On the Eigenvalue Distribution of Correlated MIMO Channels by Character Expansion of Groups

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***Abstract*—Multiple-input multiple-output (MIMO) channels have been studied from various aspects including the average of the mutual information between the transmitter and receiver (ergodic capacity) when the channel gains are known to the receiver only. A common approach for capacity analysis is to find the moment generating function (MGF) of the mutual information and by direct differentiation, the mean of the mutual information (capacity) is calculated. Recently, character expansions of groups have been used for integration over unitary matrices to obtain the joint eigenvalue distribution of the correlated Wishart matrix i.e. $\mathbf{HH}^*$ where H is the zero mean full correlated complex Gaussian random MIMO channel matrix. In this paper, we show that the previous attempt for capacity analysis of full correlated MIMO channels is correct for square channel matrices only. We modify the approach from square matrices to rectangular matrices to obtain the correct joint eigenvalue distribution of the correlated Wishart matrix. The result can be used to obtain the MGF and the capacity of full correlated MIMO channels.**

I. Introduction

Multiple-input multiple-output (MIMO) wireless systems, created by deploying antenna arrays at both the transmitter and receiver, promise high data rates and reliability [1], [2]. MIMO systems have been investigated from a variety of aspects including the ergodic capacity [1] and the outage probability [3], by exact or asymptotic analysis [4], [5]. In the case of exact analysis, several results regarding the distribution of the channel matrix have been presented in the literature. In an independent and identically distributed (i.i.d.) Rayleigh fading environment, the capacity of a MIMO system with N_t transmit antennas and N_r receive antennas scales almost linearly with $\min(N_t, N_r)$ in the high signal-to-noise ratio (SNR) regime [1].

A common method used in the literature to analyze the capacity of a MIMO channel uses the moment generating function (MGF) of the mutual information between the transmitter and the receiver. By differentiating the MGF, the moments can be generated and the first moment yields the capacity. Moreover, the probability of outage can be derived through a simple numerical integral [6]. The MGF of outage mutual information for uncorrelated MIMO channels is presented in [3], and the mutual information of Rician MIMO channels can be found in [7], and in [8] and [9] for semi-correlated MIMO channels. In particular, [8] deals with the case where the number of correlated antennas is less than or equal to the number of uncorrelated antennas and [9] with the opposite case. These works are based on the available results in the theory of Wishart random matrices [10], [11], i.e. $\mathbf{HH}^*$ where $\mathbf{H}$ is the complex Gaussian random channel matrix of a MIMO system.

Recently, the character expansion method, introduced by Balantekin [12], has been exploited in [13] to calculate the joint pdf of the eigenvalues of a correlated Wishart matrix for capacity analysis of full correlated MIMO channels. In [12], the character expansions are used for integration over unitary matrices where the coefficient matrices are nonzero-determinant square matrices. However, when the channel matrix is not a square matrix, the integrations are over unitary matrices with rectangular coefficient matrices. To handle the integrations with non-square coefficient matrices, a framework is proposed in [13]. We believe that the integration steps presented in [13] are not accurate, which result in incorrect joint eigenvalue distributions. In this paper, after a brief introduction to the character expansion of groups, we present a *modified framework* to use the character expansions for integrations over unitary matrices. We show that by using the modified framework, one can easily calculate the integrations involving general *rectangular* complex coefficient matrices appearing in the integrand. Our result is a generalization of previous classical integral over unitary matrices so that the result is not restricted to diagonal and/or real matrices. The joint pdf of the eigenvalues can be used to derive the MGF of the mutual information and capacity of the full correlated MIMO channels.

II. System Model and Capacity

Consider a narrow-band, flat-fading communication system with N_t transmit and N_r receive antennas (MIMO(N_t,N_r)). The input-output relationship between the transmit and receive antenna signals can be modeled as

$$\mathbf{x} = \sqrt{\rho}\,\mathbf{Hs} + \mathbf{n} \tag{1}$$

where $\mathbf{x} \in \mathcal{C}^{N_r}$ is the complex received vector, $\mathbf{s} \in \mathcal{C}^{N_t}$ is the transmitted vector, $\mathbf{n} \in \mathcal{C}^{N_r}$ is the additive noise and $\mathbf{H} \in \mathcal{C}^{N_r \times N_t}$ is the channel matrix. To obtain the capacity, we assume the entries of both vectors $\mathbf{s}$ and $\mathbf{n}$ are independent and identically distributed (i.i.d.) complex Gaussian random

