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ON THE EIGENVALUE SPECTRUM OF CERTAIN OPERATOR IDEALS

BY

HERMANN KÖNIG (KIEL)

The aim of this paper* is to give a survey on some recent results on the distribution of eigenvalues of certain operators in Banach spaces. All classes of operators which are considered extend the Schatten classes $S_p(H)$ on Hilbert spaces to operator ideals on general Banach spaces. The main question under study is whether and in which way Weyl's inequality for the eigenvalues of $S_p(H)$-operators may be extended to operators in Banach spaces. It turns out that, for some operator ideals extending $S_p(H)$, the summability order of the eigenvalues remains the same in Banach spaces as in Hilbert spaces, namely p, whereas in other cases only weaker summability properties can be derived in general Banach spaces which may be improved in special spaces like $L_r(\mu)$. Some applications to Hilbert space characterizations, to trace formulas and to integral operators in $L_r(\mu)$-spaces will be mentioned.

Most of the results presented here will appear somewhere else. Nevertheless, to be self-contained, this paper contains complete proofs of some of the more important new theorems and sketches of the proofs of others. The paper is a written version of some lectures given by the author at the Winter School on Functional Analysis in Nowy Sącz, Poland, in January 1978. The author would like to use this opportunity to express his thanks to the organizers of the meeting — Prof. A. Pełczyński and Prof. P. Wojtaszczyk.

1. Introduction. Unless stated differently, all Banach spaces will be complex. We shall consider only operators T which are either compact themselves or have a compact power. Hence their spectrum (except 0) will consist of a zero sequence of eigenvalues of finite multiplicity. Denoting them by $(\lambda_n(T))_{n\in N}$, we always assume that they are ordered in decreasing absolute value, and counted according to their multiplicity.

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The space of all linear [compact] operators from a Banach space X into another one Y is denoted by $\mathscr{L}(X, Y)$ $[\mathscr{K}(X, Y)]$. We are interested in the spectrum of the following ideals of operators:

(1) The *absolutely* (p, q)*-summing operators* $(\Pi_{p,q}, \pi_{p,q})$, $1 \leqslant q \leqslant p < \infty$:

$T \in \Pi_{p,q}(X, Y)$ iff there is a constant $c > 0$ such that for all $(x_i)_{i=1}^n \subseteq X$ we have

$$\Big(\sum_{i=1}^{n} \|Tx_i\|^p\Big)^{1/p} \leqslant c \sup_{\|x'\|_{X'} \leqslant 1} \Big(\sum_{i=1}^{n} |x'(x_i)|^q\Big)^{1/q}.$$

The infimum over all possible choices of c is denoted by $\pi_{p,q}(T)$. For $p = q$ we write (Π_p, π_p).

(2) The *nuclear operators* $(\mathscr{N}_p, \nu_p)$, $0 < p \leqslant 1$:

$T \in \mathscr{N}_p(X, Y)$ iff T admits a representation

$$T = \sum_{i \in N} x_i' \otimes y_i, \quad x_i' \in X', \, y_i \in Y \text{ with } \sum_{i \in N} (\|x_i'\| \, \|y_i\|)^p < \infty.$$

Let

$$\nu_p(T) = \inf \Big\{\Big(\sum_{i \in N} \|x_i'\|^p \|y_i\|^p\Big)^{1/p}\Big\},$$

where the infimum is taken over all representations of T of the above form.

(3) The *operators* (S_p^s, σ_p^s) *of type* l_p, $0 < p < \infty$:

$T \in S_p^s(X, Y)$ iff

$$\sigma_p^s(T) := \Big(\sum_{n \in N} s_n(T)^p\Big)^{1/p} < +\infty,$$

where s_n is an *s-number sequence* in the sense of Pietsch [22], i.e. s is a map associating to any continuous linear operator T between Banach spaces a decreasing sequence $s_n(T)$ of non-negative numbers with $s_1(T) = \|T\|$ such that

(i) $s_{n+m-1}(S+T) \leqslant s_n(S) + s_m(T)$ for $S, T \in \mathscr{L}(X, Y)$;

(ii) $s_n(RST) \leqslant \|R\| s_n(S) \|T\|$ for $T \in \mathscr{L}(X_0, X)$, $S \in \mathscr{L}(X, Y)$ and $R \in \mathscr{L}(Y, Y_0)$;

(iii) $s_n(T) = 0$ for any operator T with $\operatorname{rank} T < n$;

(iv) $s_n(\mathrm{Id}: l_2^m \to l_2^m) = 1$ for $m \geqslant n$, where l_2^m denotes the m-dimensional Hilbert space.

Examples of s-numbers are:

the *approximation numbers* of $T \in \mathscr{L}(X, Y)$

$$a_n(T) = \inf \{\|T - T_n\| : T_n \in \mathscr{L}(X, Y), \ \operatorname{rank} T_n < n\};$$

the *Kolmogorov numbers* of $T \in \mathscr{L}(X, Y)$

$$\delta_n(T) = \inf(\sup[\inf\{\|Tx - y\| : y \in Z \subseteq Y\} : \|x\| = 1] : \dim Z < n);$$

the *Gelfand numbers* of $T \in \mathscr{L}(X, Y)$

$$\gamma_n(T) = \inf\{\|T|_Z\| : Z \subset X,\ \operatorname{codim} Z < n\};$$

the *Hilbert numbers* of $T \in \mathscr{L}(X, Y)$

$$h_n(T) = \sup\{a_n(RTS) \colon S \in \mathscr{L}(H, X),\ R \in \mathscr{L}(Y, H),\ \|S\| = \|R\| = 1\}.$$

The operator classes (1)-(3) are stable under composition with continuous linear operators: The absolutely (p, q)-summing operators $(\Pi_{p,q}, \pi_{p,q})$ form a complete normed ideal of operators whereas the p-nuclear operators $(\mathscr{N}_p, \nu_p)$ and the operators (S_p^s, σ_p^s) of type l_p are a complete quasi-normed ideal, i.e. ν_p and σ_p^s allow only a triangle inequality with a constant factor on the right-hand side. Concerning the definition of operator ideals we refer to [27].

We shall use standard notation as $L_p(\mu)$, l_p, l_p^n for $1 \leqslant p \leqslant \infty$ and denote the conjugate index of p by p' $(1/p + 1/p' = 1)$.

In Hilbert spaces, all s-number sequences coincide with the singular numbers: $s_n(T) = \lambda_n\big((T^*T)^{1/2}\big)$ (cf. [22]). Hence we get the *Schatten classes*

$$\begin{aligned} S_p(H) &:= S_p^s(H) \\ &= \Big\{T \in \mathscr{L}(H) \colon \sigma_p(T) = \Big(\sum_n s_n(T)^p\Big)^{1/p} = \big(\operatorname{tr}(T^*T)^{p/2}\big)^{1/p}\Big\} < \infty. \end{aligned}$$

For $p \leqslant 1$ we have $S_p(H) = \mathscr{N}_p(H)$ with equality of the (quasi) norms $\sigma_p(T) = \nu_p(T)$ (cf. [23]). In general Banach spaces, only the inclusion $S_p^a(X, Y) \subseteq \mathscr{N}_p(X, Y)$ holds. Concerning the absolutely (p, q)-summing operators, we have the following identities in a Hilbert space (cf. [20] and [14]):

$$\Pi_p(H) = S_2(H),\ 1 \leqslant p < \infty, \quad \text{and} \quad \Pi_{p,2}(H) = S_p(H),\ 2 \leqslant p < \infty,$$

with $\pi_2(T) = \sigma_2(T)$.

A classical inequality of Weyl [29] gives information on the distribution of eigenvalues of S_p-operators in Hilbert spaces:

THEOREM 1. *Let* $0 < p < \infty$ *and* $T \in \mathscr{K}(H)$. *Then, for any* $n \in \boldsymbol{N}$,

$$\Big(\sum_{j=1}^{n} |\lambda_j(T)|^p\Big)^{1/p} \leqslant \Big(\sum_{j=1}^{n} s_j(T)^p\Big)^{1/p}.$$

We are interested in generalizations of the Weyl inequality to operators in Banach spaces. Note that the left-hand side makes sense also for $T \in \mathscr{K}(X)$, whereas we may think of the right-hand side — for $n \to \infty$ — as being $\sigma_p^s(T)$ or $\nu_p(T)$ if $p \leqslant 1$ or $\pi_q(T)$, $1 \leqslant q < \infty$, if $p = 2$. This poses the problem to derive summability properties for the eigenvalues of S_p^s-, $\mathscr{N}_p$- or Π_p-operators in Banach spaces, since they all extend certain $S_p(H)$-classes to operator ideals on Banach spaces.

The following notation will be useful later on:

$$\mathscr{E}_{p,q}(X) = \{T \in \mathscr{L}(X): (\lambda_j(T)) \in l_{p,q}\}, \quad 0 < p < \infty,\ 0 < q \leqslant \infty,$$

assuming that the spectrum of T consists of eigenvalues only. Here

$$l_{p,q} := \left\{(\xi_n)_n \in c_0: \begin{pmatrix} \|\xi\|_{p,q} = \left(\sum_{n\in N} \xi_n^* n^{q/p-1}\right)^{1/q} < +\infty,\ q < \infty \\ \|\xi\|_{p,\infty} = \sup_{n\in N} \xi_n^* n^{1/p} < +\infty \end{pmatrix}\right\}$$

is a Lorentz sequence space with ξ_n^* denoting the non-negative decreasing rearrangement of the zero sequence ξ_n. Then, of course, $l_p = l_{p,p}$. We have the inclusions

$$l_{p,q_1} \subsetneqq l_{p,q_2} \quad \text{for } q_1 < q_2,$$
$$l_{p_1,q_1} \subsetneqq l_{p_2,q_2} \quad \text{for } p_1 < p_2 \text{ and for all } q_1, q_2.$$

This means that the essential summability order is given by the first index. By this notation, Weyl's inequality states $S_p(H) \subseteq \mathscr{E}_p(H)$. Here, of course, $\mathscr{E}_p = \mathscr{E}_{p,p}$.

