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ON THE EFFECT OF A SEARCH UPON THE PROBABILITY DISTRIBUTION OF A TARGET WHOSE MOTION IS A DIFFUSION PROCESS

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1. Summary. We shall assume that the motion of a target in a given region is a diffusion process and that a searcher tries to find the target. If the searcher is able to remember the track he has travelled, then the probability density—from the searcher's point of view—of the target will be changing also because of the search, since the searcher knows that the target was not in the immediate proximity of his track. In the case of a stationary target the search already carried out would be taken into account by a straightforward application of Bayes' formula [4]. We shall assume the customary form for the instantaneous probability density of detection [3] and then modify the diffusion equation of the target to include the above Bayesian effect. The properties of this equation are discussed. An optimal statistical control problem of finding the best way of search is formulated.

2. Motion of the target. We shall consider the target to be a point moving in the n-dimensional space R_n. The track of the target, $\xi(t) \in R_n$, is assumed to satisfy a stochastic differential equation of the form (cf. [1], Chapter 8)

$$d\xi(t) = a(t, \xi(t))\, dt + B(t, \xi(t))\, dw(t), \tag{1}$$

where $w(t)$ is an n-dimensional Wiener process, where $a(t, x)$ is an n-dimensional vector and where $B(t, x)$ is a linear, symmetric, and positive definite mapping of R_n into R_n (in what follows $B(t, x)$ will be called more briefly a dyad). Both $a(t, x)$ and $B(t, x)$ are assumed to be, with their derivatives up to the second order with respect to the $x_1, x_2, \cdots, x_n$, bounded and continuous for all $t \in [0, \infty)$ and for all $x \in R_n$. The motion of the target is now, as is well known (see for instance [3], Chapter 7) a Markov process and there exists the transition probability $P(t, x, s, A)$ —the probability that the target is, at time s, in the Borel set $A \subset R_n$, given that the target was, at moment t at point x—such that

(1) $P(t, x, s, A)$ is, for fixed t, s, and A, where $t < s$, a Borel measurable function of x,

(2) $P(t, x, s, A)$ is, for fixed t, x, and s a measure of the Borel set A,

(3) $\int P(t, x, s, dy)P(s, y, \tau, A) = P(t, x, \tau, A)$ for all x and for all s and t such that $0 \leqq t < s < \tau < \infty$.

(Here and also in the considerations which will follow the integration will be over all R_n if the region of integration in R_n is not specified more closely.)

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(4) (a) $\int_\zeta P(t, x, s, dy) = o(s-t)$, where $\zeta = \{y: |x-y| > \varepsilon > 0\}$
(b) $\int (y-x)P(t, x, s, dy) = a(t, x)(s-t)+o(s-t)$
(c) $\int (y-s);(y-x)P(t, x, s, dy) = B(t, x)(s-t)+o(s-t)$
(d) $\int |y-x|^{\delta+2}P(t, x, s, dy) = o(s-t)$ for $\delta > 0$.

If the transition probability density exists, i.e., if $P(t, x, s, A) = \int_A p(t, x, s, dy)$, then the following Kolmogorov equation holds:

(2) $(\partial/\partial s)p(t, x, s, y) = -\nabla_y \cdot (a(s, y)p(t, x, s, y)) + \frac{1}{2}(\nabla_y;\nabla_y):(B(s, y)p(t, x, s, y))$.[1]

The purpose of our paper is to modify Equation (2) so that the effect of the search upon $p(t, x, s, y)$ is included. This will be done in Section 4.

Let us assume, for a moment, that vector $a(s, y)$ and dyad $B(x, y)$ are constants. Now Equation (2) may be solved by using Fourier transformation in the well-known manner: with

$$(3) \qquad \phi(t, x, s, \omega) = \int \exp(i\omega\cdot y)p(t, x, s, y)\,dy$$

we first obtain from Equation (2) that

$$(4) \qquad (\partial/\partial s)\phi(t, x, s, \omega) = [i(a\cdot\omega) - \tfrac{1}{2}(\omega;\omega):B]\phi(t, x, s, \omega),$$

which yields

$$(5) \qquad \phi(t, x, s, \omega) = \exp\{[ia\cdot\omega - \tfrac{1}{2}\omega;\omega:B](s-t) + i\omega\cdot x\}.$$