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variables with zero mean and variance one, $\mathcal{CN}(0,1)$. Thus, $\mathrm{E}\{\mathbf{s}\mathbf{s}^*\} = \mathbf{I}$ where $\mathrm{E}\{\cdot\}$ and $(\cdot)^*$ denote the expectation and Hermitian (transpose conjugate) and $\mathbf{I}$ is the identity matrix, and ρ will be the average transmitted power at each signaling interval from each antenna. A fully correlated MIMO channel is modeled as $\mathbf{H} = \mathbf{R}^{\frac{1}{2}}\mathbf{H}_w\mathbf{T}^{\frac{1}{2}}$ where $\mathbf{R}$ and $\mathbf{T}$ are positive definite covariance matrices that represent the channel correlations at the receiver and the transmitter respectively, and entries of $\mathbf{H}_w$ are i.i.d. $\mathcal{CN}(0,1)$ (standard Rayleigh).

Assuming that the channel matrix is known to the receiver only, the mutual information between the transmitter and the receiver is obtained by

$$\mathcal{I} = \log\left(\det\left[\mathbf{I} + \rho\mathbf{H}\mathbf{H}^*\right]\right) \quad \text{nats/s/Hz}, \tag{2}$$

where $\log(\cdot)$ denotes the natural logarithm. By defining the MGF of $\mathcal{I}$ as

$$g(z) = \mathrm{E}_{\mathbf{H}}\{e^{z\mathcal{I}}\} = \mathrm{E}_{\mathbf{H}}\{\det\left[\mathbf{I} + \rho\mathbf{H}\mathbf{H}^*\right]^z\}, \tag{3}$$

the capacity of the system is obtained by direct differentiation:

$$C = \mathrm{E}_{\mathbf{H}}\{\mathcal{I}\} = g'(0). \tag{4}$$

From (3), it is clear that the MGF can be written simply in terms of the eigenvalues λ_i of the matrix $\mathbf{H}\mathbf{H}^*$ as

$$\begin{aligned} g(z) &= \mathrm{E}_{\boldsymbol{\lambda}}\{\prod_{i=1}^{M}\left(1+\rho\lambda_i\right)^z\} \\ &= \prod_{i=1}^{M}\int_0^\infty d\lambda_i \left(1+\rho\lambda_i\right)^z P(\{\lambda_i\}) \end{aligned} \tag{5}$$

where we define $M = \min\{N_t, N_r\}$ and $N = \max\{N_t, N_r\}$, and $P(\{\lambda_i\})$ is the joint pdf of the eigenvalues of $\mathbf{H}\mathbf{H}^*$. Assuming the singular value decomposition of $\mathbf{H}$ as $\mathbf{H} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^*$ where $\mathbf{U} \in \mathcal{U}(N_t)$ (the group of unitary matrices with dimension N_t), $\mathbf{V} \in \mathcal{U}(N_r)$ and $\boldsymbol{\Sigma} = \mathrm{diag}(\{\sqrt{\lambda_i}\}) \in \mathcal{R}_+^{N_r \times N_t}$, it is shown [14] that

$$\begin{aligned} P(\{\lambda_i\}) =& K_{M,N}\Delta(\boldsymbol{\lambda})^2\prod_{i=1}^{M}\lambda_i^{N-M} \\ &\times \int D\mathbf{V}\int D\mathbf{U} p(\mathbf{H} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^*) \end{aligned} \tag{6}$$

where $K_{M,N}^{-1} = \prod_{j=1}^{M} j!(N-M+j-1)!$,

$$\Delta(\boldsymbol{\lambda}) = \det[\lambda_i^{j-1}] = \prod_{i>j}(\lambda_i - \lambda_j)$$

is the Vandermonde determinant of vector $\boldsymbol{\lambda}$ ($\det[f(i,j)]$ denotes the determinant of a matrix with the (i,j)th element given by $f(i,j)$), $D\mathbf{U}$ denotes the Haar measure of $\mathcal{U}(N_t)$ [11] and $p(\mathbf{H})$ is the pdf of $\mathbf{H}$ defined as

$$\begin{aligned} p(\mathbf{H}) &= \mathcal{N}_{\mathbf{T},\mathbf{R}}\,\mathrm{etr}\left\{-\mathbf{H}\mathbf{T}^{-1}\mathbf{H}^*\mathbf{R}^{-1}\right\} &(7)\\ &= \mathcal{N}_{\mathbf{T},\mathbf{R}}\,\mathrm{etr}\left\{-\mathbf{H}^*\mathbf{R}^{-1}\mathbf{H}\mathbf{T}^{-1}\right\} &(8) \end{aligned}$$

where $\mathrm{etr}\{\mathbf{A}\} = \exp(\mathrm{tr}\{\mathbf{A}\})$ and $\mathcal{N}_{\mathbf{T},\mathbf{R}}^{-1} = \det\left[\mathbf{T}\right]^{N_r} \times \det\left[\mathbf{R}\right]^{N_t}$.