To assess the scope of the results we can expect in Banach spaces, we shall first consider the distribution of eigenvalues of the minimal and maximal extension of $S_p(H)$: By [24] there is a minimal [maximal] operator ideal $S_p^{\min}$ [$S_p^{\max}$] on all Banach spaces which coincides on Hilbert spaces with $S_p(H)$. This is meant in the sense that for any other extension $\mathscr{A}_p$ of $S_p(H)$ to an operator ideal the inclusions

$$S_p^{\min}(X, Y) \subseteq \mathscr{A}_p(X, Y) \subseteq S_p^{\max}(X, Y)$$

hold for all X and Y. By [24], $S_p^{\min}(X, Y)$ is the set of all $T \in \mathscr{L}(X, Y)$ which can be factored over an $S_p(H)$-operator, $T = BSA$, $A \in \mathscr{L}(X, H)$, $S \in S_p(H)$ and $B \in \mathscr{L}(H, Y)$. Let

$$\sigma_p^{\min}(T) = \inf \|A\| \sigma_p(S) \|B\|,$$

the infimum extended over all possible factorizations. Similarly, $S_p^{\max}(X, Y)$ consists of all maps $T \in \mathscr{L}(X, Y)$ such that for any $A \in \mathscr{L}(H, X)$ and $B \in \mathscr{L}(Y, H)$ we have $BTA \in S_p(H)$. Let

$$\sigma_p^{\max}(T) = \inf\{\sigma_p(BTA): \|A\| = \|B\| = 1\}.$$

Both operator ideals $(S_p^{\min}, \sigma_p^{\min})$ and $(S_p^{\max}, \sigma_p^{\max})$ are complete and quasinormed by $\sigma_p^{\min}$ and $\sigma_p^{\max}$, respectively.

PROPOSITION 1. (a) *For* $0 < p < \infty$, $S_p^{\min}(X) \subseteq \mathscr{E}_p(X)$.

(b) *For* $p < 2$ *and* $1/q = 1/p - 1/2$, $S_p^{\max}(X) \subseteq \mathscr{E}_{q,\infty}(X)$.

If $X = L_r(\mu)$, *the same is true for* $1/q = 1/p - |1/2 - 1/r|$.

Proof. (a) This was shown by Pietsch [25] using the technique of related operators. We omit the proof, since (a) is contained in Theorem 3 in the sequel.

(b) Using the alternative description of $S_p^{\max}$ of [24], it is easy to see that the inclusion $S_p^{\max}(X, Y) \subseteq \Pi_{q,2}(X, Y)$ is valid $(1/q = 1/p - 1/2)$. Hence, by [19], we infer for any $T \in S_p^{\max}(X)$ that $T^N \in \mathscr{K}(X)$ if $N > (2/p - 1)^{-1}$. Note here $p < 2$. Therefore, the spectrum of any $T \in S_p^{\max}(X)$, $p < 2$, consists only of eigenvalues. Let $(\lambda_i(T))_{i=1}^n$ denote the first n eigenvalues. By a perturbation argument we may assume that their multiplicity is one. If X_n is the span of the associated eigenvectors $(x_i)_{i=1}^n$, then $\dim X_n = n$. By [5], the absolutely 2-summing and the 2-nuclear norm of the identity map I_n on X_n is $\sqrt{n}$ (which holds for real and complex X_n). I_n may be extended to $P_n\colon X \to X_n$ with 2-nuclear norm $\sqrt{n}$. This means that there are operators $A\colon X \to l_2^n$ and $B\colon l_2^n \to X_n$ with $\|A\|\,\|B\| \leqslant \sqrt{n}$ and $BA|_{X_n} = I_n$. Let $T_n\colon X_n \to X$ be the restriction of T and let $S = AT_nB \in \mathscr{L}(l_2^n)$. Then $\lambda_i(T) = \lambda_i(S)$ for $i = 1, \ldots, n$. By Theorem 1,

$$\Big(\sum_{i=1}^{n} |\lambda_i(T)|^p\Big)^{1/p} = \Big(\sum_{i=1}^{n} |\lambda_i(S)|^p\Big)^{1/p} \leqslant \sigma_p(S) = \sigma_p^{\max}(S)$$
$$\leqslant \|A\|\,\|B\|\sigma_p^{\max}(T_n) \leqslant n^{1/2}\sigma_p^{\max}(T).$$

Therefore

$$|\lambda_n(T)| \leqslant n^{-1/p}\Big(\sum_{i=1}^{n} |\lambda_i(T)|^p\Big)^{1/p} \leqslant n^{-1/q}\sigma_p^{\max}(T),$$
$$\|\lambda_i(T)\|_{q,\infty} \leqslant \sigma_p^{\max}(T).$$

If $X = L_r(\mu)$ and $r \geqslant 2$, there are maps $A\colon X \to l_2^n$ and $B\colon l_2^n \to X_n$ with $BA|_{X_n} = I_n$ and the better estimate $\|A\|\,\|B\| \leqslant n^{1/2-1/r}$ (cf. [16]). Using this we get the second part of (b) for $r \geqslant 2$ by the same argument. For $1 < r < 2$ it is enough to note that $T \in S_p^{\max}(L_r(\mu))$ iff $T' \in S_p^{\max}(L_{r'}(\mu))$.

CONJECTURE. Let $p < 2$ and $1/q = 1/p - 1/2$. Can (b) be improved slightly to yield $S_p^{\max}(X) \subseteq \mathscr{E}_q(X)$?

This is true at least for $p = 1$ and $q = 2$, since then

$$S_1^{\max}(X) \subseteq \Pi_2(X) \subseteq \mathscr{E}_2(X);$$

for the second inclusion compare the next section.

2. Absolutely p-summing operators.

LEMMA 1. *Let $R \in \mathscr{L}(X, Y)$ and $S \in \mathscr{L}(Y, X)$. Then the spectrum of $RS \in \mathscr{L}(Y)$ coincides with the spectrum of $SR \in \mathscr{L}(X)$, and the multiplicities of any non-zero eigenvalue are the same.*

This useful lemma is due to Pietsch [26]. Let $\Pi_2^{(N)}(X, Y)$ denote the set of all operators $T \in \mathscr{L}(X, Y)$ which can be factored as $T = T_N \circ \ldots \circ T_1$, where $T_i \in \Pi_2(X_{i-1}, X_i)$ with $X_0 = X$ and $X_N = Y$. Here the Banach spaces X_i may depend on T. The infimum being taken over all factorizations,

$$\pi_2^{(N)}(T) = \inf \prod_{i=1}^{n} \pi_2(T_i)$$

defines a quasinorm, under which $(\Pi_2^{(N)}, \pi_2^{(N)})$ is a quasinormed operator ideal.

PROPOSITION 2. *For any $N \in \mathbf{N}$, $\Pi_2^{(N)}(X) \subseteq \mathscr{E}_{2/N}(X)$ with*

$$\Big(\sum_{n\in \mathbf{N}} |\lambda_n(T)|^{2/N}\Big)^{N/2} \leqslant \pi_2^{(N)}(T).$$

Especially, for $n = 1$, the eigenvalues of absolutely 2-summing operators are square summable. In this way, Π_2 is a "good" extension of the Hilbert-Schmidt operators. Proposition 2 is due to Pietsch [26] for $N = 1$ and to Retherford and the author [8] for $N > 1$.

Proof. For simplicity, we give the proof only for $N = 1$, which already illustrates the method of reducing the problem to Hilbert spaces. Let $T \in \Pi_2(X)$ and $\varepsilon > 0$. By the factorization theorem [17] there are maps $R \in \Pi_2(X, H)$ and $S \in \mathscr{L}(H, X)$ with $T = SR$ and

$$\pi_2(R)\|S\| \leqslant \pi_2(T) + \varepsilon.$$

But, by Lemma 1, $Q := RS \in \Pi_2(H)$ has the same eigenvalues as T. Hence, by Weyl's inequality,

$$\Big(\sum_{i\in \mathbf{N}} |\lambda_i(T)|^2\Big)^{1/2} = \Big(\sum_{i\in \mathbf{N}} |\lambda_i(Q)|^2\Big)^{1/2} \leqslant \sigma_2(Q) = \pi_2(Q)$$
$$\leqslant \pi_2(R)\|S\| \leqslant \pi_2(T) + \varepsilon.$$

For $1 \leqslant p < 2$, the inclusion $\Pi_p(X) \subseteq \mathscr{E}_2(X)$ holds and is also optimal, since $\Pi_p(X) \subseteq \Pi_2(X)$ and $\Pi_p(H) = \Pi_2(H)$. It is an easy consequence of Proposition 2 that, for $p = 2n$ and $n \in \mathbf{N}$, $\Pi_p(X) \subseteq \mathscr{E}_p(X)$. The following theorem due to Maurey [8] generalizes this fact to $2 \leqslant p < \infty$, solving a problem of Pietsch [25].

THEOREM 2. (a) *For $2 \leqslant p < \infty$, $\Pi_p(X) \subseteq \mathscr{E}_p(X)$ with*

$$\Big(\sum_{i\in \mathbf{N}} |\lambda_i(T)|^p\Big)^{1/p} \leqslant \pi_p(T);$$

the eigenvalues of absolutely p-summing maps are p-th power summable.

(b) *For $T = T_N \circ \ldots \circ T_1 \in \mathscr{L}(X)$ with $T_j \in \Pi_{p_j}(X_{j-1}, X_j)$, where*

$$X_0 = X_N = X, \quad 2 \leqslant p_j < \infty \quad \textit{and} \quad \frac{1}{p} = \sum_{j=1}^{N} \frac{1}{p_j},$$

we have

$$\Big(\sum_{i\in N} |\lambda_i(T)|^p\Big)^{1/p} \leqslant c_p \prod_{j=1}^{N} \pi_{p_j}(T_j)$$

with a constant c_p depending only on p.

Proof. (a) Giving Maurey's first proof, we will only show

$$\Big(\sum_{i\in N} |\lambda_i(T)|^p\Big)^{1/p} \leqslant 2\pi_p(T) \quad \text{for } T \in \Pi_p(X), \tag{2.1}$$

since this argument is technically easier and conceptionally more comprehensible, and in a quick way generalizes to prove (b).

