ON THE EIGHTH COEFFICIENT OF UNIVALENT FUNCTIONS, II

By Mitsuru Ozawa and Yoshihisa Kubota

0. Let $f(z)$ be a normalized regular function univalent in the unit circle $|z|<1$

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n.$$

For the eighth coefficient a_8 several authors had proved its local maximality at the Koebe function $z/(1-z)^2$ [1], [4], [6]. One of the authors had improved the method in [4] to a great extent in [6]. Obrock [5] and Schiffer [8] proved a general result independently, which can be formulated in the present case in the following manner: If a_2, a_3 and a_4 are real, then $\Re a_8 \leq 8$ with equality holding only for $z/(1-z)^2$. In [7] we have given the following fact: If a_2 is real non-negative, then $\Re a_8 \leq 8$ with equality holding only for $z/(1-z)^2$.

In this paper we shall prove the following theorem:

THEOREM. *If $a_3-3a_2^2/4$ and $a_4-3a_2a_3/2+5a_2^3/8$ are real and $|\arg a_2| \leq \pi/7$, then $\Re a_8 \leq 8$. Equality occurs only for the Koebe function $z/(1-z)^2$.*

By the well known rotation this theorem implies the result due to Obrock and Schiffer as a simple corollary. Our original motivation in [7] and in this paper lies to investigate the status of the general a_8 problem. So the theorems are only byproducts of our original intention. We believe at the present time that the status became almost clear.

Section 1 is devoted to several preparatory lemmas and inequalities from which we start. Section 2 to 9 are concerned with the case $1 \leq \Re a_2 \leq 2$. The main part in this paper consists of sections 2, 4, 6 and 8. Section 10 is concerned with the case $0 \leq \Re a_2 \leq 1$. Sections 3, 5, 7, 9 and 10 are rather trivial parts and easy to handle.

1. We make use of the same notations as in [4]. By our assumption $y'=\eta'=0$ and $|x'/p| \leq \tan(\pi/7)$.

First we shall give here several lemmas, which will be used later on.

LEMMA 1. $11(\tau^2+\tau'^2)+9(\varphi^2+\varphi'^2)+7(\xi^2+\xi'^2)+5\eta^2+3y^2+x'^2 \leq 4x-x^2.$

Proof. This is a simple consequence of the area theorem for $f(1/z^2)^{-1/2}$.

LEMMA 2. $\qquad \eta + \left(2\beta - \dfrac{1}{2}p\right)y \leqq (2-p)\beta^2 + \dfrac{1}{12}(8-p^3) + \dfrac{1}{4}px'^2.$

Proof. By Grunsky's inequality

$$|b_{11}x_1^2 + 6b_{13}x_1x_3 + 3b_{33}x_3^2| \leqq |x_1|^2 + 3|x_3|^2.$$

Here we take $x_3 = 1/3$ and put $x_1 = \beta$. Taking the real part

$$\eta - \frac{1}{2}py + \frac{1}{12}p^3 - \frac{1}{4}px'^2 + 2\beta y + \beta^2 p \leqq \frac{2}{3} + 2\beta^2,$$

which is just the desired result.

LEMMA 3. $\qquad \dfrac{1}{2}p^3\left(\eta + \dfrac{13}{6}py\right)$

$$\leqq \frac{4}{3} - \frac{1}{48}p^6 + \frac{16}{9}(4-p^2)p^2 - \frac{2}{3}p^2(4-p^2)y - y^2 - 3\left(\eta + \frac{5}{6}py\right)^2$$

$$+ 2p^2x'^2y - \frac{1}{4}x'^4y + \frac{3}{2}px'^2\eta - \frac{3}{4}x'^2y^2 - \left(\frac{16}{9}p^2 + \frac{1}{16}p^4\right)x'^2 - \frac{1}{16}p^2x'^4 - \frac{1}{48}x'^6.$$

Proof. By Golusin's inequality we have

$$|b_{11}x_1 + b_{31}x_3|^2 + 3|b_{13}x_1 + b_{33}x_3|^2 \leqq |x_1|^2 + 3|x_3|^2.$$

Here we put $x_1 = 8p/3$, $x_3 = 2/3$. Then by a simple rearrangement of terms we have the desired result.

Grunsky's inequality with $m=7$, $x_1 = \gamma$, $x_3 = \delta/3$, $x_5 = p/5$, $x_7 = 1/7$, $x_2 = x_4 = x_6 = 0$, $y' = \eta' = 0$ leads us to the following inequality

$$\Re a_8 \leqq \frac{2}{7} + \frac{2}{5}p^2 + \frac{2}{3}\delta^2 + x\gamma^2 + \frac{27}{447}p^7 - \frac{1}{80}p^7 - \frac{1}{12}p^3\delta^2$$

$$+ \left(\frac{5}{4}p^2 - 2\delta\right)\varphi + \left(\frac{11}{8}p^3 - p\delta - 2\gamma\right)\xi + \left(\frac{7}{8}p^4 + \frac{1}{2}p^2\delta - \delta^2 - 2p\gamma\right)\eta$$

$$+ \left(\frac{11}{16}p^5 - \frac{1}{2}p^3\delta + \frac{1}{2}p\delta^2 - 2\delta\gamma\right)y + \left(\frac{3}{2}p^3 + \frac{1}{2}p\delta\right)y^2 + \frac{9}{4}p\eta^2$$

$$+ \left(\frac{27}{8}p^2 + 2\delta\right)y\eta + \frac{9}{2}py\xi + 4\eta\xi + 3y\varphi + \frac{9}{8}py^3$$

$$+ \left(-\frac{73}{64}p^5 + \frac{1}{4}p\delta^2\right)x'^2 + \left(-\frac{29}{8}p^2 - \delta\right)x'\xi' - \frac{7}{2}px'\varphi' - 2x'\tau' - \frac{3}{2}x'y\xi'$$

$$+\left(-\frac{13}{2}p^3+\frac{1}{2}p\delta\right)x'^2y+\left(-\frac{23}{4}p^2+\frac{1}{2}\delta\right)x'^2\eta-\frac{29}{8}px'^2\xi-\frac{5}{4}x'^2\varphi$$

$$-\frac{21}{4}px'^2y^2-\frac{39}{8}x'^2y\eta+\frac{11}{8}x'^3\xi'+\frac{53}{16}px'^4y+\frac{7}{8}x'^4\eta+\frac{131}{64}p^3x'^4-\frac{27}{64}px'^6.$$

Now we put

$$\delta=\frac{5}{8}p^2+Ay-\frac{5}{8}x'^2,\qquad \gamma=\frac{3}{8}p^3+Bpy+C\eta-\frac{3}{2}px'^2,$$

where A, B and C are constants to be fixed in later parts. Then we have

$$\Re a_8\leqq U+\left\{\left(\frac{13}{128}-\frac{5}{48}A-\frac{3}{4}B\right)p^5+\frac{3}{2}Bp^4+\frac{5}{6}Ap^2\right.$$

$$+\left(-\frac{251}{64}+\frac{5}{12}A+3B\right)p^3x'^2-6Bp^2x'^2-\frac{5}{6}Ax'^2+\left(\frac{169}{128}-\frac{5}{16}A\right)px'^4\bigg\}y$$

$$+\left\{\left(\frac{3}{64}-\frac{3}{4}C\right)p^4+\frac{3}{2}Cp^3+\left(-\frac{63}{32}+3C\right)p^2x'^2-6Cpx'^2+\frac{11}{64}x'^4\right\}\eta$$

$$+\left\{\left(\frac{29}{16}-\frac{1}{12}A^2-B^2-\frac{5}{8}A-\frac{5}{4}B\right)p^3+2B^2p^2+\frac{2}{3}A^2\right.$$