Because B is a positive definite dyad the integral

$$(6) \qquad q(\theta, y) = (\tfrac{1}{2}\pi)^n \int \exp(-i\omega\cdot y)\exp[-\tfrac{1}{2}\omega;\omega:B\theta]\,d\omega$$

exists and we obtain from Equations (5) and (6) that

$$(7) \qquad p(t, x, s, y) = q(s-t, y-a(s-t)-x).$$

We shall need this solution of Equation (2) in Section 5.

3. The law of detection for the searcher. We shall assume that the search is of Markovian nature in the following sense: the probability that the searcher who is proceeding along track $z(t)\in R_n$ detects the target during time (t, s), given that the target is at point $x\in R_n$, is

$$\lambda(x, z, t)(s-t)+o(s-t).$$

We shall assume that

(1) $0 \leqq \lambda(x, z, t) \leqq M < \infty$ for all $x, z\in R_n$ and for all $t\in[0, \infty)$

[1] Here and in what follows we employ, in order to simplify the writing, the notation of vector calculus: Let a, b, c and d be vectors in R_n. Then the scalar product of, say, a and b will be written as $a\cdot b$ and their dyadic product as $a;b$. Furthermore, for instance $(a;b)\cdot c = a(b\cdot c)$, $(a;b)\cdot(c;d) = (a;d)(b\cdot c)$ and $(a;b):(c;d) = (a\cdot d)(b\cdot c)$.

and that

$$\text{(2)}\ \lambda(x, z, t) = \lambda(w, z, t)+(x-w)\cdot\nabla_w\lambda(w, z, t)$$

$$+\tfrac{1}{2}(x-w);(x-w):(\nabla_w;\nabla_w)\lambda(w, z, t)+o(|x-w|^2) \tag{8}$$

for all $x, w, z \in R_n$ and for all $t \geqq 0$.

If the target is stationary at point $x \in R_n$ and if the searcher is moving, during time (t, s), along track $z(t)$, then the probability that the target will not be found during time (t, s) is given through the well-known expression [3]

$$\exp\left(-\textstyle\int_t^s \lambda(x, z(\tau), \tau)\, d\tau\right). \tag{9}$$

If the probability distribution of the location of the target in R_n is $G(x)$ then the probability that the target will be found during time (t, s) becomes

$$P\{z\} = 1-\textstyle\int \exp\left(-\int_t^s \lambda(x, z(\tau), \tau)\, d\tau\right) dG(x). \tag{10}$$

4. Effect of the search upon the transition probability.

DEFINITION 1. We shall denote through $P_z(t, x, s, A)$ the probability that the target is, at time s, within set A, given that the target was, at time t, at point $x \in R_n$ and that the search along track $z(t) \in R_n$, during time (t, s), was not successful. The probability density, if this exists, will be written $p_z(t, x, s, y)$.

DEFINITION 2. Let $t \in [0, \infty)$, $A \subset R_n$, and $x \in R_n$. Then we shall denote the event that the target is, at time t, in set A by (t, A). In the case where A reduces to a point $x \in R_n$ we shall write accordingly (t, x). Furthermore, let $0 \leqq t_1 < t_2$ and let $z(t) \in R_n$, where $t \in (t_1, t_2)$. Then we shall denote the event that the search during time (t_1, t_2), along track $z(t)$, was not successful by (t_1, t_2, z). For instance

$$P_z(t, x, s, A) = P\{(s, A)/(t, x)(t, s, z)\}. \tag{11}$$

We shall now introduce two lemmas which will be needed in the proof of Theorem 1.

LEMMA 1. *Let dy denote a set of infinitesimal measure which contains point y. Then*

$$P\{(s+\Delta s, dy)/(t, x)(t, s, z)\} = \textstyle\int P_z(t, x, s, dv)P(s, v, s+\Delta s, dy). \tag{12}$$

The proof is straightforward and it will not be given here.

