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On the Effective Hamiltonian for Pseudoscalar Meson with Pseudoscalar Coupling with Nucleon

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The method of canonical transformation is applied to the pseudo-scalar meson theory with pseudo-scalar coupling. It will be shown that the Tani[1]-Foldy[2] transformation results as the best one when regarded as a variational function among the similar family of transformation. The resulting Hamiltonian was ordered assuming some cut-off for the virtual meson momentum, and performing the mass renormalization. This Hamiltonian shows a good tendency to account for the S-phase shifts for the nucleon π-meson scattering; the L-S coupling $(\tau\cdot\varphi\times\pi)$ in the isotopic spin space, which is positive for the $T=3/2$ state and negative for the $T=1/2$ state, becomes twice as large as obtained by the perturbation theory.

§ 1. Construction of effective Hamiltonian

The Hamiltonian of the system composed of pseudo-scalar mesons interacting through pseudo-scalar coupling with the nucleon system is given by ;[4]

$$H_T = H_0^M + H_0^N + H_1, \tag{1·1}$$

$$H_0^M = (1/2)\int(\pi^2 + \phi(\mu^2 - \Delta)\phi)dx, \tag{1·2}$$

$$H_0^N = \int\psi^*(-i\rho_1\,\sigma\nabla + \rho_3 m)\psi\,dx, \tag{1·3}$$

$$H_1 = g\int\psi^*\,\rho_2(\tau\,\phi)\,\psi\,dx. \tag{1·4}$$

Here we transform the state functional as following, with the unknown function f[5];

$$\Psi = \exp[-i\int\psi^*\,\rho_1(\tau\phi)f(\sqrt{\phi^2})\psi\,dx]\Psi_1 = e^{-G(f)}\,\Psi_1, \tag{1·5}$$

We now determine the form f by requiring that, when (1·5) is regarded as a variational function with f to be determined (Ψ_1 is then replaced by one of the eigen state $(H_0^M + H_0^N)\Psi_0 = E_0\,\Psi_0$), the energy expectation value is to be stationary.

For this purpose, we first construct the transformed Hamiltonian ;

$$e^{G(f)}(H_0^M + H_0^N + H_1)e^{-G(f)}$$

*) This work was prepared for the International Conference on Theoretical Physics held in Japan (September 1953.)

$$=1/2\int(\pi^2+\phi(\mu^2-\Delta)\phi)dx+\int\psi^*(-i)\rho_1(\sigma\nabla)\psi\,dx+\int\psi^*\rho_3A_m(f)\psi\,dx.$$
$$+\int\psi^*(\rho_1\pi+(\sigma\nabla)\phi)(\phi(\tau\phi)A(f)+\tau A_{pv}(f)+(\tau\times\phi)\rho_1A_\pi(f))\psi dx$$
$$+\int\psi^*\rho_2A_{ps}(f)(\tau\phi)\,\psi\,dx$$
$$+(1/2)\int[\psi^*\rho_1(\phi(\tau\phi)A(f)+\tau A_{pv}(f)+(\tau\times\phi)\rho_1A_\pi(f))\psi]^2\,dx, \qquad (1\cdot6)^{6)}$$

where

$$A(f)=-\left(\frac{f'}{(\phi^2)^{1/2}}+\frac{f}{\phi^2}-\frac{\sin(2(\phi^2)^{1/2}f)}{2(\phi^2)^{3/2}}\right), \qquad (1\cdot7)$$

$$A_{pv}(f)=-\frac{\sin(2(\phi^2)^{1/2}f)}{2(\phi^2)^{1/2}}, \qquad (1\cdot8)$$

$$A_\pi(f)=\frac{1-\cos(2(\phi^2)^{1/2}f)}{2\phi^2}, \qquad (1\cdot9)$$

$$A_m(f)=m\cos(2(\phi^2)^{1/2}f)-g(\phi^2)^{1/2}\sin(2(\phi^2)f), \qquad (1\cdot10)$$

$$A_{ps}(f)=\frac{m}{(\phi^2)^{1/2}}\sin(2(\phi^2)^{1/2}f)+g\cos(2(\phi^2)^{1/2}f). \qquad (1\cdot11)$$

Then the variation in the energy expectation value with respect to f (functional variation)

$$\delta\,`E\,'=\delta(\Psi_0{}^*\cdot e^G(H_0{}^M+H_0{}^N+H_1)e^{-G}\Psi_0), \qquad (1\cdot12)$$

comes from the third term of $(1\cdot6)$ and from the last term, because the first and second term is independent of the f and others are odd in meson field variable.

However, the last term is, as well known, related to the derivative coupling (the 4-th term in the right hand side of $(1\cdot6)$), and represents normal dependent term. So, we put this term out of consideration, because they must be considered in combination with 4-th term (pseudo-vector term).