(A)

$$+\left(-\frac{89}{16}+\frac{1}{4}A^2+\frac{23}{8}A+\frac{5}{4}B\right)px'^2\bigg\}y^2$$

$$+\left\{\left(\frac{37}{8}-2BC-\frac{3}{4}A-2B-\frac{5}{4}C\right)p^2+4BCp+\left(-\frac{49}{8}+\frac{7}{4}A+\frac{5}{4}C\right)x'^2\right\}y\eta$$

$$+\left\{\left(\frac{9}{4}-C^2-2C\right)p+2C^2\right\}\eta^2+\left(\frac{9}{2}-A-2B\right)py\xi+(4-2C)\eta\xi+(3-2A)y\varphi$$

$$+\left(\frac{1}{2}A^2-2AB+\frac{1}{2}A+\frac{9}{8}\right)py^3+(-A^2-2AC+2A)y^2\eta$$

$$+\left(\frac{113}{64\cdot12}p^5-\frac{9}{4}p^4-\frac{25}{48}p^2-\frac{331}{64\cdot12}p^3x'^2+\frac{9}{2}p^2x'^2+\frac{25}{96}x'^2-\frac{83}{256}px'^4\right)x'^2$$

$$+\left(-\frac{17}{4}p^2+2x'^2\right)x'\xi'-\frac{7}{2}px'\varphi'-2x'\tau'-\left(A+\frac{3}{2}\right)x'y\xi',$$

$$U=\frac{2}{7}+\frac{2}{5}p^2+\frac{27}{448}p^7-\frac{1}{80}p^7+\frac{9}{64}p^6x+\frac{25}{64\cdot12}(8-p^3)p^4.$$

This is our starting inequality. U is represented as the following polynomial of x

$$8 - \frac{31}{4}x - \frac{81}{8}x^2 + \frac{1111}{48}x^3 - \frac{863}{48}x^4 + \frac{2291}{320}x^5$$

$$-\left(\frac{133}{128} + \frac{25}{96} + \frac{7}{40}\right)x^6 + \left(\frac{9}{112} + \frac{25}{64 \cdot 12} + \frac{1}{80}\right)x^7.$$

2. Case $1 \leqq p \leqq 2$, $x'^2/p^2 \leqq 1/40$ and $y \geqq 0$. In this case we put $A=1$, $B=5/4$, $C=1$ in (A). Then we have

$$\mathfrak{R}a_8 \leqq U + \left(-\frac{1}{384}p^5 + \frac{15}{16}p^4x + \frac{5}{6}p^2 - \frac{673}{192}p^3x'^2 - \frac{15}{4}p^2xx'^2 - \frac{5}{6}x'^2 + \frac{129}{128}px'^4\right)y$$

$$+ \left(\frac{3}{64}p^4 + \frac{3}{4}p^3x - \frac{63}{32}p^2x'^2 - 3pxx'^2 + \frac{11}{64}x'^4\right)\eta$$

$$+ \left(-\frac{97}{48}p^3 + \frac{25}{8}p^2 + \frac{2}{3} - \frac{7}{8}px'^2\right)y^2 + \left(-\frac{19}{8}p^2 + 5p - \frac{25}{8}x'^2\right)y\eta$$

$$+ \left(-\frac{3}{4}p + 2\right)\eta^2 + py\xi + 2\eta\xi + y\varphi - \frac{3}{8}py^3 - y^2\eta$$

$$+ \left(\frac{113}{64 \cdot 12}p^5 - \frac{9}{4}p^4 - \frac{25}{48}p^2 - \frac{331}{64 \cdot 12}p^3x'^2 + \frac{9}{2}p^2x'^2 + \frac{25}{96}x'^2 - \frac{83}{256}px'^4\right)x'^2$$

$$+ \left(-\frac{17}{4}p^2 + 2x'^2\right)x'\xi' - \frac{7}{2}px'\varphi' - 2x'\tau' - \frac{5}{2}x'y\xi'.$$

Let $\mathfrak{A}$ be a constant, to be fixed later, such that $3p^2 - \mathfrak{A}x'^2 \geqq 0$. Applying Lemma 3 to $(3p^4/64 - \mathfrak{A}p^2x'^2/64)(\eta + 13py/6)$ and Lemma 2 to $(3p^3x/4)(\eta + 5py/4)$ we have

$$\mathfrak{R}a_8 \leqq \frac{2}{7} + \frac{1}{8}p + \frac{2}{5}p^2 + \frac{5}{3}p^3 - \frac{23}{96}p^4 + \frac{409}{192}p^5 - \frac{155}{64}p^6 + \frac{9}{64}p^6x$$

$$+ \left(\frac{27}{448} - \frac{1}{80} - \frac{25}{64 \cdot 12} - \frac{1}{32 \cdot 16} + \frac{163}{256}\right)p^7 + \frac{1}{24}p^2(10 + 2p + p^2)xy$$

$$+ \left\{\left(\frac{5}{384}\mathfrak{A} + \frac{83}{192}\right)p^3 - \frac{15}{2}p^2 + \frac{\mathfrak{A}}{12}p - \frac{5}{6} + \left(\frac{63}{64} - \frac{1}{16}\mathfrak{A}\right)px'^2\right.$$

$$\left. + \frac{1}{128}\mathfrak{A}\frac{x'^4}{p}\right\}x'^2y$$

$$+ \left\{\left(\frac{1}{64}\mathfrak{A} + \frac{75}{64}\right)p^2 - 6p + \left(\frac{11}{64} - \frac{3}{64}\mathfrak{A}\right)x'^2\right\}x'^2\eta$$

$$+ \left\{-\frac{851}{384}p^3 + \frac{25}{8}p^2 - \frac{3}{32}p + \frac{2}{3} + \left(\frac{25}{32 \cdot 12}\mathfrak{A} - \frac{121}{32 \cdot 4}\right)px'^2\right.$$

$$\left. + \frac{\mathfrak{A}}{32}\frac{x'^2}{p} + \frac{3}{128}\mathfrak{A}\frac{x'^4}{p}\right\}y^2$$

$$+\left\{-\frac{91}{32}p^2+5p+\left(\frac{5}{32}\mathfrak{A}-\frac{25}{8}\right)x'^2\right\}y\eta+\left(-\frac{33}{32}p+2+\frac{3}{32}\mathfrak{A}\frac{x'^2}{p}\right)\eta^2$$

$$+py\xi+2\eta\xi+y\varphi-\frac{3}{8}py^3-y^2\eta$$

$$+\left\{\left(\frac{1}{32\cdot48}\mathfrak{A}-\frac{71}{32\cdot48}\right)p^5-\frac{45}{24}p^4+\left(\frac{1}{18}\mathfrak{A}-\frac{1}{6}\right)p^3-\frac{25}{48}p^2-\frac{2}{9}\mathfrak{A}p\right.$$

$$-\frac{1}{24}\mathfrak{A}\frac{1}{p}$$

$$+\left(\frac{1}{64\cdot8}\mathfrak{A}-\frac{671}{64\cdot24}\right)p^3x'^2+\frac{9}{2}p^2x'^2+\frac{1}{18}\mathfrak{A}px'^2+\frac{25}{96}x'^2$$

$$+\left.\left(\frac{\mathfrak{A}}{32\cdot16}-\frac{167}{32\cdot16}\right)px'^4+\frac{\mathfrak{A}}{48\cdot32}\frac{x'^6}{p}\right\}x'^2$$

$$+\left(-\frac{17}{4}p^2+2x'^2\right)x'\xi'-\frac{7}{2}px'\varphi'-2x'\tau'-\frac{5}{2}x'y\xi'.$$