LEMMA 2.

$$P\{(s, s+\Delta s, z)/(t, x)(t, s, z)(s+\Delta s, y)\} = 1-\lambda(y, z(s), s)\Delta s+o(\Delta s) \tag{13}$$

and

$$P\{(s, s+\Delta s, z)/(t, x)(t, s, z)\} = 1-\Delta s\cdot\textstyle\int\lambda(w, z, s)P_z(t, x, s, dw)+o(\Delta s). \tag{14}$$

PROOF. Equation (13) follows immediately from the properties of the function $\lambda(x, z, t)$ (cf. Section 3). In order to prove Equation (14) we first obtain, with the help of Lemma 1, that

$$\begin{aligned} & P\{(s, s+\Delta s, z)/(t, x)(t, s, z)\} \\ (15) \quad & = \int P\{(s, s+\Delta s, z)/(t, x)(t, s, z)(s+\Delta s, y)\} P\{(s+\Delta s, dy)/(t, x)(t, s, z)\} \\ & = \int [1-\lambda(y, z(s), s)\Delta s + o(\Delta s)] \int P_z(t, x, s, dv) P(s, v, s+\Delta s, dy) \\ & = 1 - \Delta s \cdot \int P_z(t, x, s, dv) \int \lambda(y, z(s), s) P(s, v, s+\Delta s, dy) + o(\Delta s). \end{aligned}$$

Now (cf. Equation (8))

$$\begin{aligned} & \int \lambda(y, z(s), s) P(s, v, s+\Delta s, dy) \\ & \quad = \int [\lambda(v, z(s), s) + (y-v) \cdot \nabla_v \lambda(v, z(s), s)) \\ (16) \quad & \qquad + \tfrac{1}{2}(y-v);(y-v) : \nabla_v ; \nabla_v \lambda(v, z(s), s) \\ & \qquad + o(|y-v|^2)] P(s, v, s+\Delta s, dy) \\ & \quad = \lambda(v, z(s), s) + \Delta s \cdot \{a(s, v) \cdot \nabla_v \lambda(v, z(s), s) \\ & \qquad + \tfrac{1}{2} B(s, w) : \nabla_v ; \nabla_v \lambda(v, z(s), s)\} + o(\Delta s). \end{aligned}$$

Equation (14) follows now from Equations (15) and (16).

THEOREM 1. *Let* $P_z(t, x, s, A) = \int_A p_z(t, x, s, y)\, dy$. *Then*

$$\begin{aligned} & (\partial/\partial s) p_z(t, x, s, w) \\ (17) \quad & = -\nabla_w \cdot [a(s, w) p_z(t, x, s, w)] + \tfrac{1}{2}(\nabla_w ; \nabla_w) : (B(s, w) p_z(t, x, s, w)) \\ & \quad + p_z(t, x, s, w)[\int \lambda(v, z(s), s) p_z(t, x, s, v)\, dv - \lambda(w, z(s), s)], \end{aligned}$$

where vector $a(s, w)$ *and dyad* $B(s, w)$, *defined in Section* 2, *belong to the diffusion process and where the scalar* $\lambda(w, z, t)$ *is the instantaneous probability density of detection, defined in Section* 3.

PROOF. By using the notation of Definition 2 it is seen that

$$\begin{aligned} & P_z(t, x, s+\Delta s, dy) \\ (18) \quad & = P\{(s+\Delta s, dy)/(t, x)(t, s+\Delta s, z)\} = P\{(s+\Delta s, dy)/(t, x)(t, s, z)(s, s+\Delta s, z)\} \\ & = \frac{P\{(s+\Delta s, dy)/(t, x)(t, s, z)\} P\{(s, s+\Delta s, z)/(t, x)(t, s, z)(s+\Delta s, y)\}}{P\{(s, s+\Delta s, z)/(t, x)(t, s, z)\}} \end{aligned}$$

which with the help of Lemmas 1 and 2 yields

$$\begin{aligned} (19) \quad P_z(t, x, s+\Delta s, dy) = & \{1 + \Delta s[\int \lambda(w, z(s), s) P_z(t, x, s, dw) - \lambda(y, z(s), s)] \\ & + o(\Delta s)\} \int P_z(t, x, s, dv) P(s, v, s+\Delta s, dy). \end{aligned}$$

