} \cap \mathrm{R}_+$. We can construct such a φ by first choosing an arc $\gamma: [0, 1] \to \mathrm{A}_0$, connecting I'_0 to I with $\gamma(1)=\gamma(0)^2$, then choosing φ on $\mathrm{S}^1_{\mathrm{R}} \cup \gamma$, then extending to $\mathrm{S}^1_{\mathrm{R}'}$ so as to commute with squaring. Now extend to Q as a homeomorphism, and to Ω by the homotopy lifting property as in chapter 1.

Clearly such a φ extends to a homeomorphism of $\overline{\Omega}$ with $\varphi|_{\mathrm{S}^1}=\mathrm{id}$. Replacing A by $\varphi(\mathrm{A})$, we may assume that $\mathrm{A} \supset \mathrm{Q} \cap \mathscr{R}_+$.

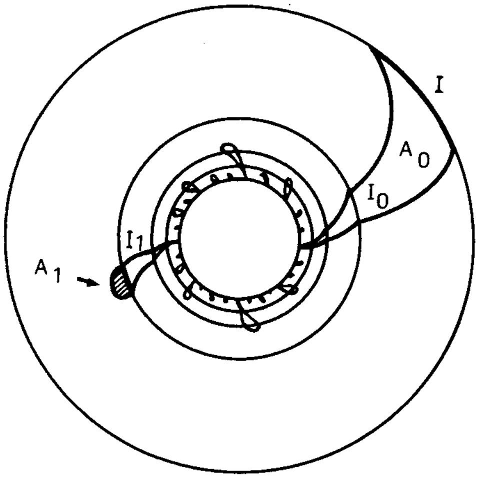

Fig. 17

Let $h_{p,k}$ be the branch of $z \mapsto z^{1/2^k}$ on A_0 such that

$$h_{p,k}(A_0 \cap \mathbb{R}_+) \subset e^{2\pi i p/2^k} \mathbb{R}_+.$$

Let $C_0 = \bigcup_k h_{0,k}(\varphi(A_0))$.

Clearly C_0 is a connected component of C, and $\bar{C}_0 \cap S^1 = \{1\}$. The other components of C are the $h_{p,k}(C_0)$, topped with a hat corresponding to the inverse image of A_1, so that

$$\bar{C}_\tau \cap S^1 = \{e^{2\pi i \tau}\}.$$

Q.E.D.

3. Carrots in a Mandelbrot-like family. — Let $\mathbf{f} = (f_\lambda : U'_\lambda \to U_\lambda)_{\lambda \in \Lambda}$ be a Mandelbrot-like family of polynomial-like mappings of degree 2 and let $\chi : M_{\mathbf{f}} \to M$ be the straightening homeomorphism.

Set

$$\mathscr{U} = \{ (\lambda, z) \mid z \in U_\lambda \},$$
$$\mathscr{K} = \{ (\lambda, z) \mid z \in K_\lambda \},$$

and let $\mathscr{C} \subset \mathscr{U} - \mathscr{K}$ be an open set such that

$$\mathbf{f}^{-1}(\mathscr{C}) = \mathscr{C} \cap \mathscr{U}'.$$

For each $\lambda \in \Lambda$, denote ω_λ the critical point of f_λ, and set

$$C_\Lambda = \{ \lambda \in \Lambda \mid (\lambda, \omega_\lambda) \in \mathscr{C} \}.$$

PROPOSITION 27. — *Suppose that there is a set* $A \subset Q$ *as in section* 2, *and a tubing*

$$T: \quad \Lambda \times Q \to \mathscr{U} - \mathscr{K}$$

such that $T^{-1}(\mathscr{C}) = \Lambda \times A$. *Then:*

(*a*) C_Λ *has an infinity of connected components.*

(*b*) *For each* $\tau \in \mathbb{Q}/\mathbb{Z}$ *of the form* $p/2^k$, *there is a component* $C_{\Lambda, \tau}$ *of* C_Λ *such that*

$$\chi^{-1}(\gamma_M(\tau)) \in \bar{C}_{\Lambda, \tau} \cap \partial M_f.$$

These sets are disjoint.

LEMMA 1. — *Extend* χ *to a map* $\chi_T: \Lambda \to \mathbb{C}$ *using* T, *and define* C *from* A *as in section* 2. *Then*

$$C_\Lambda = \chi_T^{-1}(\Phi_M^{-1}(C)).$$

Proof. — For each $\lambda \in \Lambda$, let $\varphi_\lambda: U_\lambda \to \mathbb{C}$ be the map which conjugates f_λ to $P_\lambda: z \to z^2 + \chi(\lambda)$, constructed using T_λ.

By construction, the map $\varphi_\lambda \circ T_\lambda$ is induced by the map $\psi_{\chi(\lambda)}: \mathbb{C} - D_{R'} \to \mathbb{C}$ which conjugates $z \mapsto z^2$ to P_λ.