We apply the following trivial inequality

$$2AXY\leqq|A|\alpha X^2+|A|\alpha'Y^2,\qquad\alpha'=1/\alpha,\qquad\alpha>0$$

to the term

$$\frac{1}{24}p^2(10+2p+p^2)xy.$$

Then we have

$$\Re a_8\leqq8-\frac{x}{96}\{160-(346+144\alpha)x+(2659+192\alpha)x^2-(3383.5+92\alpha)x^3$$

$$+(1845.95+20\alpha)x^4-(479.075+2\alpha)x^5\}$$

$$+\left[\left(\frac{5}{384}\mathfrak{A}+\frac{83}{192}\right)p^3-\frac{15}{2}p^2+\frac{1}{12}\mathfrak{A}p-\frac{5}{6}+\left(\frac{63}{64}-\frac{1}{16}\mathfrak{A}\right)px'^2\right.$$

$$+\left.\frac{1}{128}\mathfrak{A}\frac{x'^4}{p}\right]x'^2y$$

$$+\left\{\left(\frac{1}{64}\mathfrak{A}+\frac{75}{64}\right)p^2-6p+\left(\frac{11}{64}-\frac{3}{64}\mathfrak{A}\right)x'^2\right\}x'^2\eta$$

$$-\frac{1}{96}\left[\left\{-\frac{2}{\alpha}p^4+\left(212.75-\frac{4}{\alpha}\right)p^3-\left(300+\frac{20}{\alpha}\right)p^2+9p-64\right.\right.$$

(B) $$+(90.75-6.25\mathfrak{A})px'^2-3\mathfrak{A}\frac{x'^2}{p}-2.25\mathfrak{A}\frac{x'^4}{p}\right\}y^2$$

$$+\{273p^2-480p+(300-15\mathfrak{A})x'^2\}y\eta+\Big(99p-192-9\mathfrak{A}\frac{x'^2}{p}\Big)\eta^2$$

$$-96py\xi-192\eta\xi-96y\varphi\Big]$$

$$-\frac{1}{96}\Big[\Big\{(4.4375-0.0625\mathfrak{A})p^5+180p^4+(16-5.3334\mathfrak{A})p^3+50p^2$$

$$+21.3333\mathfrak{A}p+4\mathfrak{A}\frac{1}{p}$$

$$+(41.9375-0.1875\mathfrak{A})p^3x'^2-432p^2x'^2-5.3334\mathfrak{A}px'^2-25x'^2$$

$$+(31.3125-0.1875\mathfrak{A})px'^4-0.0625\mathfrak{A}\frac{x'^6}{p}\Big\}x'^2$$

$$+(408p^2-192x'^2)x'\xi'+336px'\varphi'+192x'\tau'\Big]-\frac{3}{8}py^3-y^2\eta-\frac{5}{2}x'y\xi'.$$

Case i) $1.8\leqq p\leqq2$.

We start from (B) with $\mathfrak{A}=45$, $\alpha=1/2$. We remark the following facts:

$$\Big(\frac{391}{384}p^3-\frac{15}{2}p^2+\frac{15}{4}p-\frac{5}{6}-\frac{117}{64}px'^2+\frac{45}{128}\frac{x'^4}{p}\Big)x'^2y+\frac{15}{8}p^3x'^2y\leqq0,$$

and by Lemma 1

$$-\frac{15}{8}p^3x'^2y-\frac{5}{2}x'y\xi'\leqq\frac{5}{6p^3}\sqrt{\frac{4-p^2}{3}}\xi'^2\leqq\frac{6.91}{96}\xi'^2$$

$$x'^2\eta\Big(\frac{15}{8}p^2-6p-\frac{31}{16}x'^2\Big)\leqq\Big(-\frac{15}{8}p^2+6p+\frac{31}{16}x'^2\Big)\sqrt{\frac{4-p^2}{5}}x'^2$$

$$\leqq\frac{1}{96}(-70.182p^2+224.5824p+72.5214x'^2)x'^2,$$

$$-\frac{3}{8}py^3-y^2\eta\leqq\frac{2}{3p}y\eta^2\leqq\frac{2}{3p}\sqrt{\frac{4-p^2}{3}}\eta^2\leqq\frac{17.904}{96}\eta^2.$$

Making use of these remarks we have

$$\Re a_8\leqq8-\frac{x}{96}(160-418x+2755x^2-3429.5x^3+1855.95x^4-480.075x^5)$$

$$-\frac{1}{96}\Big\{\Big(-4p^4+204.75p^3-340p^2+9p-64-190.5px'^2-135\frac{x'^2}{p}-101.25\frac{x'^4}{p}\Big)y^2$$

$$+(273p^2-480p-375x'^2)y\eta+\left(99p-209.904-405\frac{x'^2}{p}\right)\eta^2-96py\xi-192\eta\xi-96y\varphi\Big\}$$

$$-\frac{1}{96}\left\{\left(1.625p^5+180p^4-224.003p^3+120.182p^2+735.416p+180\frac{1}{p}\right.\right.$$

$$+33.5p^3x'^2-432p^2x'^2-240.003px'^2-97.522x'^2+22.875px'^4-2.813\frac{x'^6}{p}\Big)x'^2$$

$$+(408p^2-192x'^2)x'\xi'-6.91\xi'^2+336px'\varphi'+192x'\tau'\Big\}.$$

Applying Lemma 1 to $-145.2x/96$ we have

$$\Re a_8\leq8-\frac{x}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=14.8-381.7x+2755x^2-3429.5x^3+1855.95x^4-480.075x^5,$$

$$Q=\left(-4p^4+204.75p^3-340p^2+9p+44.9-190.5px'^2-135\frac{x'^2}{p}-101.25\frac{x'^4}{p}\right)y^2$$

$$+(273p^2-480p-375x'^2)y\eta+\left(99p-28.404-405\frac{x'^2}{p}\right)\eta^2$$

$$-96py\xi-192\eta\xi+254.1\xi^2-96y\varphi+326.7\varphi^2,$$

$$R=\left(1.625p^5+180p^4-224.003p^3+120.182p^2+735.416p+36.3+180\frac{1}{p}\right.$$

$$+33.5p^3x'^2-432p^2x'^2-240.003px'^2-97.522x'^2+22.875px'^4-2.813\frac{x'^6}{p}\Big)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+247.19\xi'^2+336px'\varphi'+326.7\varphi'^2+192x'\tau'+399.3\tau'^2.$$

It is easy to prove that $P'(x)$ is monotone increasing for $0\leq x\leq0.2$ and $P'(0.08)$ <0, $P'(0.081)>0$. Let λ be the root of $P'(x)=0$, $0.08<\lambda<0.081$. Construct $N(x)$ $=5P(x)-xP'(x)$. Then $N(x)$ is monotone decreasing for $0\leq x\leq0.1$. Further $N(0.081)>0$. Hence $N(x)>0$ for $0\leq x\leq0.081$. Especially $N(\lambda)>0$, which implies $P(\lambda)>0$. Therefore $P(x)>0$ for $0\leq x\leq0.2$.