Now let $f(y)$ be a function defined on R_n such that $f(y)$, $\nabla f(y)$, and $\nabla ; \nabla f(y)$ are bounded and continuous for all $y \in R_n$, and such that $f(y) = f(v) + (y-v) \cdot \nabla f(y) + \frac{1}{2}(y-v);(y-v) : \nabla ; \nabla f(v) + o(|y-v|^2)$ for all $y, v \in R_n$. Then, as is clear from the properties of the functions $\lambda(y, z, s)$ and $f(y)$,

$$\int P_z(t, x, s+\Delta s, du) f(u)$$

$$(20) \quad = \int P_z(t, x, s, dv) \int f(y) \{1+\Delta s[\int \lambda(w, z(s), s) P_z(t, x, s, dw) - \lambda(y, z(s), s)] + o(\Delta s)\} P(s, v, s+\Delta s, dy).$$

Therefore

$$(21) \quad \begin{aligned} &\int P_z(t, x, s+\Delta s, du) f(u) - \int P_z(t, x, s, dv) f(v) \\ &\quad = \int P_z(t, x, s, dv) \{\int f(y) P(s, v, s+\Delta s, dy) - f(v)\} \\ &\qquad + \int P_z(t, x, s, dv) \int f(y) \{\Delta s[\int \lambda(w, z(s), s) P_z(t, x, s, dw) \\ &\qquad - \lambda(y, z(s), s)] + o(\Delta s)\} P(s, v, s+\Delta s, dy). \end{aligned}$$

Now, because of the properties of the function $f(y)$ and of the transition probability $P(t, x, s, A)$ (cf. Section 2)

$$(22) \quad \begin{aligned} &\int f(y) P(s, v, s+\Delta s, dy) - f(v) \\ &= \int P(s, v, s+\Delta s, dy)[(y-v)\cdot \nabla f(v) + \tfrac{1}{2}(y-v);(y-v):\nabla;\nabla f(y) + o(|y-v|^2)] \\ &= \Delta s \cdot [a(s, v)\cdot \nabla f(v) + \tfrac{1}{2} B(s, v):\nabla;\nabla f(v)] + o(\Delta s). \end{aligned}$$

In the same way as Equation (22) was obtained it may be easily shown that

$$(23) \quad \int q(y) P(s, v, s+\Delta s, dy) = q(v) + o(1),$$

where we wrote more briefly

$$(24) \quad q(y) = f(y)[\int \lambda(w, z(s), s) P_z(t, x, s, dw) - \lambda(y, z(s), s)].$$

With

$$(25) \quad P_z(t, x, s, dv) = p_z(t, x, s, v)\, dv + o(dv).$$

Equation (17) now follows from Equations (18)–(24).

REMARK. Let $P_z(t, G, s, A)$ denote the probability that the target is, at time s, within set A, given that the target was, at time t, distributed according to the probability distribution $G(x)$ and that the search during time (t, s) was not successful, i.e., let

$$(26) \quad P_z(t, G, s, A) = \int P_z(t, x, s, A)\, dG(x).$$

Then Lemmas 1 and 2 and Theorem 1 are valid with $P_z(t, G, s, A)$ substituted for $P_z(t, x, s, A)$. This is an immediate consequence from the Markovian nature of the motion of the target, as is easily seen from the proofs of the lemmas and of the theorem.

The probability that the target will be found, during the short interval of time $(s, s+\Delta s)$, by the searcher moving along the track $z(t)$, given that the target was, at time t, at point $x \in R_n$, is now

$$\Delta s \int \lambda(v, z(s), s) P_z(t, x, s, dv) + o(\Delta s). \tag{27}$$

The probability that the target will be found by the searcher during time (t, s), given that the target was at point $x \in R_n$ at time t, becomes then

$$P\{z\} = 1 - \exp\{-\int_t^s d\tau \int \lambda(v, z(\tau), \tau) P_z(t, x, \tau, dv)\}. \tag{28}$$

In the case of a stationary target expression (28) reduces to expression (10) (cf. [2]).