Without loss of generality, we assume $N_r \geqslant N_t$ to apply (7) into (6). (For $N_t > N_r$, equation (8) is applied into (6).) Hence, for the full correlated MIMO channel

$$P(\{\lambda_i\}) = \mathcal{N}_{\mathbf{T},\mathbf{R}}\Delta(\boldsymbol{\lambda})I(\mathbf{T},\mathbf{R},\boldsymbol{\lambda}) \tag{9}$$

where

$$\begin{aligned} I(\mathbf{T},\mathbf{R},\boldsymbol{\lambda}) = &K_{M,N}\Delta(\boldsymbol{\lambda})\prod_{i=1}^{M}\left[e^{-\lambda_i}\lambda_i^{N-M}\right] \times \\ &\int D\mathbf{V}\int D\mathbf{U}\,\mathrm{etr}\left\{-\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^*\mathbf{T}^{-1}\mathbf{V}\boldsymbol{\Sigma}^*\mathbf{U}^*\mathbf{R}^{-1}\right\}. \end{aligned} \tag{10}$$

To find $P(\{\lambda_i\})$, we define the following integral which has more general form than (10):

$$I_1 = \int D\mathbf{V}\int D\mathbf{U}\,\mathrm{etr}\left\{\mathbf{U}\mathbf{A}\mathbf{V}^*\mathbf{B}\mathbf{V}\mathbf{C}\mathbf{U}^*\mathbf{D}\right\} \tag{11}$$

where $\mathbf{U} \in \mathcal{U}(N)$, $\mathbf{V} \in \mathcal{U}(M)$, $\mathbf{A} \in \mathcal{C}^{N\times M}$, $\mathbf{C} \in \mathcal{C}^{M\times N}$, $\mathbf{B} \in \mathcal{C}^{M\times M}$ and $\mathbf{D} \in \mathcal{C}^{N\times N}$ are *general rectangular complex coefficient matrices*. The authors in [13] try to solve the integral in (10) by using the character expansion of groups. We show that the attempt in [13] is not correct and solve (11) in its general form, which is our main contribution in this paper.

III. Character expansion of groups

The group of unitary matrices $\mathcal{U}(N)$ is a subgroup of the group of complex invertible matrices with dimension N denoted by $Gl(N)$. A d-dimensional representation of the group $Gl(N)$ is a homomorphism from $Gl(N)$ into the $Gl(d)$. The irreducible representations of $Gl(N)$ can be labeled by the N-dimensional ordered sets as $\mathbf{r}_N = \{r_1, r_2, \ldots, r_N\}$ where $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_N \geqslant 0$ are integers. The dimension $d_{\mathbf{r}_N}$ of the irreducible representation $\mathbf{r}_N$ is given by [15]

$$\begin{aligned} d_{\mathbf{r}_N} &= \left[\prod_{i=1}^{N}\frac{(r_i+N-i)!}{(N-i)!}\right]\det\left[\frac{1}{(r_i-i+j)!}\right] &(12)\\ &= \left[\prod_{i=1}^{N}\frac{1}{(N-i)!}\right](-1)^{\frac{N(N-1)}{2}}\Delta(\mathbf{k}) &(13) \end{aligned}$$

where the matrix elements inside the determinant in (12) with $r_i - i + j < 0$ are zero and $\mathbf{k}$ is a N-dimensional vector with $k_i = r_i - i + N$.