Next we prove the positive definiteness of Q. Since

$$7.06y^2-96y\varphi+326.7\varphi^2\geq0,$$

we may consider $Q^*=Q-(7.06y^2-96y\varphi+326.7\varphi^2)$. We make the symmetric matrix associated with Q^*:

$$\begin{pmatrix} \begin{matrix} -4p^4+204.75p^3-340p^2+9p+37.84 \\ \\ -190.5px'^2-135\dfrac{x'^2}{p}-101.25\dfrac{x'^4}{p} \end{matrix} & 136.5p^2-240p-187.5x'^2 & -48p \\ \\ 136.5p^2-240p-187.5x'^2 & 99p-28.404-405\dfrac{x'^2}{p} & -96 \\ \\ -48p & -96 & 254.1 \end{pmatrix}$$

Its principal diagonal minor determinants are

254.1,

$$254.1\left(99p-28.404-405\,\frac{x'^2}{p}\right)-(-96)^2,$$

$$-100623.6p^5+481949.6256p^4+5760763.8021p^3-10968778.908p^2+803998.1484p$$

$$-621841.990176+411642p^3x'^2-12856380.075p^2x'^2+14456263.4442px'^2$$

$$-4322241x'^2-1675616.706\,\frac{x'^2}{p}+8124212.25x'^4+1663887.4605\,\frac{x'^4}{p}$$

$$+13892917.5\,\frac{x'^4}{p^2}+10419688.125\,\frac{x'^6}{p^2}\,.$$

All of them are positive for $1.8 \leqq p \leqq 2$, $x'^2/p^2 \leqq 1/40$. This implies the non-negativity of Q^* there. Thus Q is positive definite there.

We prove the positive definiteness of R. Since

$$23.09x'^2+192x'\tau'+399.3\tau'^2 \geqq 0,$$

we may consider $R^* = R-(23.09x'^2+192x'\tau'+399.3\tau'^2)$. We make the symmetric matrix associated with R^*:

$$\begin{pmatrix} \begin{matrix} 1.625p^5+180p^4-224.003p^3+120.182p^2+735.416p \\ \\ +13.21+180\dfrac{1}{p}+33.5p^3x'^2-432p^2x'^2-240.003px'^2 \\ \\ -97.522x'^2+22.875px'^4-2.813\dfrac{x'^6}{p} \end{matrix} & 204p^2-96x'^2 & 168p \\ \\ 204p^2-96x'^2 & 247.19 & 0 \\ \\ 168p & 0 & 326.7 \end{pmatrix}.$$

Its principal diagonal minor determinants are

326.7,

247.19·326.7,

$131230.081125 p^5 + 940307.94 p^4 - 18089804.222919 p^3 + 2728843.969086 p^2$

$+ 59389970.055768 p + 1066799.61333 + 14536255.14 \dfrac{1}{p} + 2705358.5955 p^3 x'^2$

$- 22090826.736 p^2 x'^2 - 19381915.790919 px'^2 - 7875581.520906 x'^2$

$+ 1847315.757375 px'^4 - 3010867.2 x'^4 - 227169.365049 \dfrac{x'^6}{p}.$

All of them are positive for $1.8 \leq p \leq 2$, $x'^2/p^2 \leq 1/40$. Thus R is positive definite there.

Therefore we have $\Re a_8 \leq 8$ for $1.8 \leq p \leq 2$, $x'^2/p^2 \leq 1/40$ and $y \geq 0$. Equality occurs only for $p = 2$, that is, for the Koebe function.

Case ii) $1.7 \leq p \leq 1.8$.

In this case we start from (B) with $\mathfrak{A} = 39$, $\alpha = 1/2$. We remark the following facts:

$$\left(\frac{361}{384} p^3 - \frac{15}{2} p^2 + \frac{13}{4} p - \frac{5}{6} - \frac{93}{64} px'^2 + \frac{39}{128} \frac{x'^4}{p} \right) x'^2 y + 2 p^3 x'^2 y \leq 0,$$

and by Lemma 1

$$-2 p^3 x'^2 y - \frac{5}{2} x' y \xi' \leq \frac{25}{32 p^3} y \xi'^2 \leq \frac{25}{32 p^3} \sqrt{\frac{4-p^2}{3}} \xi'^2 \leq \frac{9.286}{96} \xi'^2,$$

$$\left(\frac{57}{32} p^2 - 6p - \frac{53}{32} x'^2 \right) x'^2 \eta \leq \left(-\frac{57}{32} p^2 + 6p + \frac{53}{32} x'^2 \right) \sqrt{\frac{4-p^2}{5}} x'^2$$

$$\leq \frac{1}{96} (-80.5752 p^2 + 271.4112 p + 74.9208 x'^2) x'^2,$$

$$-\frac{3}{8} py^3 - y^2 \eta \leq \frac{2}{3p} y \eta^2 \leq \frac{2}{3p} \sqrt{\frac{4-p^2}{3}} \eta^2 \leq \frac{22.906}{96} \eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96 - 52.52x^2/96$ we have

$$\Re a_8 \leqq 8 - \frac{x^2}{96} P(x) - \frac{1}{96} Q - \frac{1}{96} R,$$

$$P(x) = -430.52 + 2768.13x - 3429.5x^2 + 1855.95x^3 - 480.075x^4,$$

$$Q = \left(-4p^4 + 204.75p^3 - 340p^2 - 30.39p + 134.78 - 153px'^2 - 117\frac{x'^2}{p} - 87.75\frac{x'^4}{p} \right)y^2$$

$$+ (273p^2 - 480p - 285x'^2)y\eta + \left(33.35p + 116.394 - 351\frac{x'^2}{p} \right)\eta^2 - 96py\xi - 192\eta\xi$$

$$+ (463.82 - 91.91p)\xi^2 - 96y\varphi + (596.34 - 118.17p)\varphi^2,$$

$$R = \left(2p^5 + 180p^4 - 192.003p^3 + 130.575p^2 + 547.457p + 66.26 + 156\frac{1}{p} \right.$$

$$+ 34.625p^3x'^2 - 432p^2x'^2 - 208.003px'^2 - 99.921x'^2 + 24px'^4 - 2.438\frac{x'^6}{p} \bigg)x'^2$$

$$+ (408p^2 - 192x'^2)x'\xi' + (454.534 - 91.91p)\xi'^2 + 336px'\varphi' + (596.34 - 118.17p)\varphi'^2$$

$$+ 192x'\tau' + (728.86 - 144.43p)\tau'^2.$$

$P(x)$ is monotone increasing for $0.2 \leqq x \leqq 0.3$ and $P(0.2) > 0$. Hence $P(x) > 0$ for $0.2 \leqq x \leqq 0.3$.