Then, we have as a stationary expression for the energy,

$$\frac{\delta}{\delta f}(\Psi_0{}^*\cdot\int\psi^*\rho_3A_m(f)\psi\,dx\,\Psi_0). \qquad (1\cdot13)$$

To evaluate this expression, it is necessary to take the vacuum expectation value of $A_m(f)$. This can be given as follows;

$$\langle A_m(f)\rangle_0=\frac{1}{(2\pi)^3}\int\Big[\int(m\cos(2xf(x))-gx\sin(2xf(x)))e^{-iux}dx\Big]\cdot\langle e^{iu\phi}\rangle_0du$$

$$=\frac{1}{(2\pi)^3}\int\Big[\int(m\cos(2xf(x))-gx\sin(2xf(x))e^{-iux}dx\Big]e^{-(\langle\phi^2\rangle_0/2)u^2}\,du. \qquad (1\cdot14)$$

So that the variation with respect to $f(x)$ gives;

$$\frac{\delta}{\delta f}\langle A_m(f)\rangle_0\infty-2xm\sin(2xf(x))-2gx^2\cos(2xf(x))=0, \qquad (1\cdot15)$$

namely,

$$f(x)=-\frac{1}{2x}\tan^{-1}\left(\frac{g}{m}x\right), \tag{1·16}$$

$$f((\phi^2)^{1/2})=-\frac{1}{2(\phi^2)^{1/2}}\tan^{-1}\left(\frac{g}{m}(\phi^2)^{1/2}\right), \tag{1·17}$$

and is just the transformation function due to Foldy.[2]

Thus, the Tani-Foldy transformation is proved to be the best one (under the restriction mentioned above (1·13)) and in this case we have for A's

$$A=\frac{1}{2m\phi^2}\left(\frac{1}{1+(g/m)^2\phi^2}-\frac{1}{\sqrt{1+(g/m)^2\phi^2}}\right), \tag{1·18}$$

$$A_{pv}=\frac{g}{2m}\,\frac{1}{\sqrt{1+(g/m)^2\phi^2}}, \tag{1·19}$$

$$A_{\pi}=\frac{1}{2\phi^2}\left(1-\frac{1}{\sqrt{1+(g/m)^2\phi^2}}\right), \tag{1·20}$$

$$A_m=m\sqrt{1+(g/m)^2\phi^2}, \tag{1·21}$$

$$A_{ps}=0. \tag{1·22}$$

§ 2. Ordering of the Hamiltonian

Ordering of the A's are performed by first making their Fourier analysis, e.g., the functional Fourier analysis with respect to meson variable ϕ, and then ordering the exponential function. The following formulae are sufficient for further calculations;

$$f((\phi^2)^{1/2})=\frac{2}{(2\pi)^{1/2}}\int_0^\infty f(\langle\phi^2\rangle_0^{1/2}y)\,e^{-y^2/2}\cdot\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}y^{2n+2}\left(\frac{\bar{\phi}^2}{\langle\phi^2\rangle_0}\right)^n e^{-(\bar{\phi}^2/2\langle\phi^2\rangle_0)}dy, \tag{2·1}$$

$$(\phi e)f((\phi^2)^{1/2})=(e\phi)\frac{2}{(2\pi)^{1/2}}\int_0^\infty f(\langle\phi^2\rangle_0^{1/2}y)e^{-y^2/2}\sum_{n=0}^{\infty}\left(\frac{1}{2n!}-\frac{1}{(2n+1)!}\right)y^{2m+2}\times$$
$$\times\left(\frac{\bar{\phi}^2}{\langle\phi^2\rangle_0}\right)^{n-1}e^{-(\bar{\phi}^2/2\langle\phi^2\rangle_0)}\,dy, \tag{2·2}$$

$$(\phi e_1)(\phi e_2)f((\phi^2)^{1/2})=\frac{2}{(2\pi)^{1/2}}\langle\phi^2\rangle_0\Bigg[(e_1e_2)\sum_{n=0}^{\infty}\left(\frac{1}{2n!}-\frac{1}{(2n+1)!}\right)\times$$
$$\times\left(\frac{\bar{\phi}^2}{\langle\phi^2\rangle_0}\right)^{n-1}\int_0^\infty f(\langle\phi^2\rangle_0^{1/2}y)e^{-y^2/2}y^{2n+2}\,dy+\frac{(e_1\bar{\phi})(e_2\bar{\phi})}{\bar{\phi}^2}\sum_{n=0}^{\infty}\left(-\frac{3}{(2n+2)!}+\frac{1}{(2n+1)!}\right.$$
$$\left.+\frac{3}{(2n+3)!}\right)\left(\frac{\bar{\phi}^2}{\langle\phi^2\rangle_0}\right)^n\cdot\int_0^\infty f(\langle\phi^2\rangle_0^{1/2}y)e^{-y^2/2}y^{2n+4}\,dy.\Bigg]\,e^{-(\bar{\phi}^2/2\langle\phi^2\rangle_0)}, \tag{2·3}$$

(where $\bar{\phi}$ means the ordered operators).[7]