(i) The first step is the reduction to l_∞^n: It is enough to prove (2.1) for $T \in \mathscr{L}(l_\infty^n)$. This follows from the factorization diagram of the absolutely p-summing operators, i.e.

$$\begin{array}{ccccc} X & & \xrightarrow{\quad T \quad} & & X \\ \cap\big\downarrow i & & & & \cap\big\downarrow i \\ C(K) & \xrightarrow{j} & L_p(K,\mu) & \xrightarrow{\tilde{T}} & C(K) \end{array}$$

where K is the unit ball of X', i and j are the canonical imbeddings, μ a probability measure on K and $\|\tilde{T}\| \leqslant \pi_p(T)$. Since $\pi_p(j) = 1$, we have $\pi_p(\tilde{T}j) \leqslant \pi_p(T)$. Further, all eigenvalues of T in X are also eigenvalues of T in $C(K)$. Hence it is enough to prove (2.1) for $T \in \Pi_p(C(K))$. Let X_n be the space spanned by the first n eigenvectors of T. For any $\varepsilon > 0$, there is a finite-dimensional subspace $Y_m \subseteq C(K)$ with $X_n \subseteq Y_m$ and $d(Y_m, l_\infty^m) \leqslant 1+\varepsilon$, and onto which there is a projection $P: X \to Y_m$ of norm less than or equal to $1+\varepsilon$ (cf. [17]). Then $PT|_{Y_m}: Y_m \to Y_m$ has among its eigenvalues all first n eigenvalues of T and we get, by assumption on $\Pi_p(l_\infty^m)$,

$$\Big(\sum_{i=1}^{n} |\lambda_i(T)|^p\Big)^{1/p} \leqslant \Big(\sum_{j} |\lambda_j(PT|_{Y_m})|^p\Big)^{1/p}$$
$$\leqslant 2(1+\varepsilon)\pi_p(PT|_{Y_m}) \leqslant 2(1+\varepsilon)^2\pi_p(T).$$

(ii) We now want to show (2.1) for all $T \in \mathscr{L}(l_\infty^n)$. The ideà is, given T, to construct an equivalent Hilbert space norm on l_∞^n under which

$$\sigma_p(T : H \to H) \leqslant \pi_p(T : l_\infty^n \to l_\infty^n).$$

We write $l_\infty^n = C(K)$, $K = \{1, \ldots, n\}$. Without loss of generality, we may assume that $Te_i \neq 0$. Since the extreme points of the dual of $C(K)$ are just the $\pm$unit vectors of l_1^n, the measure characterization of the absolutely p-summing operators [17] yields that there is a probability measure λ on K such that

$$\pi_p(T) = \|T : L_p(K, \lambda) \to C(K)\|. \tag{2.2}$$

Let ϱ be an arbitrary probability measure on K. Since

$$\pi_{p'}\big(j\colon C(K)\to L_{p'}(K,\varrho)\big)\leqslant 1,$$

we get

$$\pi_1\big(T\colon C(K)\to L_{p'}(K,\varrho)\big)\leqslant \pi_p(T). \tag{2.3}$$

There is a canonical measure $\tilde{\nu}(\varrho)$ on K, given by

$$\tilde{\nu}(\varrho)_j = \|Te_j\|\Big/\Big(\sum_{k=1}^{n}\|Te_k\|\Big),$$

such that

$$\pi_1\big(T\colon C(K)\to L_{p'}(K,\varrho)\big) = \big\|T\colon L_1\big(K,\tilde{\nu}(\varrho)\big)\to L_{p'}(K,\varrho)\big\|. \tag{2.4}$$

Here $\leqslant$ is clear by the measure characterization, and $\geqslant$ follows from the triangle inequality. Let $\nu(\varrho) = \frac{1}{2}\big(\lambda+\tilde{\nu}(\varrho)\big)$. Then, for any ϱ, $\nu(\varrho)$ is a probability measure on K with $\nu(\varrho)\geqslant \frac{1}{2}\lambda$, and the map $\varrho\mapsto\nu(\varrho)$ is continuous on the compact convex set

$$\Big\{\varrho\colon\ \varrho \text{ probability measure on } K \text{ with } \varrho\geqslant\frac{1}{2}\lambda\Big\}$$

in l_∞^n. By Brouwer's fixed point theorem, there is $\mu=\nu(\mu)$ with $\mu\geqslant\frac{1}{2}\lambda$. Applying (2.4) and (2.3) for $\mu = \frac{1}{2}\big(\lambda+\tilde{\nu}(\mu)\big)$, we get

$$\|T\colon L_1(K,\mu)\to L_{p'}(K,\mu)\|\leqslant 2\pi_p(T).$$

Similarly, (2.2) implies

$$\|T\colon L_p(K,\mu)\to C(K)\|\leqslant 2\pi_p(T).$$

We now apply complex interpolation to the last two inequalities and conclude from Proposition 3 that

$$\sigma_p\big(T\colon L_2(K,\mu)\to L_2(K,\mu)\big)\leqslant 2\pi_p\big(T\colon C(K)\to C(K)\big),$$

which, by Weyl's inequality, implies (2.1).

PROPOSITION 3. *Let μ be a probability measure on $K=\{1,\ldots,n\}$ and assume*

$$\|T\colon L_p(K,\mu)\to C(K)\|\leqslant 1 \quad \textit{and} \quad \|T\colon L_1(K,\mu)\to L_{p'}(K,\mu)\|\leqslant 1.$$

Then

$$\sigma_p\big(T\colon L_2(K,\mu)\to L_2(K,\mu)\big)\leqslant 1.$$

Proof. Let $m=[p/2]$ and $1/r_j := 1/2-j/p$, $j=0,\ldots,m$. Then $r_{m-1}\leqslant p<r_m$. We get, by complex interpolation [1],

$$\|T\colon L_{r_{j-1}}(K,\mu)\to L_{r_j}(K,\mu)\|\leqslant 1,\quad j=1,\ldots,m,$$

since $1/r_j = 1/r_{j-1} - 1/p$. A similar statement holds for the Hilbert space adjoint T^*:

$$\|T^*: L_{r_{j-1}}(K, \mu) \to L_{r_j}(K, \mu)\| \leqslant 1, \quad j = 1, \ldots, m.$$

Thus, if m is even, then

$$\|(T^*T)^{m/2}: L_2 \to L_{r_m}\| \leqslant 1.$$

If m is odd, then

$$\|(T^*T)^{(m-1)/2}T: L_2 \to L_{r_m}\| \leqslant 1.$$

There is a unitary operator U in L_2 such that $(T^*T)^{1/2} = TU$. Hence for any m, even or odd,

$$\|(T^*T)^{m/2}: L_2(K, \mu) \to L_2(K, \mu)\| \leqslant 1.$$

Since $p < r_m$, complex interpolation yields

$$\|(T^*T)^{s/2}: L_2(K, \mu) \to L_p(K, \mu)\| \leqslant 1 \tag{2.5}$$

with

$$\frac{1}{p} = \frac{1-s/m}{2} + \frac{s/m}{r_m}, \quad \text{i.e.} \quad s = \frac{p}{2} - 1 < m = \left[\frac{p}{2}\right].$$

Consider

$$L_2(K, \mu) \xrightarrow{(T^*T)^{s/2}} L_p(K, \mu) \xrightarrow{T} C(K) \xrightarrow{I} L_2(K, \mu),$$

where $\pi_2(I) = 1$. Using (2.5), the assumption and ideal norm properties, we get

$$\begin{aligned}\sigma_p(T: L_2 \to L_2) &= \sigma_2\big((T^*T)^{p/2}: L_2 \to L_2\big)\\ &= \pi_2\big(T(T^*T)^{s/2}: L_2 \to L_2\big) \leqslant 1.\end{aligned}$$

This proves Proposition 3.

(iii) Part (b) of Theorem 2 can be proved by using the following extension of part (ii) due to Johnson [8]. Let $T = T_N \circ \ldots \circ T_1$, where $T_j \in \mathscr{L}(l_\infty^n)$. Then there is a probability measure μ on $K = \{1, \ldots, n\}$ such that, for $i = 1, \ldots, N$,

$$\|T_i: L_{p_i}(K, \mu) \to C(K)\| \leqslant 2N\pi_{p_i}(T_i),$$

$$\|T_i: L_1(K, \mu) \to L_{p_i'}(K, \mu)\| \leqslant 2N\pi_{p_i}(T_i).$$

This, Proposition 3 and Weyl's inequality yield (b) with constant $c_p = (2N)^N$ which seems to depend on N. But composing the operators T_i together in a suitable way shows that c_p depends only on p,

$$\frac{1}{p} = \sum_{j=1}^{N} \frac{1}{p_j}.$$

Part (a) of Theorem 2 is easily seen to be equivalent to the following matrix inequality for which no direct proof is known:

PROPOSITION 4. *For* $2 \leqslant p \leqslant \infty$ *and any* $(n \times n)$*-matrix* $T = (t_{ij})$,

$$\Big(\sum_i |\lambda_i(T)|^p\Big)^{1/p} \leqslant \Big(\sum_j \Big(\sum_k |t_{jk}|^{p'}\Big)^{p/p'}\Big)^{1/p}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

This follows from Theorem 2, since $\pi_p(T: l_p^n \to l_p^n)$ is smaller than the double sum on the right.

Starting from the inequality, we get (2.1) like this: It is a consequence of (2.2) that there is a $\delta \in l_2^n \setminus \{0\}$ such that

$$\pi_p(T: l_\infty^n \to l_\infty^n) \geqslant \Big(\sum_j \Big(\sum_k \Big|\frac{\delta_j}{\delta_k} t_{jk}\Big|^{p'}\Big)^{p/p'}\Big)^{1/p}.$$

But $\big((\delta_j/\delta_k)t_{jk}\big)_{j,k}$ has the same eigenvalues as T.

COROLLARY. *Let* (Ω, μ) *be a measure space and let* $K: \Omega^2 \to C$ *be a measurable kernel with*

$$c_p := \Big(\int_\Omega \Big(\int_\Omega |K(x, y)|^{p'} d\mu(y)\Big)^{p/p'} d\mu(x)\Big)^{1/p} < \infty, \quad \text{where } 2 \leqslant p < \infty.$$

Then

$$Tf(x) = \int_\Omega K(x, y) f(y)\, dy$$

defines a map $T: L_p(\mu) \to L_p(\mu)$ *with* p*-th power summable eigenvalues* $\|\lambda_j(T)\|_p \leqslant c_p$.

An immediate corollary to this is the Hausdorff-Young inequality $\|\hat{f}\|_p \leqslant \|f\|_{p'}$ for $p \geqslant 2$ and $f \in L_{p'}(0, 1)$. Just consider the kernel $K(x, y) = f(x-y)$.

The weaker form $\Pi_p(X) \subseteq \mathscr{E}_{p,\infty}(X)$, $p \geqslant 2$, of Theorem 2 follows from Proposition 1 (b) and from the fact that Π_p extends $S_2(H)$. Thus, considering Π_p, we loose some order of summability of the eigenvalues in Banach spaces starting from Hilbert spaces. Since $\Pi_{p,2}(H) = S_p(H)$ for $2 \leqslant p < \infty$ — with of course $\Pi_p(X) \subseteq \Pi_{p,2}(X)$ — it is more natural to ask for the solution of the following

PROBLEM (**P 1192**). Are the eigenvalues of absolutely $(p, 2)$-summing operators still p-th power summable with

$$\Big(\sum_j |\lambda_j(T)|^p\Big)^{1/p} \leqslant \pi_{p,2}(T)?$$

Note that the spectrum is a pure eigenvalue spectrum since, by [19], T^N is compact for $T \in \Pi_{p,2}(X)$ and for some $N \in \mathbf{N}$. The answer is open, we can only give the following weaker result:

PROPOSITION 5. *Let* $2 < p < 4$ *and* $1/q = 2/p - 1/2$. *Then the eigenvalues of* $T \in \Pi_{p,2}(X)$ *are of order* $j^{-1/q}$, *i.e.*

$$\Pi_{p,2}(X) \subseteq \mathscr{E}_{q,\infty}(X) \quad \textit{with } \|\lambda_j(T)\|_{q,\infty} \leqslant c\pi_{p,2}(T).$$

Proposition 5 is implied by Lemmas 2 and 3 given in the sequel.