Since $(2.6 + 1.9p)y^2 - 96y\varphi + (596.34 - 118.17p)\varphi^2 \geqq 0$ for $1.7 \leqq p \leqq 1.8$ we may consider $Q^* = Q - (2.6 + 1.9p)y^2 + 96y\varphi - (596.34 - 118.17p)\varphi^2$. We make the symmetric matrix associated with Q^*:

$$\begin{pmatrix} -4p^4 + 204.75p^3 - 340p^2 - 32.29p & 136.5p^2 - 240p - 142.5x'^2 & -48p \\ + 132.18 - 153px'^2 - 117\dfrac{x'^2}{p} - 87.75\dfrac{x'^4}{p} & & \\ 136.5p^2 - 240p - 142.5x'^2 & 33.35p + 116.394 - 351\dfrac{x'^2}{p} & -96 \\ -48p & -96 & 463.82 - 91.91p \end{pmatrix}.$$

We can prove the positive definiteness of this matrix for $1.7 \leqq p \leqq 1.8$, $x'^2/p^2 \leqq 1/40$ by taking its principal diagonal minor determinants. Thus Q is positive definite for $1.7 \leqq p \leqq 1.8$, $x'^2/p^2 \leqq 1/40$.

Since $(8.428 + 6.259p)x'^2 + 192x'\tau' + (728.86 - 144.43p)\tau'^2 \geqq 0$ for $1.7 \leqq p \leqq 1.8$ we may consider $R^* = R - (8.428 + 6.259p)x'^2 - 192x'\tau' - (728.86 - 144.43p)\tau'^2$. We make the symmetric matrix associated with R^*:

$$\left(\begin{array}{ccc}
\begin{aligned}& 2p^5+180p^4-192.003p^3+130.575p^2+541.198p \\& +57.832+156\,\frac{1}{p}+34.625p^3x'^2-432p^2x'^2 \\& -208.003px'^2-99.921x'^2+24px'^4-2.438\,\frac{x'^6}{p}\end{aligned} & 204p^2-96x'^2 & 168p \\[2em]
204p^2-96x'^2 & 454.534-91.91p & 0 \\[1em]
168p & 0 & 596.34-118.17p
\end{array}\right).$$

By an easy calculation we can prove the positive definiteness of this matrix for $1.7\leqq p\leqq1.8$, $x'^2/p^2\leqq1/40$. Thus R is positive definite there.

Therefore we have $\Re a_8<8$ for $1.7\leqq p\leqq1.8$, $x'^2/p^2\leqq1/40$, $y\geqq0$.

Case iii) $1.5\leqq p\leqq1.7$.

In this case we start from (B) with $\mathfrak{A}=29$, $\alpha=1/2$. We remark the following facts:

$$\left(\frac{311}{384}p^3-\frac{15}{2}p^2+\frac{29}{12}p-\frac{5}{6}-\frac{53}{64}px'^2+\frac{29}{128}\,\frac{x'^4}{p}\right)x'^2y+2.93p^3x'^2y\leqq0,$$

and by Lemma 1

$$-2.93p^3x'^2y-\frac{5}{2}x'y\xi'\leqq\frac{25}{46.88p^3}\,y\xi'^2\leqq\frac{25}{46.88p^3}\,\sqrt{\frac{4-p^2}{3}}\,\xi'^2\leqq\frac{11.59}{96}\,\xi'^2,$$

$$\left(\frac{13}{8}p^2-6p-\frac{19}{16}x'^2\right)x'^2\eta\leqq\left(-\frac{13}{8}p^2+6p+\frac{19}{16}x'^2\right)\sqrt{\frac{4-p^2}{5}}\,x'^2$$

$$\leqq\frac{1}{96}\,(-92.3052p^2+340.8192p+67.4538x'^2)x'^2,$$

$$-\frac{3}{8}py^3-y^2\eta\leqq\frac{2}{3p}\,y\eta^2\leqq\frac{2}{3p}\,\sqrt{\frac{4-p^2}{3}}\,\eta^2\leqq\frac{32.592}{96}\,\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96$ $-201.08x^2/96$ we have

$$\Re a_8\leqq8-\frac{x^2}{96}\,P(x)-\frac{1}{96}Q-\frac{1}{96}\,R,$$

$$P(x)=-579.08+2805.27x-3429.5x^2+1855.95x^3-480.075x^4,$$

$$Q=\left(-4p^4+204.75p^3-340p^2-141.81p+357.62-90.5px'^2-87\,\frac{x'^2}{p}-65.25\,\frac{x'^4}{p}\right)y^2$$

$$+(273p^2-480p-135x'^2)y\eta+\left(-152.35p+478.108-261\frac{x'^2}{p}\right)\eta^2-96py\xi-192\eta\xi$$

$$+(983.78-351.89p)\xi^2-96y\varphi+(1264.86-452.43p)\varphi^2,$$

$$R=\left(2.625p^5+180p^4-138.669p^3+142.305p^2+227.576p+140.54+116\frac{1}{p}\right.$$

$$\left.+36.5p^3x'^2-432p^2x'^2-154.669px'^2-92.454x'^2+25.875px'^4-1.813\frac{x'^6}{p}\right)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+(972.19-351.89p)\xi'^2+336px'\varphi'$$

$$+(1264.86-452.43p)\varphi'^2+192x'\tau'+(1545.94-552.97p)\tau'^2.$$

$P(x)$ is monotone increasing for $0.3\leq x\leq0.5$ and $P(0.3)>0$. Hence $P(x)>0$ for $0.3\leq x\leq0.5$.

Since $(4.9p-3.4)y^2-96y\varphi+(1264.86-452.43p)\varphi^2\geq0$ for $1.5\leq p\leq1.7$, we may consider $Q^*=Q-(4.9p-3.4)y^2+96y\varphi-(1264.86-452.43p)\varphi^2$. We make the symmetric matrix associated with Q^*:

$$\begin{pmatrix} -4p^4+204.75p^3-340p^2-146.71p+361.02 & 136.5p^2-240p-67.5x'^2 & -48p \\ -90.5px'^2-87\dfrac{x'^2}{p}-65.25\dfrac{x'^4}{p} & & \\ 136.5p^2-240p-67.5x'^2 & -152.35p+478.108-261\dfrac{x'^2}{p} & -96 \\ -48p & -96 & 983.78-351.89p \end{pmatrix}.$$

It is easy to prove that this matrix is positive definite for $1.5\leq p\leq1.7$, $x'^2/p^2\leq1/40$. Hence Q is positive definite there.