The character of a group element $\mathbf{X} \in Gl(N)$ in its representation $\mathbf{r}_N$ is defined by Weyl's character formula as [16]

$$\chi_{\mathbf{r}_N}(\mathbf{X}) = \mathrm{tr}\left\{\mathbf{X}^{(\mathbf{r}_N)}\right\} = \frac{\det\left[x_i^{r_j+N-j}\right]}{\Delta(x_1,\ldots,x_N)} \tag{14}$$

where $\mathbf{X}^{(\mathbf{r}_N)}$ denotes the $d_{\mathbf{r}_N}$ dimensional representation matrix of $\mathbf{X}$ and $\{x_1,\ldots,x_N\}$ are the eigenvalues of $\mathbf{X}$. In this case, the following equation holds for $\mathbf{X}$ [12]:

$$\mathrm{etr}\{\mathbf{X}\} = \sum_{\mathbf{r}_N}\alpha_{\mathbf{r}_N}\chi_{\mathbf{r}_N}(\mathbf{X}) \tag{15}$$

where the summation is over all irreducible representations of $Gl(N)$ and the expansion factor $\alpha_{\mathbf{r}_N}$ is defined as

$$\alpha_{\mathbf{r}_N} = \det\left[\frac{1}{(r_i - i + j)!}\right] = \left[\prod_{i=1}^{N} \frac{(N-i)!}{(r_i + N - i)!}\right] d_{\mathbf{r}_N} \quad (16)$$

Lemma 1: The orthogonality relation between unitary matrix group elements implies that [17]

$$\int D\mathbf{U} U_{ij}^{(\mathbf{r}_N)} U_{kl}^{(\mathbf{r}'_N)*} = \frac{1}{d_{\mathbf{r}_N}} \delta_{\mathbf{r}_N \mathbf{r}'_N} \delta_{ik} \delta_{jl} \quad (17)$$

where $U_{ij}^{(\mathbf{r}_N)}$ denotes the (i,j)th element of the representation matrix of $\mathbf{U}$ and $d_{\mathbf{r}_N}$ is the dimension of the representation.

Proposition 1: Assuming $\mathbf{A}, \mathbf{B} \in Gl(N)$, $\mathbf{U} \in \mathcal{U}(N)$ and $\mathbf{r}_N$ a representation of $Gl(N)$, then

$$\int D\mathbf{U}\, \chi_{\mathbf{r}_N}(\mathbf{U}\mathbf{A}\mathbf{U}^*\mathbf{B}) = \frac{1}{d_{\mathbf{r}_N}} \chi_{\mathbf{r}_N}(\mathbf{A}) \chi_{\mathbf{r}_N}(\mathbf{B}).$$

Proof: From (14), we have $\chi_{\mathbf{r}_N}(\mathbf{U}\mathbf{A}\mathbf{U}^*\mathbf{B}) = \mathrm{tr}\{(\mathbf{U}\mathbf{A}\mathbf{U}^*\mathbf{B})^{(\mathbf{r}_N)}\}$. Since a representation is a homomorphism, i.e. $(\mathbf{U}\mathbf{A}\mathbf{U}^*\mathbf{B})^{(\mathbf{r}_N)} = \mathbf{U}^{(\mathbf{r}_N)}\mathbf{A}^{(\mathbf{r}_N)}\mathbf{U}^{*(\mathbf{r}_N)}\mathbf{B}^{(\mathbf{r}_N)}$, we obtain

$$\begin{aligned}\chi_{\mathbf{r}_N}(\mathbf{U}\mathbf{A}\mathbf{U}^*\mathbf{B}) &= \mathrm{tr}\left\{\mathbf{U}^{(\mathbf{r}_N)}\mathbf{A}^{(\mathbf{r}_N)}\mathbf{U}^{*(\mathbf{r}_N)}\mathbf{B}^{(\mathbf{r}_N)}\right\} \\ &= \sum_{k_4=1}^{N}\sum_{k_3=1}^{N}\sum_{k_2=1}^{N}\sum_{k_1=1}^{N} U_{k_4k_3}^{(\mathbf{r}_N)} A_{k_3k_2}^{(\mathbf{r}_N)} U_{k_1k_2}^{*(\mathbf{r}_N)} B_{k_1k_4}^{(\mathbf{r}_N)}\end{aligned}$$