For each $c \in \mathbb{C}$, the map ψ_c conjugating $z \mapsto z^2$ to $z \mapsto z^2 + c$ in a neighborhood of ∞ extends analytically to $\mathbb{C} - D_{r(c)}$, with $r(c) = 1$ if $c \in M$ and $r(c) > 1$ if $c \notin M$, but still small enough so that c is in the image of ψ_c. The isomorphism $\Phi: \mathbb{C} - M \to \mathbb{C} - \bar{D}$ is given by $\Phi_M(c) = \psi_c^{-1}(c)$ ([D-H], [D]).

For $\lambda \in \Lambda$, and sufficiently close to M_f to have $\omega(\lambda)$ and $f_\lambda(\omega_\lambda)$ in the region enclosed by $T_\lambda(Q)$, the following conditions are equivalent:

$$\begin{gathered} \lambda \in C_\Lambda, \\ (\exists n) \quad f_\lambda^n(\omega_\lambda) \in T_\lambda(A), \\ (\exists n) \quad P_\lambda^n(0) \in \psi_\lambda(A), \\ (\exists m) \quad P_\lambda^m(\chi(\lambda)) \in \psi_\lambda(A), \\ (\exists m) \quad (\psi_{\chi(\lambda)}^{-1}(\chi(\lambda)))^{2^m} \in A, \\ \Phi_M(\chi(\lambda)) \in C. \end{gathered}$$

Q.E.D.

LEMMA 2. — *Let* W *and* W′ *be open sets in* $\mathbb{C}$, M *a closed set in* W *and* $\chi: W' \to W$ *a continuous map which is topologically holomorphic above* M. *Let* $x \in M$ *and* $x' \in \chi^{-1}(x)$. *Let* C *be a connected open set in* W *such that* x *is accessible by an arc in* C. *Then there is a connected component* C′ *of* $\chi^{-1}(C)$ *such that* $x' \in \bar{C}'$.

Proof. — Let η be an arc in $C \cup \{x\}$, ending at x and not reduced to $\{x\}$. Let Δ' be a topological closed disc in W′, such that

$$\Delta' \cap \chi^{-1}(x) = \{x'\}$$

and that $\chi(\Delta') \not\supset \eta$. Set $\gamma' = \partial\Delta'$.

Let η' be the connected component of $\chi^{-1}(\eta)$ containing x'. Then η' intersects γ'; indeed, if it did not, one could find a loop $\tilde{\gamma}'$ enclosing η' and avoiding $\chi^{-1}(\eta)$. Then $\tilde{\gamma}=\chi\circ\tilde{\gamma}'$ would have a positive winding number around x and winding number 0 around some point of η, without meeting η and that is impossible. Then the component C′ of $\chi^{-1}(C)$ containing $\eta'-\{x'\}$ satisfies the condition.

Q.E.D.

Remark. — The hypothesis of topological holomorphy is necessary to avoid the counter-example shown in *Fig.* 18.

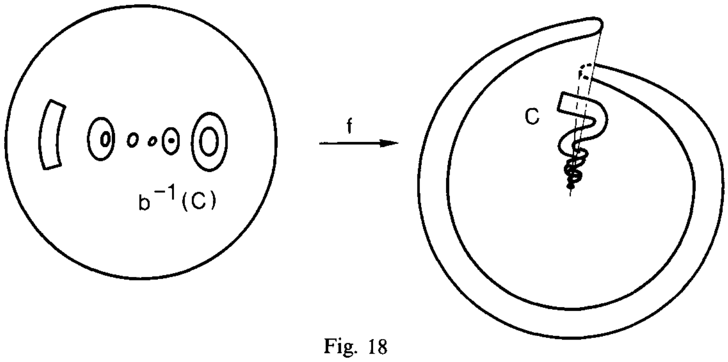

Fig. 18

Proof of Proposition 27. — Replacing T by $T\circ\varphi$, where φ is constructed as in section 2, we may assume that $\mathbb{R}_+\cap Q\subset A$. Then $\mathbb{R}_+\cap\Omega\subset C$, and for $\tau=p/2^k$, setting $\Omega_k=\bar{D}_{R^{1/2^k}}-\bar{D}$, the segment $e^{2\pi i\tau}\mathbb{R}_+\cap\Omega_k$ is contained in a component C_τ of C.

Now $\gamma_M(\tau)$ is accessible by an arc in

$$C_{M,\tau}=\Phi_M^{-1}(C_\tau)$$

and by Lemma 2 there exists a connected component $C_{\Lambda,\tau}$ of $C_\Lambda=\chi^{-1}(C_{M,\tau})$ such that

$$\chi^{-1}(\gamma_M(\tau))\in\bar{C}_{\Lambda,\tau}.$$

Q.E.D.

Remarks. — One can prove that $\bar{C}_{\Lambda,\tau}\cap M_f$ is reduced to $\chi^{-1}(\gamma_M(\tau))$.

(2) If T is horizontally analytic, then χ is quasi-conformal, and then one can prove that for each $\tau=p/2^k$, the component $C_{\Lambda,\tau}$ is unique.

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