Since $(16.166p-11.385)x'^2+192x'\tau'+(1545.94-552.97p)\tau'^2\geq0$ for $1.5\leq p\leq1.7$, we may consider $R^*=R-(16.166p-11.385)x'^2-192x'\tau'-(1545.94-552.97p)\tau'^2$. We make the symmetric matrix associated with R^*:

$$\begin{pmatrix} 2.625p^5+180p^4-138.669p^3+142.305p^2 & 204p^2-96x'^2 & 168p \\ +211.41p+151.925+116\dfrac{1}{p}+36.5p^3x'^2 & & \\ -432p^2x'^2-154.669px'^2-92.454x'^2 & & \\ +25.875px'^4-1.813\dfrac{x'^6}{p} & & \\ 204p^2-96x'^2 & 972.19-351.89p & 0 \\ 168p & 0 & 1264.86-452.43p \end{pmatrix}.$$

It is easy to prove that this matrix is positive definite for $1.5 \leq p \leq 1.7$, $x'^2/p^2 \leq 1/40$. Hence R is positive definite there.

Therefore we have $\Re a_8 < 8$ for $1.5 \leq p \leq 1.7$, $x'^2/p^2 \leq 1/40$, $y \geq 0$.

Case iv) $1.3 \leq p \leq 1.5$.

In this case we start from (B) with $\mathfrak{A} = 60$, $\alpha = 1/2$. We remark the following facts:

$$\left(\frac{233}{192} p^3 - \frac{15}{2} p^2 + 5p - \frac{5}{6} - \frac{177}{64} px'^2 + \frac{15}{32} \frac{x'^4}{p} \right) x'^2 y + \frac{8}{5} p^3 x'^2 y \leq 0,$$

$$- \frac{8}{5} p^3 x'^2 y - \frac{5}{2} x' y \xi' \leq \frac{125}{128 p^3} y \xi'^2 \leq \frac{125}{128 p^3} \sqrt{\frac{4-p^2}{3}} \xi'^2 \leq \frac{37.445}{96} \xi'^2,$$

$$\left(\frac{135}{64} p^2 - 6p - \frac{169}{64} x'^2 \right) x'^2 \eta \leq \left(-\frac{135}{64} p^2 + 6p + \frac{169}{64} x'^2 \right) \sqrt{\frac{4-p^2}{5}} x'^2$$

$$\leq \frac{1}{96} (-137.6595 p^2 + 391.5648 p + 172.3293 x'^2) x'^2,$$

$$- \frac{3}{8} py^3 - y^2 \eta \leq \frac{2}{3p} y \eta^2 \leq \frac{2}{3p} \sqrt{\frac{4-p^2}{3}} \eta^2 \leq \frac{43.2}{96} \eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96 - 393.2x^2 /96$ we have

$$\Re a_8 \leq 8 - \frac{x^2}{96} P(x) - \frac{1}{96} Q - \frac{1}{96} R,$$

$$P(x) = -771.2 + 2853.3x - 3429.5x^2 + 1855.95x^3 - 480.75x^4,$$

$$Q = \left(-4p^4 + 204.75p^3 - 340p^2 - 285.9p + 645.8 - 284.25 px'^2 - 180 \frac{x'^2}{p} - 135 \frac{x'^4}{p} \right) y^2$$

$$+ (273p^2 - 480p - 600x'^2) y\eta + \left(-392.5p + 947.8 - 540 \frac{x'^2}{p} \right) \eta^2 - 96 py\xi$$

$$- 192 \eta\xi + (1656.2 - 688.1p) \xi^2 - 96 y\varphi + (2129.4 - 884.7p) \varphi^2,$$

$$R = \left(0.6875p^5 + 180p^4 - 304.004p^3 + 187.659p^2 + 790.133p + 236.6 + 240 \frac{1}{p} \right.$$

$$\left. + 30.687 p^3 x'^2 - 432 p^2 x'^2 - 320.004 px'^2 - 197.33 x'^2 + 20.062 px'^4 - 3.75 \frac{x'^6}{p} \right) x'^2$$

$$+ (408p^2 - 192 x'^2) x' \xi' + (1618.755 - 688.1p) \xi'^2 + 336 px' \varphi'$$

$$+ (2129.4 - 884.7p) \varphi'^2 + 192 x' \tau' + (2602.6 - 1081.3p) \tau'^2.$$

$P(x)$ is monotone increasing for $0.5 \leq x \leq 0.7$ and $P(0.5) > 0$. Hence $P(x) > 0$ for $0.5 \leq x \leq 0.7$.

Since $(5.7p-5)y^2 - 96y\varphi + (2129.4 - 884.7p)\varphi^2 \geq 0$ for $1.3 \leq p \leq 1.5$, we may consider $Q^* = Q - (5.7p-5)y^2 + 96y\varphi - (2129.4 - 884.7p)\varphi^2$. We make the symmetric matrix associated with Q^*:

$$\begin{pmatrix} \begin{matrix} -4p^4 + 204.75p^3 - 340p^2 - 291.6p + 650.8 \\ -284.25px'^2 - 180\,\dfrac{x'^2}{p} - 135\,\dfrac{x'^4}{p} \end{matrix} & 136.5p^2 - 240p - 300x'^2 & -48p \\ 136.5p^2 - 240p - 300x'^2 & -392.5p + 947.8 - 540\,\dfrac{x'^2}{p} & -96 \\ -48p & -96 & 1656.2 - 688.1p \end{pmatrix}.$$

It is easy to prove that this matrix is positive definite for $1.3 \leq p \leq 1.5$, $x'^2/p^2 \leq 1/40$. Hence Q is positive definite there.

Since $(18.923p - 16.899)x'^2 + 192x'\tau' + (2602.6 - 1081.3p)\tau'^2 \geq 0$ for $1.3 \leq p \leq 1.5$, we may consider $R^* = R - (18.923p - 16.899)x'^2 - 192x'\tau' - (2602.6 - 1081.3p)\tau'^2$. We make the symmetric matrix associated with R^*:

$$\begin{pmatrix} \begin{matrix} 0.6875p^5 + 180p^4 - 304.004p^3 + 187.659p^2 \\ + 771.21p + 253.499 + 240\,\dfrac{1}{p} + 30.687p^3x'^2 \\ - 432p^2x'^2 - 320.004px'^2 - 197.33x'^2 \\ + 20.062px'^4 - 3.75\,\dfrac{x'^6}{p} \end{matrix} & 204p^2 - 96x'^2 & 168p \\ 204p^2 - 96x'^2 & 1618.755 - 688.1p & 0 \\ 168p & 0 & 2129.4 - 884.7p \end{pmatrix}.$$

It is easy to prove that this matrix is positive definite for $1.3 \leq p \leq 1.5$, $x'^2/p^2 \leq 1/40$. Hence R is positive definite there.

Therefore we have $\Re a_8 < 8$ for $1.3 \leq p \leq 1.5$, $x'^2/p^2 \leq 1/40$, $y \geq 0$.

Case v) $1 \leq p \leq 1.3$.