Therefore,

$$\begin{aligned}&\int D\mathbf{U}\, \chi_{\mathbf{r}_N}(\mathbf{U}\mathbf{A}\mathbf{U}^*\mathbf{B}) \\ &= \sum_{k_4=1}^{N}\sum_{k_3=1}^{N}\sum_{k_2=1}^{N}\sum_{k_1=1}^{N} A_{k_3k_2}^{(\mathbf{r}_N)} B_{k_1k_4}^{(\mathbf{r}_N)} \int D\mathbf{U}\, U_{k_4k_3}^{(\mathbf{r}_N)} U_{k_1k_2}^{*(\mathbf{r}_N)} \\ &= \frac{1}{d_{\mathbf{r}_N}} \sum_{k_2=1}^{N}\sum_{k_1=1}^{N} A_{k_2k_2}^{(\mathbf{r}_N)} B_{k_1k_1}^{(\mathbf{r}_N)} = \frac{1}{d_{\mathbf{r}_N}} \chi_{\mathbf{r}_N}(\mathbf{A})\chi_{\mathbf{r}_N}(\mathbf{B})\end{aligned}$$

where the second equality comes from Lemma 1. ■

IV. Calculation of I_1

Considering the rank of matrices in (11), if we define $\mathbf{E} = \mathbf{A}\mathbf{V}^*\mathbf{B}\mathbf{V}\mathbf{C}$, the N-dimensional matrix $\mathbf{E}$ is of the rank M and is not a member of $Gl(N)$. Thus, we need more assumptions to use the character expansions. *We assume the matrix* $\mathbf{E}$ *has a full rank of* N*, to use the* N*-dimensional representations and the character expansions* and obtain:

$$\begin{aligned}I_2 &= \int D\mathbf{V} \int D\mathbf{U}\, \mathrm{etr}\{\mathbf{U}\mathbf{E}\mathbf{U}^*\mathbf{D}\} \\ &= \sum_{\mathbf{r}_N} \alpha_{\mathbf{r}_N} \int D\mathbf{V} \int D\mathbf{U}\, \chi_{\mathbf{r}_N}(\mathbf{U}\mathbf{E}\mathbf{U}^*\mathbf{D}) \\ &= \sum_{\mathbf{r}_N} \frac{\alpha_{\mathbf{r}_N}}{d_{\mathbf{r}_N}} \int D\mathbf{V}\, \chi_{\mathbf{r}_N}(\mathbf{E})\chi_{\mathbf{r}_N}(\mathbf{D}) \\ &= \sum_{\mathbf{r}_N} \frac{\alpha_{\mathbf{r}_N}}{d_{\mathbf{r}_N}} \chi_{\mathbf{r}_N}(\mathbf{D}) \int D\mathbf{V}\, \chi_{\mathbf{r}_N}(\mathbf{A}\mathbf{V}^*\mathbf{B}\mathbf{V}\mathbf{C}) \quad (18)\end{aligned}$$

where the second equality comes from (15) and the third equality is obtained by applying the Proposition 1. Note that the above assumptions allowed us to take the integration over $\mathcal{U}(N)$. Clearly, we must introduce the following limits to make sure that the above assumptions hold for I_1:

$$I_1 = \lim_{\{\eta_1,\ldots,\eta_{N-M}\}\to 0} \; \lim_{\{r_{M+1},\ldots,r_N\}\to 0} I_2 \quad (19)$$

where the first limit guarantees that the matrix $\mathbf{E}$ or equivalently the matrix $\mathbf{A}\mathbf{V}^*\mathbf{B}\mathbf{V}\mathbf{C}$ has only M nonzero eigenvalues; and the second limit is because we used N-dimensional representations to be able to integrate over $\mathcal{U}(N)$ while we were allowed to use M-dimensional representations only. For instance, we could use the M-dimensional representations for character expansion of $\mathrm{etr}\{\mathbf{U}\mathbf{A}\mathbf{V}^*\mathbf{B}\mathbf{V}\mathbf{C}\mathbf{U}^*\mathbf{D}\}$ in (11) by using the substitution of $\mathrm{etr}\{\mathbf{U}\mathbf{A}\mathbf{V}^*\mathbf{B}\mathbf{V}\mathbf{C}\mathbf{U}^*\mathbf{D}\}$ with $\mathrm{etr}\{\mathbf{V}^*\mathbf{B}\mathbf{V}\mathbf{C}\mathbf{U}^*\mathbf{D}\mathbf{U}\mathbf{A}\}$. Because $\mathrm{tr}\{\mathbf{X}\mathbf{Y}\} = \mathrm{tr}\{\mathbf{Y}\mathbf{X}\}$ [18] and despite the matrix $\mathbf{U}\mathbf{A}\mathbf{V}^*\mathbf{B}\mathbf{V}\mathbf{C}\mathbf{U}^*\mathbf{D}$ which is not full rank, the matrix $\mathbf{V}^*\mathbf{B}\mathbf{V}\mathbf{C}\mathbf{U}^*\mathbf{D}\mathbf{U}\mathbf{A}$ has the full rank M. But in this case, we could not perform the integration over $\mathcal{U}(N)$. We emphasize that the second limit in (19) is critical. The authors in [13] does not consider this point and by adding rows and columns to matrices, they practically take both integrations over $\mathcal{U}(N)$ and thus the final result is not correct. Interested readers may examine equation (52) in [13] for a one transmit and two receive antenna system or refer to [19] and [7] for more details.