5. On the solution of equation (17). Equation (17) for the probability density $p_z(t, x, s, w)$ is a nonlinear integro-differential equation. It turns out, however, that the problem of solving Equation (17) for $p_z(t, x, s, w)$ may be reduced to the solution of a linear partial differential equation. We shall consider the more general case where the location of the target, at time t, is given through the probability distribution $G(x)$ (see the Remark after Theorem 1).

THEOREM 2. *Let $p_z(t, G, s, y)$ be the probability density of the location of the target at time s, given that the probability distribution of the location of the target was, at time t, given through the probability distribution $G(x)$ and that the search during time (t, s) was not successful. Furthermore, let $P\{z\}$ be the probability that the target will be found during time (t, s). Then*

$$p_z(t, G, s, y) = X_z(t, G, s, y)\{\int X_z(t, G, s, v)\, dv\}^{-1} \tag{29}$$

and

$$P\{z\} = 1 - \int X_z(t, G, s, v)\, dv, \tag{30}$$

where function $X_z(t, G, s, v)$ satisfies initial condition $X_z(t, G, t, x) = G(x)$, appropriate boundary conditions and the following partial differential equation

$$\begin{aligned}(\partial/\partial s) X_z(t, G, s, y) &= \tfrac{1}{2}\nabla_y ; \nabla_y : (B(s, y) X_z(t, G, s, y)) \\ &\quad - \nabla_y \cdot [a(s, y) X_z(t, G, s, y)] - \lambda(y, z(s), s) X_z(t, G, s, y).\end{aligned} \tag{31}$$

If $a(s, w)$ and $B(s, w)$ are constants and if $q(t, x)$ defined by (7) *is such that*

$$\sup_{s \in [t, \infty)}, \sup_{y \in R_n} \int_t^s d\tau \int q(s-\tau, y - \xi - a(s-\tau)) \lambda(\xi, z(\tau), \tau)\, d\xi < 1, \tag{32}$$

then Equation (31) *has a unique solution which is, for all probability distributions $G(x)$ and for all t and τ, such that $0 < t < \tau$ and for all $y \in R_n$ a bounded and continuous function of s and y.*

PROOF. Equation (31) is immediately obtained from Equations (17) through substitution

$$p_z(t, x, s, w) = X_z(t, x, s, w) \exp\{\int_t^s d\tau \int \lambda(v, z(\tau), \tau) p_z(t, x, \tau, v)\, dv\}. \tag{33}$$

Since $\int p_z(t, x, s, w)\, dw = 1$, it follows from (33) that

$$\int X_z(t, x, s, w)\, dw = \exp\{-\int_t^s d\tau \int \lambda(v, z(s), s) p_z(t, x, s, v)\, dv\}, \tag{34}$$

which together with (28) implies (29) and (30). Now let

$$\phi_z(t, G, s, \omega) = \int \exp(i\omega \cdot y) X_z(t, G, s, y)\, dy \tag{35}$$

so that

$$X_z(t, G, s, y) = (\tfrac{1}{2}\pi)^n \int \exp(-i\omega \cdot y) \phi_z(t, G, s, \omega)\, d\omega, \tag{36}$$

and let $a(s, y)$ and $B(s, y)$ be constants. We then obtain from Equation (31) that

$$(\partial/\partial s)\phi_z(t, G, s, \omega) = (ia \cdot \omega - \tfrac{1}{2}\omega; \omega : B)\phi_z(t, G, s, \omega) - \int \exp(i\omega \cdot y)\lambda(y, z(s), s) X_z(t, G, s, y)\, dy. \tag{37}$$