In this case we start from (B) with $\mathfrak{A} = 60$, $\alpha = 1/2$. We remark the following facts:

$$\left(\frac{233}{192}p^3 - \frac{15}{2}p^2 + 5p - \frac{5}{6} - \frac{177}{64}px'^2 + \frac{15}{32}\frac{x'^4}{p} \right)x'^2 y + \frac{8}{5}p^3x'^2 y \leq 0,$$

$$-\frac{8}{5}p^3x'^2 y - \frac{5}{2}x'y\xi' \leq \frac{125}{128p^3}y\xi'^2 \leq \frac{125}{128p^3}\sqrt{\frac{4-p^2}{3}}\xi'^2 \leq \frac{93.75}{96}\xi'^2,$$

$$\left(\frac{135}{64}p^2-6p-\frac{169}{64}x'^2\right)x'^2\eta\leqq\left(-\frac{135}{64}p^2+6p+\frac{169}{64}x'^2\right)\sqrt{\frac{4-p^2}{5}}\,x'^2$$

$$\leqq\frac{1}{96}(-156.8565p^2+446.1696p+196.3611x'^2)x'^2,$$

$$-\frac{3}{8}py^3-y^2\eta\leqq\frac{2}{3p}\,y\eta^2\leqq\frac{2}{3p}\sqrt{\frac{4-p^2}{3}}\,\eta^2\leqq\frac{64}{96}\,\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96$ $-428x^2/96$ we have

$$\Re a_8\leqq8-\frac{x^2}{96}\,P(x)-\frac{1}{96}\,Q-\frac{1}{96}\,R,$$

$$P(x)=-806+2862x-3429.5x^2+1855.95x^3-480.075x^4,$$

$$Q=\left(-4p^4+204.75p^3-340p^2-312p+698-284.25px'^2-180\,\frac{x'^2}{p}-135\,\frac{x'^4}{p}\right)y^2$$

$$+(273p^2-480p-600x'^2)y\eta+\left(-436p+1014-540\,\frac{x'^2}{p}\right)\eta^2-96py\xi-192\eta\xi$$

$$+(1778-749p)\xi^2-96y\varphi+(2286-963p)\varphi^2,$$

$$R=\left(0.6875p^5+180p^4-304.004p^3+206.856p^2+726.828p+254+240\,\frac{1}{p}\right.$$

$$+30.687p^3x'^2-432p^2x'^2-320.004px'^2-221.361x'^2+20.062px'^4-3.75\,\frac{x'^6}{p}\left.\right)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+(1684.25-749p)\xi'^2+336px'\varphi'+(2286-963p)\varphi'^2$$

$$+192x'\tau'+(2794-1177p)\tau'^2.$$

$P'(x)$ is monotone decreasing for $0.7\leqq x\leqq1$ and $P'(0.7)>0$, $P'(1)<0$. Further $P(0.7)$ >0, $P(1)>0$. Hence $P(x)>0$ for $0.7\leqq x\leqq1$.

Since $(4.6p-2.8)y^2-96y\varphi+(2286-963p)\varphi^2\geqq0$ for $1\leqq p\leqq1.3$, we may consider $Q^*=Q-(4.6p-2.8)y^2+96y\varphi-(2286-963p)\varphi^2$. We make the symmetric matrix associated with Q^*:

$$\left(\begin{array}{ccc}
-4p^4+204.75p^3-340p^2-316.6p+700.8 & 136.5p^2-240p-300x'^2 & -48p \\
-284.25px'^2-180\,\dfrac{x'^2}{p}-135\,\dfrac{x'^4}{p} & & \\
136.5p^2-240p-300x'^2 & -436p+1014-540\,\dfrac{x'^2}{p} & -96 \\
-48p & -96 & 1778-749p
\end{array}\right).$$

It is easy to prove that this matrix is positive definite for $1 \leq p \leq 1.3$, $x'^2/p^2 \leq 1/40$. Hence Q is positive definite there.

Since $(15.24p - 9.53)x'^2 + 192x'\tau' + (2794 - 1177p)\tau'^2 \geq 0$ for $1 \leq p \leq 1.3$, we may consider $R^* = R - (15.24p - 9.53)x'^2 - 192x'\tau' - (2794 - 1177p)\tau'^2$. We make the symmetric matrix associated with R^*:

$$\begin{pmatrix} \begin{aligned} & 0.6875p^5 + 180p^4 - 304.004p^3 + 206.856p^2 \\ & +711.588p + 263.53 + 240\,\frac{1}{p} + 30.687p^3x'^2 \\ & -432p^2x'^2 - 320.004px'^2 - 221.361x'^2 \\ & +20.062px'^4 - 3.75\,\frac{x'^6}{p} \end{aligned} & 204p^2 - 96x'^2 & 168p \\ \\ 204p^2 - 96x'^2 & 1684.25 - 749p & 0 \\ \\ 168p & 0 & 2286 - 963p \end{pmatrix}.$$

It is easy to prove that this matrix is positive definite for $1 \leq p \leq 1.3$, $x'^2/p^2 \leq 1/40$. Hence R is positive definite there.

Therefore we have $\Re a_8 < 8$ for $1 \leq p \leq 1.3$, $x'^2/p^2 \leq 1/40$, $y \geq 0$.

3. Case $1 \leq p \leq 2$, $x'^2/p^2 \leq 1/40$ and $y \leq 0$. In this case we put $A = 1/2$, $B = 3/2$, $C = 1$ in (A).

Applying the trivial inequality $2AXY \leq |A|X^2 + |A|Y^2$ and Lemma 1

$$\left(-\frac{413}{384}p^5 + \frac{9}{4}p^4 + \frac{5}{12}p^2 + \frac{151}{192}p^3x'^2 - 9p^2x'^2 - \frac{5}{12}x'^2 + \frac{149}{128}px'^4\right)y - 2x'y\xi'$$

$$\leq -\frac{y}{384}(413p^5 - 864p^4 - 160p^2 - 302p^3x'^2 + 3456p^2x'^2 + 160x'^2 - 447px'^4 + 384x'^2 + 384\xi'^2)$$

$$\leq -\frac{y}{384}(413p^5 - 864p^4 - 214.86p^2 + 219.5 - 302p^3x'^2 + 3456p^2x'^2 + 489.14x'^2 - 447px'^4)$$

$$\leq 0$$

for $1 \leq p \leq 2$, $x'^2/p^2 \leq 1/40$, $y \leq 0$. Then we have

$$\Re a_8 \leq U + \left(-\frac{45}{64}p^4 + \frac{3}{2}p^3 + \frac{33}{32}p^2x'^2 - 6px'^2 + \frac{11}{64}x'^4\right)\eta$$

$$+ \left(-\frac{127}{48}p^3 + \frac{9}{2}p^2 + \frac{1}{6} - \frac{35}{16}px'^2\right)y^2 + (-3p^2 + 6p - 4x'^2)y\eta + \left(-\frac{3}{4}p + 2\right)\eta^2$$

$$(C) \qquad + py\xi + 2\eta\xi + 2y\varphi - \frac{1}{4}y^2\eta$$

$$+\left(\frac{113}{64\cdot12}p^5-\frac{9}{4}p^4-\frac{25}{48}p^2-\frac{331}{64\cdot12}p^3x'^2+\frac{9}{2}p^2x'^2+\frac{25}{96}x'^2-\frac{83}{256}px'^4\right)x'^2$$

$$+\left(-\frac{17}{4}p^2+2x'^2\right)x'\xi'-\frac{7}{2}px'\varphi'-2x'\tau'.$$

Case i) $\eta\geqq0$.   By Lemma 2

$$\eta\leqq-\frac{7}{48}p^3+\frac{1}{8}p^2+\frac{2}{3}+\frac{1}{4}px'^2,$$

whence we have

$$\left(-\frac{45}{64}p^4+\frac{3}{2}p^3+\frac{33}{32}p^2x'^2-6px'^2+\frac{11}{64}x'^4\right)\eta$$

$$\leqq\left\{\left(\frac{3}{64}p^4-\frac{60}{32}p^2x'^2\right)+\left(\frac{3}{4}p^3x-3pxx'^2\right)\right\}\eta$$

$$\leqq\frac{105}{64\cdot16}p^7-\frac{157}{64\cdot8}p^6+\frac{3}{16}p^5-\frac{15}{32}p^4+p^3-\frac{87}{64\cdot4}p^5x'^2+\frac{89}{64}p^4x'^2$$

$$-\frac{3}{4}p^3x'^2+\frac{3}{4}p^2x'^2-4px'^2+\frac{9}{32}p^3x'^4-\frac{3}{2}p^2x'^4.$$