To calculate (19), we present the following propositions:

Proposition 2:

$$\begin{aligned}\lim_{\{r_{M+1},\ldots,r_N\}\to 0} \frac{\alpha_{\mathbf{r}_N}}{d_{\mathbf{r}_N}} &= \lim_{\{r_{M+1},\ldots,r_N\}\to 0} \left[\prod_{i=1}^{N} \frac{(N-i)!}{(r_i+N-i)!}\right] \\ &= \frac{\prod_{i=1}^{N}(N-i)!}{\prod_{i=1}^{M}(r_i+N-i)! \prod_{i=M+1}^{N}(N-i)!} \\ &= \frac{\prod_{i=N-M}^{N-1} i!}{\prod_{i=1}^{M}(r_i+N-i)!}.\end{aligned}$$

Lemma 2: If we define the following ratio

$$R(x_1,\ldots,x_N) = \frac{\det\left[f_i(x_j)\right]}{\Delta(x_1,\ldots,x_N)}$$

where $i,j = 1,\ldots,N$, then

$$\begin{aligned}&\lim_{\{x_1,\ldots,x_p\}\to 0} R(x_1,\ldots,x_N) \\ &= \frac{\det[\mathbf{Z}]}{\Delta(x_{p+1},\ldots,x_N)\prod_{i=p+1}^{N} x_i^p \prod_{j=1}^{p-1} j!}\end{aligned}$$

where

$$\mathbf{Z} = [f_i(0), f_i^{(1)}(0), \ldots, f_i^{(p-1)}(0), f_i(x_{p+1}), \ldots, f_i(x_N)]$$

where $i = 1,\ldots,N$ generates all rows of $\mathbf{Z}$ and $f^{(k)}$ denotes the kth derivative of the function f. (See Lemma 6 in [13] for proof.)

Proposition 3: Assuming $\mathbf{A}_{N\times M}$, $\mathbf{G}_{M\times M}$ and $\mathbf{C}_{M\times N}$ are of the rank M ($N \geqslant M$), and x_i's $i=1,\ldots,N$ are the eigenvalues of the matrix $\mathbf{AGC}$, then

$$\lim_{\{x_1,\ldots,x_{N-M}\}\to 0}\ \lim_{\{r_{M+1},\ldots,r_N\}\to 0} \chi_{\mathbf{r}_N}(\mathbf{AGC}) = \chi_{\mathbf{r}_M}(\mathbf{GCA})$$

where $\{x_1,\ldots,x_{N-M}\}$ represent the $N-M$ zero eigenvalues of $\mathbf{AGC}$.

Proof: From (14) and noting that $\det\left[x_i^{r_j+N-j}\right] = \det\left[x_j^{r_i+N-i}\right]$, we have

$$\begin{aligned}\lim_{\{r_{M+1},\ldots,r_N\}\to 0} \chi_{\mathbf{r}_N}(\mathbf{AGC}) &= \lim_{\{r_{M+1},\ldots,r_N\}\to 0} \frac{\det\left[x_i^{r_j+N-j}\right]}{\Delta(x_1,\ldots,x_N)}\\ &= \frac{\det[\mathbf{X}]}{\Delta(x_1,\ldots,x_N)}\end{aligned}$$

where

$$X_{ij} = \begin{cases} x_j^{r_i+N-i}, & i \leqslant M \\ x_j^{N-i}, & i > M\end{cases}$$

