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ON THE EFFICIENT COMPUTATION OF RECURRENCE RELATIONS

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A new parallel algorithm for the solution of a general linear recurrence is described. Its relation to the work of Kogge and Stone is discussed.		

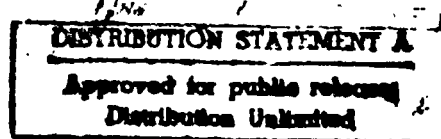
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## ON THE EFFICIENT COMPUTATION OF RECURRENCE RELATIONS

Recently much progress has been made in the formulation of parallel algorithms which compute the terms of a sequence  $(y_i)$  defined by

$$(1) \quad \begin{aligned} y_0 & \text{ given,} \\ y_i & = f_i(y_0, y_1, \dots, y_{i-1}), \quad i = 1, \dots, N. \end{aligned}$$

The germinal point of this work is the now well-known "log-sum" algorithm which computes  $\sum_{i=1}^N a_i$  in  $\lceil \log_2 N \rceil$  parallel addition steps, given  $\lceil N/2 \rceil$  processors. Here the underlying recurrence is

$$\begin{aligned} y_0 & = 0 \\ y_i & = y_{i-1} + a_i, \quad i = 1, \dots, N; \end{aligned}$$

$y_N$  is the desired result.

Two apparently distinct generalizations of the log-sum algorithm have appeared. Kogge and Stone [1] have considered the case

$$(2) \quad \begin{aligned} y_0 & = b_0 \\ y_i & = f(b_i, g(a_i, y_{i-1})), \quad i = 1, \dots, N, \end{aligned}$$

where  $f$  is associative,  $g$  distributes over  $f$ , and there is a function  $h$  such that  $g(x, g(y, z)) = g(h(x, y), z)$ . Seemingly restricted to first order recurrences, by a suitable mapping  $m^{\text{th}}$  order recurrences are also treated.

Heller [2] has studied the case

$$(3) \quad \begin{aligned} y_0 & = h_0 \\ y_i & = \sum_{j=0}^{i-1} a_{ij} y_j + h_i, \quad i = 1, \dots, N. \end{aligned}$$

This problem is equivalent to the solution of a lower triangular linear system of equations. In this note we give an improved parallel algorithm for (3) and show a relationship between the two generalizations.

Rewrite (3) as  $(I-L)y=h$ , where  $L$  is a strictly lower triangular matrix, and  $I$  is the identity.  $y$  and  $h$  are  $(N+1)$ -vectors. Since  $L^{N+1} = 0$ ,

$$\begin{aligned}(I-L)^{-1} &= (I+L+L^2+\dots+L^N) \\ &= (I+L^{2^m})(I+L^{2^{m-1}})\dots(I+L)\end{aligned}$$

where  $2^m \leq N < 2^{m+1}$ . Thus we have the algorithm:

```
{x0 ← h; LI ← I};
for i ← 0 step 1 until m-1 do
  {xi+1 ← (I+L2i) xi;
   L2i+1 ← L2i L2i;
   LI ← (I+L2i) LI};
{y ← (I+L2m) xm; LI ← (I+L2m) LI};
```

The algorithm is sequential in  $i$ , and within the braces operations are performed concurrently. When completed, we have the desired  $y$ , and  $(I-L)^{-1}$  is stored in  $LI$ .  $LI$  may now be used to compute  $y'$  given  $h'$ . It is easily shown that, with  $O(N^3)$  processors, the calculation may be done in  $m^2 + 3m + 1$  parallel steps of addition and multiplication. (We use the fact that matrix products may be computed in logarithmic time with sufficiently many processors.) The previous result required  $O(N^4)$  processors and  $m^2 + 4m + 2$  operation steps.

We now turn to the Kogge - Stone results. Rewrite (2) as

$$(2') \quad \begin{aligned}y_0 &= b_0 \\ y_i &= a_i \otimes y_{i-1} \oplus b_i, \quad i=1, \dots, N.\end{aligned}$$