A quasinormed operator ideal $(\mathscr{A}, A)$ is called *injective* if, for any $T \in \mathscr{L}(X, Y)$ and any isometric imbedding $i: Y \to C(K)$, $iT \in \mathscr{A}$ implies $T \in \mathscr{A}$ with $A(iT) = A(T)$. The ideal $\Pi_{p,q}$ is injective. The *adjoint ideal* of $\mathscr{A}$ is defined by

$$\mathscr{A}^*(X, Y) = \{S \in \mathscr{L}(X, Y): A^*(S) < \infty\},$$

where

$$A^*(S) = \sup\{|\mathrm{tr}(ST)|: A(T: Y \to X) = 1, \ T \text{ of finite rank}\}.$$

Here $\mathrm{tr}(ST)$,

$$\mathrm{tr}(ST) = \sum_i \lambda_i(ST),$$

is the trace of the finite rank operator ST.

LEMMA 2. *Let* $1 < r < \infty$ *and let* $(\mathscr{A}, A)$ *be an injective quasinormed operator ideal with pure eigenvalue spectrum. Then the following are equivalent:*

(1) $\forall_X \mathscr{A}(X) \subseteq \mathscr{E}_{r,\infty}(X)$;

(2) $\exists_{c>0} \ \forall_{X_n, \dim X_n = n} \ A^*(\mathrm{Id}: X_n \to X_n) \leqslant cn^{1/r'}$.

Proof. By an indirect argument of Pietsch [25], (1) is equivalent to

(1′) $\exists_{d>0} \ \forall_X \ \forall_{T \in \mathscr{A}(X)} \ \|\lambda_i(T)\|_{r,\infty} \leqslant dn^{1/r}$.

To show that (1′) implies (2), let $T \in \mathscr{L}(X_n)$, $\dim X_n = n$. Then

$$|\mathrm{tr}(T)| = \Big|\sum_{i=1}^{n} \lambda_i(T)\Big| \leqslant \sum_{i=1}^{n} |\lambda_i(T)| \leqslant \|\lambda_i(T)\|_{r,\infty}\Big(\sum_{i=1}^{n} i^{-1/r}\Big) \leqslant r'dn^{1/r'}A(T)$$

which implies (2) with $c := r'd$. To derive (1′) from (2), let $n \in N$ and decompose the eigenvalues $(\lambda_i(T))_{i=1}^n$ into real and imaginary parts with positive and negative components to find that there exists a subset $I \subseteq \{1, \ldots, n\}$ such that

$$\sum_{i=1}^{n} |\lambda_i(T)| \leqslant 4\Big|\sum_{i \in I} \lambda_i(T)\Big|.$$

Let X_I be the span of the eigenvectors $(x_i)_{i \in I}$ belonging to the eigenvalues $(\lambda_i(T))_{i \in I}$ which we may assume to have multiplicity one. Denoting by T_I the restriction of T, we get

$$\sum_{i=1}^{n} |\lambda_i(T)| \leqslant 4|\mathrm{tr}(T_I\colon X_I \to X_I)|$$
$$\leqslant 4A^*(\mathrm{Id}\colon X_I \to X_I)A(T_I\colon X_I \to X_I) \leqslant 4cn^{1/r'}A(T\colon X \to X).$$

In the last inequality, we used the assumption and the injectivity of $(\mathscr{A}, A)$. This implies

$$\|\lambda_i(T)\|_{r,\infty} \leqslant 4c\sup_n \Big(n^{-1/r'}\sum_{i=1}^{n} |\lambda_i(T)|\Big) \leqslant 4cA(T).$$

Lemma 3. *For $2 < p < 4$ and for every n-dimensional space X_n,*

$$\pi_{p,2}^*(\mathrm{Id}\colon X_n \to X_n) \leqslant 2n^{3/2-2/p}.$$

Proof. By [5], the identity map on a real Banach space X_n of dimension n can be written as

$$\mathrm{Id}_n = \sum_{j=1}^{s} \lambda_j^{1/2}x_j' \otimes \lambda_j^{1/2}x_j, \qquad s \leqslant n^2,$$

where

$$\|x_j'\|_{X_n'} = \|x_j\|_{X_n} = 1, \qquad \sum_{j=1}^{s} \lambda_j = n$$

and

$$\varepsilon_2(\lambda_j^{1/2}x_j) := \sup_{\|x'\|_{X_n'} \leqslant 1} \Big(\sum_{j=1}^{s} \lambda_j |x'(x_j)|^2\Big)^{1/2} \leqslant 1.$$

Using Hölder's inequality, we get

$$a_{p'}(\lambda_j^{1/2}x_j') := \Big(\sum_{j=1}^{s} \|\lambda_j^{1/2}x_j'\|^{p'}\Big)^{1/p'} = \Big(\sum_{j=1}^{s} \lambda_j^{p'/2}\Big)^{1/p'} \leqslant s^{1/p'-1/2}\Big(\sum_{j=1}^{s} \lambda_j\Big)^{1/2} \leqslant n^{3/2-2/p}.$$

Hence any $S\colon X_n \to X_n$ can be written as

$$S = \sum_{j=1}^{s} \lambda_j^{1/2}x_j' \otimes S(\lambda_j^{1/2}x_j),$$

and using Hölder's inequality again, we obtain

$$|\mathrm{tr}(S)| \leqslant \sum_{j=1}^{s} \|\lambda_j^{1/2}x_j'\|_{X_n'} \|S(\lambda_j^{1/2}x_j)\|_{X_n}$$
$$\leqslant a_{p'}(\lambda_j^{1/2}x_j')\pi_{p,2}(S)\varepsilon_2(\lambda_j^{1/2}x_j) \leqslant n^{3/2-2/p}\pi_{p,2}(S),$$

which implies the lemma for complex spaces with an additional constant 2.

PROBLEM (**P 1193**). For any n-dimensional space X_n, do there exist $x'_j \in X'_n$, $x_j \in X_n$, $j = 1, \ldots, s$ with $s \leqslant cn$, such that

$$\mathrm{Id}_n = \sum_{j=1}^{s} x'_j \otimes x_j, \quad \|x'_j\|_{X'_n} \leqslant 1 \quad \text{and} \quad \varepsilon_2(x_j) \leqslant 1?$$

If the answer were positive, the argument above would imply $\Pi_{p,2}(X) \subseteq \mathscr{E}_{p,\infty}(X)$ such that the eigenvalues of $T \in \Pi_{p,2}(X)$ would be "almost" p-th power summable.

3. Operators of type l_p. In Hilbert spaces, all s-number sequences coincide with the singular numbers. However, Weyl's inequality

$$\Big(\sum_j |\lambda_j(T)|^p\Big)^{1/p} \leqslant \Big(\sum_j s_j(T)^p\Big)^{1/p}, \quad T \in S_p(H), \tag{3.1}$$

does not generalize to all s-number ideals S_p^s on general Banach spaces. For any s-number sequence, at least an additional constant c_p in (3.1) is necessary, as easy examples [8] show. Markus and Macaev [18] proved, using methods of complex analysis, the weaker result $S_p^a(X) \subseteq \mathscr{E}_q(X)$ for $p < q$ in Banach spaces. The following theorem due to the author [8] strengthens their result to a direct generalization of (3.1), answering a problem of [18].

THEOREM 3. *Let s_n denote either the approximation or the Gelfand or the Kolmogorov numbers, $s \in \{a, \gamma, \delta\}$. Then, for any p $(0 < p < \infty)$, $S_p^s(X) \subseteq \mathscr{E}_p(X)$. In fact, there is a c_p such that, for all X, $T \in \mathscr{K}(X)$ and $n \in N$,*

$$\Big(\sum_{j=1}^{n} |\lambda_j(T)|^p\Big)^{1/p} \leqslant c_p \Big(\sum_{j=1}^{n} s_j(T)^p\Big)^{1/p}.$$

We give now a simplification due to Johnson of the original proof. First of all we need

LEMMA 4. *Let $0 < p < \infty$ and $T \in S_p^a(X, Y)$. Then there exist operators $D_j \in \mathscr{L}(X, Y)$ with*

$$d_j := \operatorname{rank} D_j \leqslant 3 \cdot 2^j, \quad T = \sum_{j=0}^{\infty} D_j$$

and

$$\Big(\sum_{j=0}^{\infty} d_j \|D_j\|^p\Big)^{1/p} \leqslant c_p \sigma_p^a(T) \tag{3.2}$$

with some constant depending only on p.

This lemma is due to Pietsch [23] and contained in his proof of $S_p^a \subseteq \mathscr{N}_p$. The procedure is to choose operators T_j of rank less than 2^j such that

$$\|T - T_j\| \leqslant 2a_{2^j}(T)$$

and let $D_j = T_{j+1} - T_j$, $j = 0, 1, \dots$ (with $T_0 = 0$). Then

$$T = \sum_{j=0}^{\infty} D_j.$$

Inequality (3.2) can be checked using the monotonicity of the approximation numbers.

Proof of Theorem 3. (i) Since the Gelfand numbers are dual to the Kolmogorov numbers, $\delta_n(T) = \gamma_n(T')$ for $T \in \mathscr{K}(X)$, and since they are smaller than the approximation numbers, $\gamma_n(T) \leqslant a_n(T)$, it is enough to give the proof for the Gelfand number ideals S_p^γ. We remark that the Gelfand numbers are injective and that $\gamma_n(T) = a_n(iT)$ for $T \in \mathscr{L}(X, Y)$ and for any isometric imbedding $i\colon Y \to C(K)$. For these facts see [22].

If $0 < p < \infty$ is given, choose $N \in \mathbf{N}$ with $s := 2/N < p$. Since $\pi_2^{(N)}$ is a quasinorm, it is equivalent to an r-norm for some $0 < r \leqslant 1$ (cf. [13]), which means

$$\pi_2^{(N)}\Big(\sum_j S_j\Big) \leqslant c_N \Big(\sum_j \pi_2^{(N)}(S_j)^r\Big)^{1/r}, \tag{3.3}$$

where c_N depends only on N. We may assume without loss of generality that also $r < p$.