Now by taking $f_i(x_j) = X_{ij}$ as defined above and applying Lemma 2, it is easy to see that

$$\begin{aligned}&\lim_{\{x_1,\ldots,x_{N-M}\}\to 0} \frac{\det[\mathbf{X}]}{\Delta(x_1,\ldots,x_N)}\\ &= \frac{\det\begin{bmatrix}\mathbf{0}_{M\times(N-M)} & \mathbf{Q}_{M\times M}\\ \mathbf{P}_{(N-M)\times(N-M)} & \mathbf{T}_{(N-M)\times M}\end{bmatrix}}{\Delta(x_{N-M+1},\ldots,x_N)\prod_{i=N-M+1}^{N} x_i^{N-M}\prod_{j=1}^{N-M-1} j!}\end{aligned}$$

where

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & \cdots & (N-M-1)!\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 2! & \cdots & 0\\ 0 & 1! & 0 & \cdots & 0\\ 0! & 0 & 0 & \cdots & 0\end{bmatrix}$$

and

$$\mathbf{Q} = \begin{bmatrix} x_{N-M+1}^{r_1+N-1} & \cdots & x_N^{r_1+N-1}\\ \vdots & \ddots & \vdots\\ x_{N-M+1}^{r_M+N-M} & \cdots & x_N^{r_M+N-M}\end{bmatrix}.$$

By column factoring of $\mathbf{Q}$, we obtain

$$\begin{aligned}\det\begin{bmatrix}\mathbf{0} & \mathbf{Q}\\ \mathbf{P} & \mathbf{T}\end{bmatrix} &= \prod_{j=0}^{N-M-1} j!\,\det[\mathbf{Q}]\\ &= \prod_{j=1}^{N-M-1} j!\prod_{i=N-M+1}^{N} x_i^{N-M}\,\det\left[x_{N-M+j}^{r_i+M-i}\right]\end{aligned}$$

Therefore,

$$\begin{aligned}&\lim_{\{x_1,\ldots,x_{N-M}\}\to 0}\ \lim_{\{r_{M+1},\ldots,r_N\}\to 0} \chi_{\mathbf{r}_N}(\mathbf{AGC})\\ &= \frac{\det\left[x_{N-M+j}^{r_i+M-i}\right]}{\Delta(x_{N-M+1},\ldots,x_N)} = \chi_{\mathbf{r}_M}(\mathbf{GCA})\end{aligned}$$

where the last equality comes from the fact that the nonzero eigenvalues of the matrices $\mathbf{AGC}$ and $\mathbf{GCA}$ are equal [18]. ∎

To continue the calculation of I_1, from (19) we have

$$\begin{aligned}I_1 = \int D\mathbf{V} &\lim_{\{\eta_1,\ldots,\eta_{N-M}\}\to 0}\\ &\lim_{\{r_{M+1},\ldots,r_N\}\to 0}\sum_{\mathbf{r}_N}\left[\frac{\alpha_{\mathbf{r}_N}}{d_{\mathbf{r}_N}}\right]\chi_{\mathbf{r}_N}(\mathbf{D})\,\chi_{\mathbf{r}_N}(\mathbf{AV}^*\mathbf{BVC})\end{aligned}$$

By applying Propositions 2 and 3, we have

$$\begin{aligned}I_1 &= \sum_{\mathbf{r}_M}\frac{\prod_{i=N-M}^{N-1} i!}{\prod_{i=1}^{M}(N-i+r_i)!}\,\overline{\chi}_{\mathbf{r}_M}(\mathbf{D})\int D\mathbf{V}\chi_{\mathbf{r}_M}(\mathbf{V}^*\mathbf{BVCA})\\ &= \sum_{\mathbf{r}_M}\frac{\prod_{i=N-M}^{N-1} i!}{\prod_{i=1}^{M}(N-i+r_i)!}\,\overline{\chi}_{\mathbf{r}_M}(\mathbf{D})\,\frac{1}{d_{\mathbf{r}_M}}\,\chi_{\mathbf{r}_M}(\mathbf{B})\,\chi_{\mathbf{r}_M}(\mathbf{CA})\end{aligned} \tag{20}$$

where the second equality comes from Proposition 1. Here

$$\overline{\chi}_{\mathbf{r}_M}(\mathbf{D}) = \frac{\det_M\left[x_i^{r_j+N-j}\right]}{\Delta(x_1,\ldots,x_N)} \tag{21}$$

where $\{x_1,\ldots,x_N\}$ are the eigenvalues of $\mathbf{D}$ and

$$\det_M\left[x_i^{r_j+N-j}\right] = \det\begin{bmatrix} x_i^{r_j+N-j}, & j \leqslant M\\ x_i^{N-j}, & M < j \leqslant N\end{bmatrix}.$$