It now follows from Equation (37), since $X_z(t, G, t, y) = p_z(t, G, t, y) = G(y)$, that

$$\phi_z(t, G, s, \omega) = \exp[(ia \cdot \omega - \tfrac{1}{2}\omega; \omega : B)(s-t)] \cdot \{C(\omega) - \int_t^s \exp[-(ia \cdot \omega - \tfrac{1}{2}\omega; \omega : B)(\tau - t)] \cdot \{\int \exp(i\omega \cdot y)\lambda(y, z(\tau), \tau) X_z(t, G, \tau, y)\, dy\}\, d\tau\}, \tag{38}$$

where

$$C(\omega) = \int \exp(i\omega \cdot y)\, dG(y). \tag{39}$$

Equation (38) implies the following integral equation for $X_z(t, G, s, y)$:

$$X_z(t, G, s, y) = \int q(s-t, y-\xi-a(s-t))\, dG(\xi) - \int_t^s d\tau \int d\eta\, q(s-\tau, y-\eta-a(s-\tau))\lambda(\eta, z(\tau), \tau) X_z(t, G, \tau, \eta)\, d\eta. \tag{40}$$

Now let C_t be the space of real functions $f(\tau, x)$ which are, for all $\tau \in (t, \infty)$ and for all $x \in R_n$, bounded and continuous functions of τ and x. With the norm defined as $||f|| = \sup_{\tau \in [t,\infty)} \sup_{x \in R_n} |f(\tau, x)|$ space C_t becomes a Banach space. Obviously

$$\int q(s-t, y-\xi-a(s-t))\, dG(\xi) \in C_t.$$

Furthermore, by writing

$$Af = \int_t^s d\tau \int q(s-\tau, y-\xi-a(s-\tau))\lambda(\xi, z(\tau), \tau) f(\tau, \xi)\, d\xi \tag{41}$$

it is immediately seen that

$$||Af|| \leqq ||A||\, ||f|| \tag{42}$$

where

$$||A|| = \sup_{s \in [t,\infty)} \sup_{y \in R_n} \int_t^s d\tau \int q(s-t, y-\xi-a(s-\tau)\lambda(\xi, z(\tau), \tau)\, d\xi \tag{43}$$

and that Af is a continuous function of s and y for all $s \in [t, \infty)$ and for all $y \in R_n$. Therefore, $f \in C_t$ implies $Af \in C_t$. Now, because C_t is a Banach space and because $||A|| < 1$, the equation $x = f - Ax$ has for every $f \in C_t$, as is well known, a unique solution $x \in C_t$. This solution may be obtained as the limit of a sequence $\{x_n\}$ such that $x_{n+1} = f - Ax_n$, where $x_0 \in C_t$ is arbitrary.

REMARK 1. Equation (31) may be solved in a closed form in the important special case where

$$\lambda(y, z, t) = \lambda_0 \quad = \text{const.} \quad \text{for} \quad y \in \hat{R}, \qquad = 0 \quad \text{for} \quad y \notin \hat{R} \tag{44}$$

where $\hat{R}$ is a region of finite measure in R_n. Now

$$X_z(t, G, s, y) = \exp(-\lambda_0(s-t)) \int q(s-t, y-\xi-a(s-t))\, dG(\xi) \quad \text{for} \quad y \in \hat{R}, \qquad = \int q(s-t, y-\xi-a(s-t))\, dG(\xi) \quad \text{for} \quad y \notin \hat{R}. \tag{45}$$

The probability density $p_z(t, G, s, y)$ would then be obtained from Equation (29).

REMARK 2. When the search for the target is to be optimized we have the following optimal control problem to solve: let us assume that the motion of the searcher is described by the system of ordinary differential equations

$$(d/dt)_z^{\,i} = f^i(z^1, z^2, \cdots, z^n; v^1, \cdots, v^r), \tag{46}$$

$(i=1, 2, \cdots, n)$, where the $v^1, \cdots, v^r$ are the control parameters. To be found is the control vector $(v^1, \cdots, v^r)$ at each time $\tau \in [t, s]$ such that

$$P\{z\} = 1 - \int X_z(t, G, s, \xi)\, d\xi = \max, \tag{47}$$

where the function $X_z(t, G, \tau, y)$ satisfies the differential equation

$$(\partial/\partial s) X_z(t, G, s, y) = \tfrac{1}{2}\nabla_y; \nabla_y : (B(s, y) X_z(t, G, s, y)) - \nabla_y \cdot [a(s, y) X_z(t, G, s, y)] - \lambda(y, z(s), s) X_z(t, G, s, y) \tag{48}$$

and appropriate initial and boundary conditions.

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