(ii) Let $T \in S_p^\gamma(X)$. Then $iT \in S_p^a\big(X, C(K)\big)$ for an isometric imbedding $i\colon X \to C(K)$. By perturbation we can assume that the first 2^n eigenvalues of T have multiplicity one. Let X_n denote the 2^n-dimensional subspace of X spanned by the corresponding eigenvectors. If $T_n\colon X_n \to X_n$ is the restriction of T and $k\colon X_n \to X \xrightarrow{i} C(K)$, we have

$$\sigma_p^a(kT_n) \leqslant \sigma_p^a(iT) = \sigma_p^\gamma(T).$$

By Lemma 4, there are operators $D_j \in \mathscr{L}\big(X, C(K)\big)$ of rank equal to $d_j \leqslant 3\cdot 2^{j+1}$, $j = 0, 1, \dots, n+1$, such that

$$kT_n = \sum_{j=0}^{n+1} D_j, \quad \Big(\sum_{j=0}^{n+1} d_j \|D_j\|^p\Big)^{1/p} \leqslant a_p \sigma_p^\gamma(T). \tag{3.4}$$

Note here that it is enough to extend the sum up to $n+1$, since $\operatorname{rank} kT_n = 2^n$; all operators D_j in the construction proving Lemma 4 may be chosen to be zero if $j > n+1$. Let I_j be the identity map on $D_j(X) \subseteq C(K)$. Since $\dim D_j(X) = d_j$, by [5] we have $\pi_2(I_j) \leqslant d_j^{1/2}$. Hence, in view of the ideal property of Π_2 and the injectivity of the π_2-norm,

$$\pi_2^{(N)}(D_j) \leqslant \pi_2^{(N)}(I_j)\|D_j\| \leqslant d_j^{N/2}\|D_j\|.$$

Using this, (3.3) and (3.4), we can estimate the eigenvalues of T by Proposition 2:

$$|\lambda_{2^n}(T)| \leqslant 2^{-n/s}\Big(\sum_{i=1}^{2^n} |\lambda_i(T)|^s\Big)^{1/s} \leqslant 2^{-n/s}\pi_2^{(N)}(T: X_n \to X_n)$$

$$\leqslant c_N\cdot 2^{-n/s}\Big(\sum_{j=0}^{n+1} \pi_2^{(N)}(D_j)^r\Big)^{1/r} \leqslant c_N\cdot 2^{-n/s}\Big(\sum_{j=0}^{n+1} d_j^{rN/2}\|D_j\|^r\Big)^{1/r}, \quad s = \frac{2}{N}.$$

We write each term in the last sum as

$$d_j^{rN/2}\|D_j\|^r = d_j^{r/s-r/p}(d_j^{r/p}\|D_j\|^r)$$

and apply Hölder's inequality with the exponent $Q := p/r > 1$ to the sum to conclude

$$|\lambda_{2^n}(T)| \leqslant c_N\cdot 2^{-n/s}\Big(\sum_{j=0}^{n+1} d_j^a\Big)^{1/r-1/p}\Big(\sum_{j=0}^{n+1} d_j\|D_j\|^p\Big)^{1/p}$$

with $a := (1/s-1/p)/(1/r-1/p)$. Since $s < p$ and $r < p$, the first sum is of order $(2^n)^{1/s-1/p}$. Hence, for some other constant d_p depending only on p (and the choice of s, N and r), we have

$$|\lambda_{2^n}(T)| \leqslant d_p\cdot 2^{-n/p}\Big(\sum_{j=0}^{n+1} d_j\|D_j\|^p\Big)^{1/p} \leqslant c_p'\cdot 2^{-n/p}\sigma_p^\gamma(T),$$

where we used (3.4), $c_p' := a_p d_p$. This inequality extends easily from the subsequence $\{2^n\}_{n\in\mathbf{N}}$ of $\{m\}_{m\in\mathbf{N}}$ to

$$|\lambda_m(T)| \leqslant c_p m^{-1/p}\sigma_p^\gamma(T).$$

Applying this to the m-dimensional eigenvector spaces, we conclude — in view of the injectivity of the Gelfand numbers — that it is enough to extend the sum in $\sigma_p^\gamma(T)$ only up to the m-th term,

$$|\lambda_m(T)| \leqslant c_p\left(\frac{\sum_{j=1}^{m}\gamma_j(T)^p}{m}\right)^{1/p}. \tag{3.5}$$

(iii) A classical inequality of Hardy [7] states for $0 < r < p < \infty$ and sequences $a \in l_p$ that

$$\left[\sum_{m=1}^{\infty}\left(\frac{\sum_{j=1}^{m} a_j^r}{m}\right)^{p/r}\right]^{1/p} \leqslant \left(\frac{p}{p-r}\right)^{1/p}\Big(\sum_{j=1}^{\infty} a_j^p\Big)^{1/p}. \tag{3.6}$$

Replacing p in (3.5) by $r := p/2$ and using Hardy's inequality, we get

$$\Big(\sum_{m=1}^{\infty}|\lambda_m(T)|^p\Big)^{1/p} \leqslant c_{p/2}\left[\sum_{m=1}^{\infty}\left(\frac{\sum_{j=1}^{m}\gamma_j(T)^r}{m}\right)^{p/r}\right]^{1/p}$$

$$\leqslant 2^{1/p}c_{p/2}\Big(\sum_{m=1}^{\infty}\gamma_m(T)^p\Big)^{1/p} = \tilde{c}_p\sigma_p^\gamma(T).$$

This shows that $S_p^\gamma(X) \subseteq \mathscr{E}_p(X)$. The second statement of Theorem 3 is an immediate consequence if the last inequality is applied to the restriction of $T \in \mathscr{K}(X)$ to n-dimensional eigenvector spaces. This proves Theorem 3.

The original proof of Theorem 3 (see [8]) also yields $c_p \leqslant \max(c, c^{1/p})$ for some absolute constant c and the following decomposition result:

PROPOSITION 6. *Let $0 < p_i < \infty$ for $i = 1, \ldots, N$ and*

$$\frac{1}{p} = \sum_{i=1}^{N} \frac{1}{p_i}, \quad \textit{where } 0 < p \leqslant 1.$$

Then the following statements are equivalent:

(1) $T \in S_p^a(X, Y)$.

(2) *There are Banach spaces X_i and operators $T_i \in S_{p_i}^a(X_{i-1}, X_i)$ with $X_0 = X$, $X_N = Y$ such that*

$$T = T_N \circ \ldots \circ T_1 \quad \textit{and} \quad \sum_{i=1}^{N} \sigma_{p_i}^a(T_i) \leqslant a_p \sigma_p^a(T).$$

We now mention some applications of Theorem 3; first — an isomorphic characterization of Hilbert spaces [8]:

THEOREM 4. *X is isomorphic to a Hilbert space if and only if the nuclear operators on X coincide with the operators $\mathscr{N}_1(X) = S_1^a(X)$ of type l_1.*

Proof. By [23], $\mathscr{N}_1(H) = S_1^a(H)$. If, on the other hand, $\mathscr{N}_1(X) = S_1^a(X)$ for a complex Banach space X, then the eigenvalues of nuclear operators are absolutely summable by Theorem 3. But then a theorem of Johnson and Retherford [8] implies that X is isomorphic to a Hilbert space.

If X is a real Banach space, consider its complexification $Y = X \oplus iX$. Then $T \in \mathscr{N}_1(Y)$ implies $T \in S_1^a(Y)$ if $\mathscr{N}_1(X) = S_1^a(X)$. This can be seen by decomposing T into real and imaginary parts. Hence Y is (complex) isomorphic to a Hilbert space and X is (real) isomorphic to a Hilbert space.

THEOREM 5. *Let $0 < p < \infty$ and $0 < r \leqslant \infty$. Then there is a $c_{p,r} > 0$ such that, for $s_j \in \{a_j, \gamma_j, \delta_j\}$ and for any $T \in \mathscr{K}(X)$ with $\|s_j(T)\|_{p,r} < \infty$,*

$$\|\lambda_j(T)\|_{p,r} \leqslant c_{p,r} \|s_j(T)\|_{p,r}.$$

This can easily be derived from (3.5) by a generalization of Hardy's inequality [9]. The analogon of Theorem 5 in general sequence spaces replacing $l_{p,r}$ is false. An example for this can be given (cf. [9]) in

$$\{\xi \in c_0 : \sup_n \xi_n^* \cdot 2^n < \infty\}.$$

By [2], the asymptotic order of the approximation numbers of Sobolev-Besov imbeddings over bounded domains $\Omega \subseteq \mathbf{R}^N$ is given by

$$a_n\left(B^{\lambda}_{p,\infty}(\Omega) \to L_p(\Omega)\right) \sim j^{-\lambda/N}, \quad 1 \leqslant p < \infty.$$

Hence we get, using Theorem 5 for $r = \infty$, the following

COROLLARY. *Let $T \in \mathscr{L}(L_p(\Omega))$ with* Image$(T) \subseteq B^{\lambda}_{p,\infty}(\Omega)$. *Then the eigenvalues of T in $L_p(\Omega)$ are of order*

$$|\lambda_n(T)| \lesssim n^{-\lambda/N}.$$

The third application of Theorem 3 concerns the trace formula for S^a_1-operators. For any $T \in S^a_1(X)$, X being an arbitrary Banach space, a trace can be defined as follows:

For any finite rank operator $T \in \mathscr{L}(X)$ the *trace* is given by

$$\operatorname{tr}(T) = \sum_i \lambda_i(T)$$

with only finitely many non-zero eigenvalues $\lambda_i(T)$.

But, by [23], the finite rank operators are σ^a_1-dense in $S^a_1(X)$ and $|\operatorname{tr}(T)| \leqslant 8\,\sigma^a_1(T)$ for any finite rank map T. Hence there exists a unique σ^a_1-continuous extension of $\operatorname{tr}(\cdot)$ to all of $S^a_1(X)$, denoted again by $\operatorname{tr}(\cdot)$. For the so-defined trace we have the following theorem which answers a problem of [18] positively.

THEOREM 6. (a) *Let $T \in S^a_1(X)$. Then*

$$\operatorname{tr}(T) = \sum_{i \in \mathbf{N}} \lambda_i(T).$$

(b) *For $S, T \in S^a_1(X)$,*

$$\sum_{i \in \mathbf{N}} \lambda_i(S+T) = \sum_{i \in \mathbf{N}} \lambda_i(S) + \sum_{i \in \mathbf{N}} \lambda_i(T).$$

(c) *If $\{e_j\}$ is an unconditional basis in X with coefficient functionals $\{f_j\}$, then for any $T \in S^a_1(X)$*

$$\sum_{j \in \mathbf{N}} f_j(Te_j) = \sum_{i \in \mathbf{N}} \lambda_i(T).$$

For the proof we refer to [9]. The main point is to show (a). For this, the estimates of the proof of Theorem 3 are essential. Note that (a) makes sense, since the sum $\sum_{i \in \mathbf{N}} \lambda_i(T)$ is unconditionally convergent for $T \in S^a_1(X)$.