After the ordering, the effective Hamiltonian becomes;

$$
\begin{aligned}
{}^{\backprime}H_T{}' = & H_0^M + \int \psi^* (-i\,\rho_1\,\sigma\nabla + A\cdot m\,\rho_3)\psi\, dx \\
& + \int \psi^* \rho_3 \left(\frac{B}{m}\,\bar{\phi}^2 + \frac{C}{m^3}(\bar{\phi}^2)^2 + \cdots \right)\psi\, dx + \int \psi^* D \frac{(\bar{\pi}\times\tau\cdot\bar{\phi})}{m^2}\,\psi\, dx + \cdots \\
& + \int \psi^* E \frac{1}{m}(\rho_1\,\bar{\pi} + (\sigma\nabla)\bar{\phi})\cdot\tau\,\psi\, dx + \cdots.
\end{aligned} \tag{2·4}
$$

The evaluation of the numerical coefficients are performed by putting the cutoff momentum at about nucleon mass m.

$g^2/4\pi$	A	B	C	D	E
4π	2.559	27.61	−91.49	7.082	2.543
6π	2.985	34.18	−119.35	8.049	2.545
8π	3.389	40.04	−143.55	8.598	2.566

Table I. Numerical coefficients appearing in the equation (2.4). The cut-off momentum is at nucleon rest mass m.

By using the abbreviation $mA = m^*$,

$$
\begin{aligned}
{}^{\backprime}H_T{}' = & H_0^M + \int \psi^* (-i\rho_a(\sigma\nabla) + m^*\rho^3)\psi dx \\
& + \int \psi^* \rho_3 \left(\frac{g^2}{2m^*} B' \bar{\phi}^2 + \frac{g^4}{8m^{*3}}\, C'\, (\bar{\phi}^2)^2 + \cdots \right)\psi dx \\
& + \frac{g^2}{4m^{*2}} D' \int \psi^* (\bar{\pi}\times\tau\cdot\bar{\phi})\,\psi dx + \cdots \\
& + \frac{g}{2m^*} E' \int \psi (\rho_1\,\bar{\pi} + (\sigma\nabla)\bar{\phi})\,\tau\psi dx + \cdots.
\end{aligned} \tag{2·5}
$$

$g^2/4\pi$	B'	C'	D'	E'
4π	0.895	−0.492	1.175	1.036
6π	0.862	−0.453	1.211	0.995
8π	0.859	−0.446	1.251	0.979
	$8/3\pi$ $=0.849$	$-64/15\pi^2$ $=0.432$	$16\|3\pi$ $=1.70$	$8/3\pi$ $=0.849$

Table II. The numerical coefficients appearing in equation (2·5). The figures are the relative ratio with the coefficients which is obtained by expanding (1.18)-(1.22), and if one does not take into account the ordering, the lowest order term gives Hamiltonian (2·5) with all coefficients 1. The last column is obtained by assuming $\frac{g^2}{m^2}\langle\phi^2\rangle_0\rangle\rangle 1$ (strong limit) with $m^* = \frac{4g\sqrt{\langle\phi^2\rangle_0}}{(2\pi)^{1/2}}$ and in this case the result is not dependent on the cut-off.

It will be seen from Table II that the core term and pseudo-vector coupling reduces slightly but the L–S coupling term $(\tau\cdot\phi\times\pi)$ is increased by a factor two in the strong limit. This is a very interesting result in order to explain the S-wave phase shift in the pion-nucleon scattering.

§ 3. Discussion

The Tani-Foldy transformation function was found to be the best one in the similar classes of transformations which preserves the relativistic invariance of the transformed Hamiltonian. But it is to be shown whether the remaining interaction of the Hamiltonian thus derived is small or not. As to this point we cannot say anything at present, but we may expect that the main features of the low-momentum meson reactions are well established in our effective Hamiltonian. The discussion on this point will be more carefully given in a separate paper.

Of course, if it is found that the higher powers of the transformed Hamiltonian is not so weak, we should replace the transformation functions by the more general ones; what we found is that the Foldy type of transformation is the best among the similar type of transformations.

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References

1) S. Tani, Prog. Theor. Phys. **6** (1950), 267.

2) L. L. Foldy, Wouthuysen, Phys. Rev. **78** (1950), 29.

3) The similar effective Hamiltonian for the Dyson transformation was obtained by S. D. Drell and E. M. Henley, Phys. Rev. **88** (1952), 1053.

4) We take natural units, $c=1$, $\hbar=1$.

5) Here we take, for simplicity, as arguments of the function f, the form $(\phi^2)^{1/2}$.

6) The expression $\pi\cdot f(\phi)$ means $\frac{1}{2}\{\pi, f(\phi)\}_+$ and its vacuum expectation value $\langle\frac{1}{2}\{\pi, f(\phi)\}_+\rangle=0$.

7) $\langle\phi^2\rangle_0$ means $\langle\phi_i^2\rangle_0$ with i any one of 1, 2, 3, $\bar{\phi}^2=\bar{\phi}_1^2+\bar{\phi}_2^2+\bar{\phi}_3^2$..