By applying (13), (14) and (21) in (20) we obtain

$$\begin{aligned}I_1 = \left[\prod_{n=N-M}^{N-1} n!\right]&\left[\prod_{m=1}^{M-1} m!\right]\sum_{\mathbf{k}_M}\frac{(-1)^{\frac{M(M-1)}{2}}}{\prod_{i=1}^{M}(k_i+N-M)!}\times\\ &\frac{\det_M\left[x_i^{k_j+N-M}\right]\det\left[y_i^{k_j}\right]\det\left[z_i^{k_j}\right]}{\Delta(\mathbf{x})\Delta(\mathbf{y})\Delta(\mathbf{z})\Delta(\mathbf{k})}\end{aligned} \tag{22}$$

where $k_i \triangleq r_i + M - i$ and vectors $\mathbf{x} \in \mathcal{C}^N$ and $\mathbf{y},\mathbf{z} \in \mathcal{C}^M$ are the eigenvalues of matrices $\mathbf{D}$, $\mathbf{B}$ and $\mathbf{CA}$, respectively.

V. Joint PDF of Eigenvalues of $\mathbf{H}$

Recalling (10) and by defining $\mathbf{D} = \mathbf{R}^{-1}$, $\mathbf{B} = -\mathbf{T}^{-1}$, $\mathbf{CA} = \mathbf{\Sigma}^*\mathbf{\Sigma}$, $\boldsymbol{\lambda} = \mathrm{diag}(\mathbf{\Sigma}^*\mathbf{\Sigma})$, $\mathbf{x} = \mathrm{diag}(\mathrm{eig}(\mathbf{R}^{-1}))$ and $\mathbf{t} = \mathrm{diag}(\mathrm{eig}(\mathbf{T}^{-1}))$, applying (22) in (10) results in

$$\begin{aligned}I(\mathbf{T},\mathbf{R},\boldsymbol{\lambda}) = K_{M,N}&\left[\prod_{n=N-M}^{N-1} n!\right]\left[\prod_{m=1}^{M-1} m!\right]\prod_{i=1}^{M}\left[e^{-\lambda_i}\lambda_i^{N-M}\right]\\ \times\sum_{\mathbf{k}_M}&\frac{\det_M\left[x_i^{k_j+N-M}\right]\det\left[t_i^{k_j}\right]\det\left[\lambda_i^{k_j}\right]}{\Delta(\mathbf{x})\Delta(\mathbf{t})\Delta(\mathbf{k})\prod_{i=1}^{M}(k_i+N-M)!}\end{aligned} \tag{23}$$

Note that $\Delta(-\mathbf{t}) = (-1)^{\frac{M(M-1)}{2}}\Delta(\mathbf{t})$.

By defining

$$C_{M,N} = K_{M,N}\,\mathcal{N}_{\mathbf{T},\mathbf{R}}\left[\prod_{n=N-M}^{N-1} n!\right]\left[\prod_{m=1}^{M-1} m!\right]$$

and applying (23) in (9), the joint eigenvalue distribution of the full correlated Wishart matrix in obtained as follows:

$$P(\{\lambda_i\}) = C_{M,N}\Delta(\boldsymbol{\lambda})\prod_{i=1}^{M}\left[e^{-\lambda_i}\lambda_i^{N-M}\right] \times \sum_{\mathbf{k}_M} \frac{\underset{M}{\det}\left[x_i^{k_j+N-M}\right]\det\left[t_i^{k_j}\right]\det\left[\lambda_i^{k_j}\right]}{\Delta(\mathbf{x})\Delta(\mathbf{t})\Delta(\mathbf{k})\prod_{i=1}^{M}(k_i+N-M)!} \quad (24)$$

The result in (24) is different from the joint pdf of the eigenvalues of the correlated Wishart matrix presented in [13]. As we explained before, the authors in [13] assume $M = N$ and calculate I_1 and then for $M \neq N$, they apply limits on the eigenvalues. Since they ignore the fact that by assuming $M = N$, both integrations in I_1 are performed over unitary matrices with the same size, the final result for the case that $M \neq N$ is not valid.

To calculate the MGF, $g(z)$, one can apply (24) in (5) and perform part by part integrations over λ_i's. By direct differentiation of the MGF, the capacity of the full correlated MIMO channels is obtained. Interested readers can follow these steps as presented in [13] with the joint eigenvalue distribution obtained in (24). Due to the page limits, we omit the MGF and capacity calculations which will appear in the journal version of the paper.

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