We now turn to the study of the eigenvalue distribution of general s-number ideals S^s_p, where s is an s-number sequence possibly smaller than $s_n \in \{a_n, \gamma_n, \delta_n\}$. Hence the result which we can derive will be weaker. An s-number sequence is called *multiplicative* if for any $S \in \mathscr{L}(X, Y)$

and $T \in \mathscr{L}(Y, Z)$

$$s_{n+m-1}(ST) \leqslant s_n(S)s_m(T), \qquad n, m \in \mathbf{N}.$$

The sequences a_n, γ_n and δ_n are multiplicative. The following useful result is due to Carl [3]:

Proposition 7. *Let $0 < p < \infty$ and let s_n be a multiplicative s-number sequence. Then for any $T \in \mathscr{K}(X)$ and any $n \in \mathbf{N}$*

$$\Big(\sum_{j=1}^{n} |\lambda_j(T)|^p\Big)^{1/p} \leqslant c_p\big(\ln(n+1)\big)^{1/p}\Big(\sum_{j=1}^{n} s_j(T)^p\Big)^{1/p},$$

where c_p is a constant depending only on p. Hence, for any $q > p$, $S_p^s(X) \subseteq \mathscr{E}_q(X)$.

Proof. We will assume again that the eigenvalues have multiplicity one. Let X_n be the span of the eigenvectors belonging to the first n eigenvalues. Similarly as in the proof of Proposition 1, there are operators $A: X \to l_2^n$ and $B: l_2^n \to X_n$ such that $BA|_{X_n} = \mathrm{Id}$ and $\|A\|\|B\| \leqslant \sqrt{n}$. Let $T_n: X_n \to X$ be the restriction of T and let

$$S := AT_nB: l_2^n \to l_2^n.$$

Then $\lambda_i(S) = \lambda_i(T)$ for $1 \leqslant i \leqslant n$. For any Hilbert space map R, by [29] we have

$$|\lambda_1(R)\ldots\lambda_n(R)| \leqslant s_1(R)\ldots s_n(R).$$

Applying this to $R = S^j$ with $j \in \mathbf{N}$, we get

$$\begin{aligned} (3.7)\qquad |\lambda_n(T)| = |\lambda_n(S)| &= |\lambda_n(S^j)|^{1/j} \leqslant |\lambda_1(S^j)\ldots\lambda_n(S^j)|^{1/nj} \\ &\leqslant \big(s_1(S^j)\ldots s_n(S^j)\big)^{1/nj} \leqslant n^{1/2j}\big(s_1(T^j)\ldots s_n(T^j)\big)^{1/nj}, \end{aligned}$$

using for the last inequality $S^j = A(T^j)_nB$ and

$$s_i(S^j) \leqslant \|A\|\,\|B\|\,s_i(T^j) \leqslant n^{1/2}s_i(T^j).$$

The geometric form of Hardy's inequality, derived from (3.6) letting $r \to 0$, states for $a \in l_1$

$$\sum_{n\in\mathbf{N}} |a_1\ldots a_n|^{1/n} \leqslant e\sum_{n\in\mathbf{N}} |a_n|.$$

This inequality, applied to (3.7), yields

$$\begin{aligned} \Big(\sum_{n=1}^{k} |\lambda_n(T)|^p\Big)^{1/p} &\leqslant n^{1/nj}\Big(\sum_{n=1}^{k} [s_1(T^j)\ldots s_n(T^j)]^{p/nj}\Big)^{1/p} \\ &\leqslant e^{1/p}n^{1/2j}\Big(\sum_{n=1}^{k} s_n(T^j)^{p/j}\Big)^{1/p}. \end{aligned}$$

By a standard monotonicity argument, using the multiplicativity of the s-numbers s_n, the sum on the right-hand side is smaller than

$$j^{1/p}\Big(\sum_{n=1}^{k} s_n(T)^p\Big)^{1/p}.$$

Choosing $j = [\ln(n+1)]$, we therefore get

$$\Big(\sum_{n=1}^{k} |\lambda_n(T)|^p\Big)^{1/p} \leqslant c_p[\ln(k+1)]^{1/p}\Big(\sum_{n=1}^{k} s_n(T)^p\Big)^{1/p}.$$

Under even weaker assumptions on s_n, only the following holds:

PROPOSITION 8. *Let* $0 < p < 2$ *and let* s_n *be an arbitrary s-number sequence. Let* $1/q = 1/p - 1/2$. *Then*

$$S_p^s(X) \subseteq \mathscr{E}_{q,p}(X) \qquad \textit{with } \|\lambda_n(T)\|_{q,p} \leqslant e^{1/p}\sigma_p^s(T).$$

If $X = L_r(\mu)$, *then the same holds with* $1/q = 1/p - |1/2 - 1/r|$.

Proof. Apply the same technique as in the previous proof with $j = 1$. Thus

$$|\lambda_n(T)|n^{-1/2} \leqslant \big(s_1(T)\ \ldots\ s_n(T)\big)^{1/n},$$

$$\|\lambda_n(T)\|_{q,p} = \|\lambda_n(T)n^{-1/2}\|_p \leqslant \big\|\big(s_1(T)\ \ldots\ s_n(T)\big)_n\big\|_p \leqslant e^{1/p}\|s_n(T)\|_p.$$

For $X = L_r(\mu)$ and $r \geqslant 2$, apply Lewis' result [16] that A and B can be chosen with $\|A\|\,\|B\| \leqslant n^{1/2-1/r}$. If $r < 2$, use this estimate for T'.

Since $l_{q,p} \subsetneq l_q$, Proposition 8 yields a slightly better eigenvalue space for S_p^s than the worst possible space $l_{q,\infty}$ (or l_q, respectively) for $S_p^{\max}$ (cf. Proposition 1). In general, Proposition 8 is optimal with respect to Lorentz spaces: Let $0 < p < 2$ and let h_n denote the Hilbert numbers. Then there is an operator $T \in S_p^h(l_1)$ such that the eigenvalues of T belong to $l_{q,p}$ but to no smaller Lorentz space $l_{q,s}$ (with $s < p$, $1/q = 1/p - 1/2$). This map can be constructed similarly to the example in the next section using Walsh matrices. The estimates proceed similarly to the ones of Section 4 of [9].

Proposition 7 also holds for the so-called entropy numbers [4]. This has been applied successfully by Carl [3] to the study of Sobolev-Carleman operators, and especially to the study of the eigenvalues of certain integral operators.

Let us mention that, in general, there is no estimate for the single eigenvalues $|\lambda_n(T)|$, $T \in \mathscr{K}(X)$, against the single s-numbers $s_n(T)$. However, we have the following formula [10] which generalizes the spectral radius formula for compact operators in Banach spaces:

PROPOSITION 9. *Let* s_n *be an s-number sequence. Then, for any* $T \in \mathscr{K}(X)$ *and* $n \in N$,

$$|\lambda_n(T)| = \lim_{j\to\infty} s_n(T^j)^{1/j}.$$

However, for a special class of $\mathscr{H}$-operators, which generalize the self-adjoint maps to Banach spaces, one can prove the asymptotic equivalence of $|\lambda_n(T)|$ and $s_n(T)$ (cf. [9]).

4. p-nuclear operators. In this section the previous results are applied to the study of the distribution of eigenvalues of p-nuclear operators in Banach spaces. In view of $\mathscr{N}_1(X) \subseteq \Pi_2(X)$, nuclear operators have square-summable eigenvalues. This is optimal:

LEMMA 5. *For any sequence $\sigma \in l_2$ there exists a nuclear operator T on l_∞ such that the σ_i are (among others) eigenvalues of T.*

Proof. Let $Q: l_1 \to l_2$ be a quotient map and let $D_\sigma: l_2 \to l_2$ be the diagonal map $(x_n) \mapsto (\sigma_n x_n)$. The map Q is absolutely 2-summing by [17], and D_σ is absolutely 2-summing since $\sigma \in l_2$. Hence their composition $D_\sigma Q$ is nuclear, $D_\sigma Q \in \mathscr{N}_1(l_1, l_2)$ (cf. [21]). Thus also the dual map $Q'D_\sigma$ is nuclear, $Q'D_\sigma \in \mathscr{N}_1(l_2, l_\infty)$. Consider $Q': l_2 \to l_\infty$ as an isometric imbedding. By the nuclear extension property, $Q'D_\sigma$ admits a nuclear extension to an operator on l_∞ with σ_i among its eigenvalues.

For p-nuclear operators, the loss of summability of the eigenvalues in Banach spaces compared with Hilbert spaces is similar to the general s-number ideals S_p^s. The situation is studied in the next two theorems.

THEOREM 7. *Let $0 < p \leqslant 1$. Then*

(1) *$\mathscr{N}_p(X) \subseteq \mathscr{E}_q(X)$ with $1/q = 1/p - 1/2$ and*

$$\Big(\sum_{i \in N} |\lambda_i(T)|^q\Big)^{1/q} \leqslant c_p \nu_p(T).$$

(2) *If $X = L_r(\mu)$, then the same is true for $1/q = 1/p - |1/2 - 1/r|$.*

Proof. Theorem 7 will follow from Theorem 2. Part (1) is due to Grothendieck [6]. By definition, $T \in \mathscr{N}_p(X)$ has a representation

$$T = \sum_{j \in N} a_j x_j' \otimes y_j \quad \text{with } \|x_j'\| = \|y_j\| = 1 \text{ and } \|a\|_p \leqslant 2\nu_p(T).$$

We may assume that $a_n \geqslant 0$ is decreasing. Let

$$n := \left[2\left(\frac{1}{p} - 1\right)\right], \qquad \delta := 1 - \frac{(n+2)p}{2}, \qquad u := \frac{p}{\delta}.$$

Then $n \geqslant 0$, $\delta \geqslant 0$ and $2 \leqslant u \leqslant \infty$. Let $O: X \to l_u$ be the map given by $x \mapsto \big(a_j^\delta x_j'(x)\big)_{j \in N}$, further let $P: l_u \to l_2$, $Q: l_2 \to l_2$ and $R: l_2 \to l_1$ be the diagonal maps induced by $(a_i^{p/2})_{j \in N}$ and, finally, $S: l_1 \to X$,

$$(\xi_j)_{j \in N} \to \sum_{j \in N} \xi_j y_j.$$

Then $T = SRQ^n PO$. The operator O is absolutely u-summing, whereas P and Q are absolutely 2-summing. Hence, by Theorem 2,

$$\big(\lambda_i(T)\big) \in l_q, \quad \text{where } \frac{1}{q} = \frac{1}{u} + \frac{n+1}{2} = \frac{1}{p} - \frac{1}{2},$$

with the corresponding estimate. Since $R' \in \Pi_2(l_\infty, l_2)$, for $r \geqslant 2$ we have $R'S' \in \Pi_2(X', l_2) \subseteq \Pi_r(X', l_2)$. For $X = L_r(\mu)$ this implies, by [15], $SR \in \Pi_r(l_2, X)$. Hence

$$\big(\lambda_i(T)\big) \in l_q \quad \text{with } \frac{1}{q} = \frac{1}{p} - \frac{1}{2} + \frac{1}{r} = \frac{1}{p} - \Big(\frac{1}{2} - \frac{1}{r}\Big).$$

For $r < 2$ one can argue by dualization.