Here  $g$  is replaced by the binary operation  $\otimes$ , and  $f$  by  $\oplus$ . Assume that  $\otimes$  is associative,  $\oplus$  distributes over  $\otimes$ , and there is a  $\otimes'$  such that  $a \otimes (b \otimes c) = (a \otimes' b) \otimes c$ . Let  $\alpha$  be a symbol distinct from all others, and define  $\alpha \otimes x = x \otimes \alpha = x$ ,  $\alpha \oplus x = x \oplus \alpha = \alpha$  for all  $x$ . Define an

operator  $L$  on  $(N+1)$ -vectors by

$$(Lz)_0 = \alpha$$

$$(Lz)_i = a_i \oplus z_{i-1}, \quad i = 1, \dots, N.$$

Then  $y = Ly \oplus b$  by (2'). It is observed that  $L$  is an additive operator since  $\oplus$  distributes over  $\oplus$  and by the definition of  $\alpha$ . Moreover,  $L^{N+1} = \alpha$ , since for any  $z$ , and  $i = 1, \dots, N+1$ ,  $(L^i z)_0 = \alpha$  and for  $1 \leq j < i$ ,  $(L^i z)_j =$

$$(L(L^{i-1} z))_j = a_j \oplus (L^{i-1} z)_{j-1} = a_j \oplus \alpha = \alpha. \quad \text{Therefore,}$$

$$\begin{aligned} y &= Ly \oplus b = L(Ly \oplus b) \oplus b = L^2 y \oplus (L \oplus I)b \\ &= \dots = L^{N+1} y \oplus (L^N \oplus L^{N-1} \oplus \dots \oplus I)b \\ &= (L^N \oplus L^{N-1} \oplus \dots \oplus I)b \\ &= (L^{2^m} \oplus I)(L^{2^{m-1}} \oplus I) \dots (L \oplus I)b. \end{aligned}$$

Since  $L^3 = (L^2)L = L(L^2)$ ,  $\oplus'$  behaves as an associative operation, and so

$$\begin{aligned} (L^{2^i} y)_j &= \alpha, \quad 0 \leq j < 2^i \\ &= a_j \oplus (a_{j-1} \oplus (\dots \oplus (a_{j-2^{i+1}} \oplus y_{j-2^i}) \dots)) \\ &= (a_j \oplus' a_{j-1} \oplus' \dots \oplus' a_{j-2^{i+1}}) \oplus y_{j-2^i}, \quad 2^i \leq j \leq N. \end{aligned}$$

Similarly,

$$\begin{aligned} (L^{2^{i+1}} y)_j &= \alpha, \quad 0 \leq j < 2^{i+1} \\ &= [(a_j \oplus' \dots \oplus' a_{j-2^i+1}) \\ &\quad \oplus' (a_{j-2^i} \oplus' \dots \oplus' a_{j-2^{i+1}+1})] \oplus y_{j-2^{i+1}}, \quad 2^{i+1} \leq j \leq N, \end{aligned}$$

and the "coefficients" of  $L^{2^{i+1}}$  may be computed from the "coefficients" of  $L^{2^i}$  in one  $\oplus'$  operation step. Thus an algorithm similar to the previous one may be used to compute  $y$ . If the operator  $(L^N \oplus \dots \oplus I)$  is not formed, the computation time is  $O(\log_2 N)$  with  $O(N)$  processors. In fact, if  $y' = Ly' \oplus b'$ , it is less efficient to directly apply  $(L^N \oplus \dots \oplus I)$  than to use the above method.

The general recurrence (1) may be written as  $y = L_1 y$ , where  $L_1$  is a strictly lower triangular operator in the sense that, for any  $z$ ,  $(L_1 z)_i$  is independent of  $z_i, z_{i+1}, \dots, z_N$ . By an induction argument  $L_1^{N+1}$  is a constant operator, and so the solution may be found by  $y = L_1^{N+1} z$  for any  $z$ . The special cases (2) and (3) allow the simple computation of the powers of  $L_1$  when  $L_1 z = Lz \oplus b$ , and  $L$  is linear. Kung [3] has shown that for non-linear recurrences, it is not possible, in general, to decrease the computation time by more than a constant factor by the use of parallelism.

#### REFERENCES

1. Kogge, P. M. and Stone, H. S., "A Parallel Algorithm for the Efficient Solution of a General Class of Recurrence Equations." IEEE Trans. on Computers, C-22 (1973), pp. 786-793.
2. Heller, D., "A Determinant Theorem with Applications to Parallel Algorithms." To appear, SIAM Journal Num. Anal., 1974.
3. Kung, H. T., "New Algorithms and Lower Bounds for the Parallel Evaluation of Certain Rational Expressions." Proceedings, ACM Symposium on Theory of Computing, 1974.