As is indicated by Lemma 5, Theorem 7 is optimal for $p = 1$. However, if $0 < p < 1$, Theorem 7 can slightly be improved to

THEOREM 8. *Let* $0 < p < 1$. *Then*

$$\mathcal{N}_p(X) \subseteq \mathcal{E}_{q,p}(X) \quad \textit{with } \|\lambda_i(T)\|_{q,p} \leqslant c_p \nu_p(T),$$

where

(a) $1/q = 1/p - 1/2$ *for general Banach spaces* X,

(b) $1/q = 1/p - |1/2 - 1/r|$ *for* $X = L_r(\mu)$.

We will only give a very rough idea of the proof. The complete proof can be found in [11]. Instead of using Theorem 2 of Section 2, it is a modification of the proof of Theorem 3 in Section 3. For $S \in \mathcal{N}_1(X, Y)$ let

$$a_n^{\nu}(S) := \inf\{\nu_1(S - S_n)\colon \operatorname{rank} S_n < n\}, \quad n \in \mathbf{N},$$

denote the approximation numbers relative to the nuclear norm. Starting with a nuclear representation of $T \in \mathcal{N}_p(X)$,

$$T = \sum_{j \in \mathbf{N}} a_j x_j' \otimes x_j, \quad \|x_j'\| = \|x_j\| = 1, \quad \|a_j\|_p \leqslant 2\nu_p(T),\ a_j \geqslant 0,$$

we clearly have

$$a_n^{\nu}(T) \leqslant \sum_{j>n} a_j.$$

Using this, one can show for $1/t = 1/p - 1$ that

$$\|a_j^{\nu}(T)\|_{t,p} \leqslant c_p \nu_p(T), \quad 0 < p < 1.$$

Roughly, we lost a summability order of one. Instead of considering the ideal $S_{t,p}^{a}$ we now consider similar ideals defined by the approximation numbers relative to ν_1 with quasinorm $\|a_j^{\nu}(T)\|_{t,p}$. For the normal approximation numbers, the summability order would be t, but the nuclear norm allows an additional summability order $1/2$ so that the final loss will be only of order $1 - 1/2 = 1/2$, namely yielding summability q with $1/q = 1/p - 1/2$. The point in the proof of Theorem 3 which has to be changed is the estimate

$$\pi_2^{(N)}(D_j) \leqslant d_j^{N/2}\|D_j\|.$$

Instead of this, use now

$$\pi_2^{(N)}(D_j) \leqslant d_j^{(N-1)/2}\pi_2(D_j) \leqslant d_j^{(N-1)/2}\nu_1(D_j),$$

where the operators D_j are now constructed relatively to the new approximation number ideal. The method yields

$$|\lambda_n(T)|n^{1/2} \leqslant a_s\Big(\sum_{j=1}^{n} a_j^{\nu}(T)^s n^{-1}\Big)^{1/s}.$$

For $s < p/(1-p)$, Hardy's inequality implies

$$\|\lambda_n(T)\|_{q,p} \leqslant b_p\|a_j^{\nu}(T)\|_{t,p} \leqslant d_p\nu_p(T).$$

For $X = L_r(\mu)$, a similar technique can be applied.

Example. Theorem 8 is optimal in general: For any $0 < p \leqslant 1$, we now construct an operator $T \in \mathcal{N}_p(l_1)$ such that the eigenvalues $(\lambda_i(T))$ of T belong to $l_{q,p}$ but to no smaller Lorentz sequence space $l_{q,s}$ for $s < p$ and $1/q = 1/p - 1/2$. A similar example exists in l_r with $1/q = 1/p - -|1/2 - 1/r|$.

Let

$$A_{2^0} = (1), \qquad A_{2^{n+1}} = \begin{pmatrix} A_{2^n} & A_{2^n} \\ A_{2^n} & -A_{2^n} \end{pmatrix}, \qquad n \in N_0,$$

be the Walsh matrices. Then $A_{2^n}^2 = 2^n\,\mathrm{Id}$ and $|\lambda_i(A_{2^n})| = 2^{n/2}$ for all $1 \leqslant i \leqslant 2^n$. Choose $\sigma \in l_p \setminus l_s$ and define

$$A = \sum_{n\in N} \sigma_n(2^n)^{-1/p}A_{2^n}\colon (\bigoplus_n l_1^{2^n})_1 \to (\bigoplus_n l_1^{2^n})_1$$

as a blockwise sum of multiples of the A_{2^n}'s. Thus $A\colon l_1 \to l_1$. Letting a_j denote the rows of A_{2^n}, we get

$$A_{2^n} = \sum_{j=1}^{2^n} a_j \otimes e_j, \qquad \nu_p(A_{2^n})^p \leqslant \sum_{j=1}^{2^n} \|a_j\|_\infty^p = 2^n,$$

$$\nu_p(A)^p \leqslant \sum_{n=1}^{\infty} \sigma_n^p \cdot 2^{-n}\nu_p(A_{2^n})^p \leqslant \|\sigma_n\|_p^p < \infty.$$

Therefore, A is p-nuclear in l_1. However, we have the following igenvalue estimation:

$$\|\lambda_i(T)\|_{q,s}^s = \sum_{i\in N} |\lambda_i(T)|^s i^{s/q-1} \sim \sum_{n\in N} 2^n(\sigma_n \cdot 2^{-n/p}\cdot 2^{n/2})^s(2^n)^{s/q-1}$$
$$= \|\sigma_n\|_s^s = \infty.$$

This means that $A \notin \mathcal{E}_{q,s}(l_1)$. To modify the example in l_r — showing the optimality of (b) — use for $1 < r < 2$ the factor $\sigma_n(2^n)^{-(1/p-1/r')}$ instead of $\sigma_n(2^n)^{-1/p}$.

Remark. Grothendieck [6] conjectured that the following composition formula is valid for p-nuclear operators:

$$\mathcal{N}_p \circ \mathcal{N}_q \subseteq \mathcal{N}_r \quad \text{if } \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{1}{2},\ 0 < p, q, r \leqslant 1.$$

As recognized by Pisier, it follows from Theorem 8 that this is false in general: Choose $\sigma \in l_1 \setminus l_{1,2/3}$ and split the sequence as $\sigma = \mu\nu$, where $\nu \to 0$ and $\mu \in l_1 \setminus l_{1,2/3}$. This is always possible. Then the induced diagonal maps $D_\nu: l_1 \to c_0$ and $D_\mu: c_0 \to l_1$ are nuclear. However, their composition $D_\sigma: l_1 \to l_1$ is not (2/3)-nuclear, since otherwise the eigenvalues σ_i would form an $l_{1,2/3}$-sequence. A more natural conjecture for a composition formula seems to be

$$\mathcal{N}_p \circ \mathcal{N}_q \subseteq \mathcal{N}_r \quad \text{if } \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1, \quad 0 < p, q, r \leqslant 1.$$

We have seen in Theorem 7 that the summability order q of the eigenvalues of nuclear operators in X is somewhere between 1 and 2 and depends on X. So it is natural to ask for the geometric properties of X such that the summability order is some fixed q, $1 < q < 2$.

PROPOSITION 10. *Let* $1 < q < 2$. *Then the following statements are equivalent*:

(1) $\mathcal{N}_1(X) \subseteq \mathcal{S}_{q,\infty}(X)$.

(2) *There is a constant* $c > 0$ *such that for any* n*-dimensional subspace* X_n *of* X *there exists a projection* P *of* X *onto* X_n *of norm* $\|P\| \leqslant cn^{1/q}$.

Proof. We shall only prove the easy direction (2) $\Rightarrow$ (1). The implication (1) $\Rightarrow$ (2) is due to Johnson; the proof uses methods similar to those of the proof of Theorem 3.11 in [8]. To show that (2) implies (1), assume that $T \in \mathcal{N}_1(X)$ has only eigenvalues of multiplicity one. Given $n \in N$, there is a subset $I \subseteq \{1, \ldots, n\}$ such that

$$\sum_{j=1}^{n} |\lambda_j(T)| \leqslant 4\Big|\sum_{i\in I} \lambda_i(T)\Big| \leqslant 4|\mathrm{tr}(T_I: X_I \to X_I)|,$$

where X_I is spanned by eigenvectors belonging to the eigenvalues $(\lambda_i(T))_{i\in I}$. Since $\# I \leqslant n$, by assumption we get

$$\sum_{j=1}^{n} |\lambda_j(T)| \leqslant 4\nu_1(T_I: X_I \to X_I) \leqslant 4\nu_1(T: X \to X)\|P: X \to X_I\| \leqslant 4cn^{1/q'}\nu_1(T),$$

which implies

$$\|\lambda_n(T)\|_{q,\infty} \leqslant \sup_{n\in N} n^{-1/q'} \sum_{j=1}^{n} |\lambda_j(T)| \leqslant 4c\nu_1(T).$$

Part (2) of Theorem 7 and part (b) of Theorem 8 remain valid for subspaces or quotients of $L_r(\mu)$. Hence, by a result of Rosenthal [28], we infer that for any reflexive subspace X of $L_1(0,1)$ there is a $q<2$ with $\mathcal{N}_1(X) \subseteq \mathcal{E}_q(X)$.

5. Product ideals. In Sections 2 and 3 we considered the eigenvalue distribution of absolutely p-summing operators and operators of type l_p separately. The next theorem gives information on the eigenvalues of operators which can be factored as products of these two different types of operators. If $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are operator ideals, we write $\mathcal{A}_1 \circ \ldots \circ \mathcal{A}_n(X, Y)$ for the set of all operators $T \in \mathcal{L}(X, Y)$ which can be factored — over possibly different Banach spaces — as $T = T_1 \circ \ldots \circ T_n$ with $T_i \in \mathcal{A}_i$. Then $\mathcal{A}_1 \circ \ldots \circ \mathcal{A}_n$ again is an operator ideal. The results in this section are extensions of the ones of [12].

THEOREM 9. *Assume that* $2 \leqslant p_i < \infty$, $i = 1, \ldots, n$, *and* $0 < q_i < \infty$, $i = 1, \ldots, m$; *moreover, let*

$$\frac{1}{p} = \sum_{i=1}^{n} \frac{1}{p_i} + \sum_{i=1}^{m} \frac{1}{q_i} \quad \textit{and} \quad \frac{1}{q} = \sum_{i=1}^{m} \frac{1}{q_i}.$$

If $T \in \Pi_{p_1} \circ \ldots \circ \Pi_{p_n} \circ S^{\gamma}_{q_1} \circ \ldots \circ S^{\gamma}_{q_m}(X)$, *where the order of the operator ideals is not essential and may be arbitrary, then the eigenvalues* $(\lambda_j(T))_{j\in N}$ *belong to the Lorentz space* $l_{p,q}$ *with*

$$\|\lambda_j(T)\|_{p,q} \leqslant c(p_i, q_i) A(T),$$

where A is the product ideal quasinorm.

Proof (idea). Let us choose $N \in \mathbf{N}$ and $\tilde{q}_j$ with $2/N < \tilde{q}_j < q_j$ for $1 \leqslant j \leqslant m$. Let

$$\frac{1}{r} = \sum_{i=1}^{n} \frac{1}{p_i} + \frac{mN}{2} \quad \text{and} \quad \frac{1}{s} = \sum_{i=1}^{n} \frac{1}{p_i}.$$

Let X_l denote again the first l-dimensional eigenvector space of T. If

$$T = S_1 \ldots S_n T_1 \ldots T_m, \qquad S_i \in \Pi_{p_i},\ T_j \in S^{\gamma}_{q_j},$$

we get by Theorem 2, part (b),

$$\left\|\left(\lambda_i(T)\right)_{i\leqslant l}\right\|_r \leqslant c_1 \prod_{i=1}^{n} \pi_{p_i}(S_i) \prod_{j=1}^{m} \pi_2^{(N)}(T_j^l),$$

where the T_i^l are restrictions of rank less than or equal to l of the T_i's to certain image spaces of X^l. Similarly as in part (ii) of the proof of Theorem 3 one can show that there is a constant $c_{\tilde{q}_i,N}$ such that, for any operator S of rank l,

$$\pi_2^{(N)}(S) \leqslant c_{\tilde{q}_j(N)} l^{N/2-1/\tilde{q}_j} \sigma^{\gamma}_{\tilde{q}_j}(S), \qquad \frac{N}{2} > \frac{1}{\tilde{q}_j}.$$

Applying this to the operators T_i^l, we get

$$|\lambda_l(T)| l^{1/r} \leqslant \|(\lambda_i(T))_{i\leqslant l}\|_r \leqslant c_2 \prod_{i=1}^{n} \pi_{p_i}(S_i) \prod_{j=1}^{m} \left(\sigma^{\gamma}_{\tilde{q}_j}(T_j^l) l^{N/2-1/\tilde{q}_j}\right),$$

$$|\lambda_l(T)| l^{1/s} \leqslant c_2 \prod_{i=1}^{n} \pi_{p_i}(S_i) \prod_{j=1}^{m} \left(\sigma^{\gamma}_{\tilde{q}_j}(T_j^l) l^{-1/\tilde{q}_j}\right). \tag{5.1}$$

Note that $1/p - 1/q = 1/s$. Thus

$$\|\lambda_l(T)\|_{p,q} = \|\lambda_l(T) l^{1/s}\|_q.$$

Hence, using Hölder's inequality for

$$\frac{1}{q} = \sum_{j=1}^{m} \frac{1}{q_j},$$

from (5.1) we get

$$\|\lambda_l(T)\|_{p,q} \leqslant c_2 \prod_{i=1}^{n} \pi_{p_i}(S_i) \prod_{j=1}^{m} \left\|\left(\sigma^{\gamma}_{\tilde{q}_j}(T_j^l) l^{-1/\tilde{q}_j}\right)_l\right\|_{q_j}.$$

Each of the last m factors can be estimated by Hardy's inequality in view of $\tilde{q}_j < q_j$. This yields

$$\|\lambda_l(T)\|_{p,q} \leqslant c(p_i, q_i) \prod_{i=1}^{n} \pi_{p_i}(S_i) \prod_{j=1}^{m} \sigma^{\gamma}_{q_j}(T_j)$$

which is the statement of Theorem 9.

Without further knowledge on the order of the operators and the spaces concerned, Theorem 9 is optimal: In general, the eigenvalues of T are not absolutely p-summable, as will be seen by Lemma 6. However, since they belong to $l_{p,q}$, they are absolutely r-summable for any $r > p$. Thus, the summability of the eigenvalues of the operators considered in Theorem 9 is slightly worse than the best possible, whereas for the p-nuclear and S_p^h-operators we found the summability to be slightly better than the worst possible. For the ideal $\Pi_2 \circ S_2$, Theorem 9 is optimal:

LEMMA 6. *We have*

$$\Pi_2 \circ S_2(H) = S_{1,2}(H) \subseteq \mathscr{E}_{1,2}(H) \supsetneq \mathscr{E}_1(H),$$

where, of course, $S_{1,2}(H) = \{T \in \mathscr{K}(H): (s_n(T)) \in l_{1,2}\}$.

Proof. Let $T \in \Pi_2 \circ S_2(H)$. Then $(T^*T)^{1/2} = UT \in \Pi_2 \circ S_2(H)$ with some unitary operator U. By Theorem 9, the eigenvalues $s_n(T) = \lambda_n((T^*T)^{1/2})$ belong to $l_{1,2}$, $T \in S_{1,2}(H)$.

If $T \in S_{1,2}(H)$, we can use a trick of Kwapień [14] and write $T = UD_\sigma V$, where V is a unitary map, D_σ a diagonal map in $H = l_2$ with the sequence $(\sigma_n) = (s_n(T)) \in l_{1,2}$, and U an isometry on Image$(D_\sigma V)$. Hence it is enough to show that $D_\sigma\colon l_2 \to l_2$, $(x_n) \to (\sigma_n x_n)$, belongs to the factorization ideal $\Pi_2 \circ S_2(l_2)$ if $\sigma \in l_{1,2}$. Without loss of generality we can assume that $\sigma_n \geqslant 0$, σ_n being decreasing. Then, by [22],

$$a_j(D_\sigma\colon l_2 \to l_1) = \Big(\sum_{n\geqslant j} \sigma_n^2\Big)^{1/2},$$

$$\sigma_2^a(D_\sigma\colon l_2 \to l_1) = \Big(\sum_{j\in N} a_j(D_\sigma\colon l_2 \to l_1)^2\Big)^{1/2} \leqslant \Big(\sum_j \sum_{n\geqslant j} \sigma_n^2\Big)^{1/2}$$
$$= \Big(\sum_{n\in N} n\sigma_n^2\Big)^{1/2} = \|\sigma\|_{1,2} < \infty.$$

Hence $D_\sigma \in S_2(l_2, l_1)$. Since the identity map $I\colon l_1 \to l_2$ is absolutely 2-summing, $D_\sigma \in \Pi_2 \circ S_2(l_2)$.

On the other hand, $S_2 \circ \Pi_2(H) = S_1(H) \subseteq \mathscr{E}_1(H)$. Hence the second Lorentz index q in Theorem 9 may depend on the order of the operators whereas the first and main index does not.

By an argument similar to the one just used we can now give a counter-example to a conjecture of Retherford, which could have enabled a simple proof of $S_1^a(X) \subseteq \mathscr{E}_1(X)$:

Example. In general, operators $T \in S_1^a(X, Y)$ of type l_1 cannot be written as the product of two absolutely 2-summing operators. Let $X = l_2$ and $Y = l_1$ and consider the diagonal map

$$D_\sigma\colon l_2 \to l_1, \quad \text{where } \sigma_n := [n\ln(n+1)]^{-3/2}.$$

Then $D_\sigma\colon l_2 \to l_1 \in S_1^a$, since

$$\sum_{j\in N} a_j(D_\sigma) \leqslant \sum_{j\in N}\Big(\sum_{n\geqslant j}\sigma_n^2\Big)^{1/2} \sim \sum_{j\in N} j^{-1}[\ln(j+1)]^{-3/2} < \infty.$$

However, if $D_\sigma\colon l_2 \to l_1$ were in $\Pi_2^{(2)}(l_2, l_1)$, then $D_\sigma\colon l_2 \to l_2$, as a map in l_2, would be in $\Pi_2^{(3)}(l_2)$, since the identity $l_1 \to l_2$ is in Π_2. But then the eigenvalues (σ_n) of D_σ in l_2 would have to belong to $l_{2/3}$ (cf. Proposition 2), which is false. Hence

$$D_\sigma\colon l_2 \to l_1 \in S_1^a \setminus \Pi_2^{(2)}.$$

However, the conjecture is "almost" correct: For $p < 1$, any $T \in S_p^a(X, Y)$ can be written as the product of two absolutely 2-summing operators.

The use of Hardy's inequality in the proof of Theorems 3, 8 and 9 reflects interpolation properties of the operator ideals $\mathscr{N}_p$ and S_p^a (cf. [12]). Denoting by $(\,,\,)_{\theta,r}$ the real interpolation method of Peetre, for any X, Y

and $1/r = (1-\theta)/p + \theta/q$ we have

$$\mathcal{N}_r(X, Y) \subseteq (\mathcal{N}_p(X, Y), \mathcal{N}_q(X, Y))_{\theta,r}, \quad 0 < p, q, r \leqslant 1, \ 0 < \theta < 1,$$

$$\mathcal{S}_r^a(X, Y) = (\mathcal{S}_p^a(X, Y), \mathcal{S}_q^a(X, Y))_{\theta,r}, \quad 0 < p, q, r < \infty, \ 0 < \theta < 1.$$

The absolutely p-summing operators on c_0 are stable under the complex interpolation method $(\,,\,)_{[\theta]}$:

For $1/r = (1-\theta)/p + \theta/q$,

$$\Pi_r(c_0) \subseteq (\Pi_p(c_0), \Pi_q(c_0))_{[\theta]}, \quad 1 \leqslant p, q, r \leqslant \infty, \ 0 < \theta < 1.$$

However, $\Pi_r(c_0)$ is no real interpolation space which is one reason why Theorem 9 does not yield $T \in \mathcal{E}_p(X)$.

Added in proof. Since the redaction of the paper new results have been obtained.

1. The eigenvalues of any map $T \in \Pi_{p,2}(X)$ are in $l_{p,\infty}$ but in general not in l_p, improving Proposition 5 and solving the problem P 1192 mentioned there (cf. [31]).

2. Theorem 3 has been improved to hold for the so-called Weyl numbers instead of the larger Gelfand numbers (cf. [32]). This concept yields an easier proof of Theorem 9.

3. Concerning Proposition 7 and the comments there it should be mentioned that estimates of single eigenvalues against single entropy numbers have been derived (cf. [30]).

4. In Proposition 6, the assumption $p \leqslant 1$ is unnecessary.

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