By Lemma 1

$$-\frac{654}{96}x\leqq-\frac{163.5}{96}(x^2+x'^2+3y^2+5\eta^2+7\xi^2+7\xi'^2+9\varphi^2+9\varphi'^2+11\tau'^2).$$

Making use of these inequalities and (C) we have

$$\Re a_8\leqq8-\frac{x^2}{96}P_0(x)-\frac{1}{96}Q_0-\frac{1}{96}R_0,$$

$$P_0(x)=649.5-963.5x+601x^2-195.675x^3+33.175x^4-2.1956x^5,$$

$$Q_0=(254p^3-432p^2+474.5+210px'^2)y^2+(288p^2-576p+384x'^2)y\eta$$

$$+(72p+625.5)\eta^2-96py\xi+1144.5\xi^2-192\eta\xi-192y\varphi+1471.5\varphi^2,$$

$$R_0=(18.5p^5+82.5p^4+72p^3-22p^2+384p+163.5+14.375p^3x'^2$$

$$-288p^2x'^2-25x'^2+31.125px'^4)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+1144.5\xi'^2+336px'\varphi'+1471.5\varphi'^2+192x'\tau'+1798.5\tau'^2.$$

$P_0(x)$ is monotone decreasing for $0\leqq x\leqq1$ and $P_0(1)>0$.   Hence $P_0(x)>0$ for $0\leqq x\leqq1$.

Q_0 is positive definite for $1\leqq p\leqq2$, $x'^2/p^2\leqq1/40$.   Indeed we have

$$6.3y^2 - 192y\varphi + 1471.5\varphi^2 \geqq 0,$$

$$29.4\eta^2 - 192\eta\xi + 314\xi^2 \geqq 0,$$

$$2.8p^2y^2 - 96py\xi + 830.5\xi^2 \geqq 0$$

and

$$(254p^3 - 434.8p^2 + 468.2 + 210px'^2)y^2 + (288p^2 - 576p + 384x'^2)y\eta + (72p + 596.1)\eta^2 \geqq 0$$

for $1 \leqq p \leqq 2$, $x'^2/p^2 \leqq 1/40$. This implies the positive definiteness of Q_0 for $1 \leqq p \leqq 2$, $x'^2/p^2 \leqq 1/40$.

R_0 is positive definite for $1 \leqq p \leqq 2$, $x'^2/p^2 \leqq 1/40$. Indeed we have

$$5.13x'^2 + 192x'\tau' + 1798.5\tau'^2 \geqq 0,$$

$$19.2p^2x'^2 + 336px'\varphi' + 1471.5\varphi'^2 \geqq 0$$

and

$$(18.5p^5 + 82.5p^4 + 72p^3 - 41.2p^2 + 384p + 158.37 + 14.375p^3x'^2 - 288p^2x'^2 - 25x'^2$$

$$+ 31.125px'^4)x'^2 + (408p^2 - 192x'^2)x'\xi' + 1144.5\xi'^2 \geqq 0$$

for $1 \leqq p \leqq 2$, $x'^2/p^2 \leqq 1/40$, which imply the desired result.

Thus we have $\Re a_8 \leqq 8$ in the present case with equality holding only for $p = 2$.

Case ii) $\eta \leqq 0$. Applying Lemma 1

$$\left(-\frac{45}{64}p^4 + \frac{3}{2}p^3 + \frac{33}{32}p^2x'^2 - 6px'^2 + \frac{11}{64}x'^4\right)\eta - \frac{1}{4}y^2\eta$$

$$= \left\{\left(\frac{3}{64}p^4 - \frac{60}{32}p^2x'^2\right) + \left(\frac{3}{4}p^3x - 3pxx'^2 - \frac{1}{4}y^2\right) + \frac{11}{64}x'^4 - \frac{3}{32}p^2x'^2\right\}\eta$$

$$\leqq \left(\frac{11}{64}x'^2 - \frac{3}{32}p^2\right)x'^2\eta \leqq \frac{1}{96}(6.972p^2 - 12.78x'^2)x'^2.$$

By Lemma 1

$$-\frac{744}{96}x \leqq -\frac{186}{96}(x^2 + x'^2 + 3y^2 + 5\eta^2 + 7\xi^2 + 7\xi'^2 + 9\varphi^2 + 9\varphi'^2 + 11\tau'^2).$$

Making use of these inequalities and (C) we have

$$\Re a_8 \leqq 8 - \frac{x^2}{96}P_1(x) - \frac{1}{96}Q_1 - \frac{1}{96}R_1,$$

$$P_1(x) = 1158 - 2222x + 1726x^2 - 687.3x^3 + 141.55x^4 - 12.04x^5,$$

$$Q_1 = (254p^3 - 432p^2 + 542 + 210px'^2)y^2 + (288p^2 - 576p + 384x'^2)y\eta$$

$$+ (72p + 738)\eta^2 - 96py\xi - 192\eta\xi + 1302\xi^2 - 192y\varphi + 1674\varphi^2,$$

$$R_1 = (-14.125p^5 + 216p^4 + 43.028p^2 + 186 + 41.375p^3x'^2 - 432p^2x'^2$$

$$-12.22x'^2 + 31.125px'^4)x'^2 + (408p^2 - 192x'^2)x'\xi' + 1302\xi'^2 + 336px'\varphi'$$

$$+1674\varphi'^2 + 192x'\tau' + 2046\tau'^2.$$

$P_1(x)$ is monotone decreasing for $0 \leq x \leq 1$ and $P_1(1) > 0$. Hence $P_1(x) > 0$ for $0 \leq x \leq 1$.

Since $Q_1 \geq Q_0$ for $1 \leq p \leq 2$, $x'^2/p^2 \leq 1/40$, Q_1 is positive definite for $1 \leq p \leq 2$, $x'^2/p^2 \leq 1/40$.

R_1 is positive definite for $1 \leq p \leq 2$, $x'^2/p^2 \leq 1/40$. Indeed we have

$$4.6x'^2 + 192x'\tau' + 2046\tau'^2 \geq 0,$$

$$16.9p^2x'^2 + 336px'\varphi' + 1674\varphi'^2 \geq 0$$

and

$$(-14.125p^5 + 216p^4 + 26.128p^2 + 181.4 + 41.375p^3x'^2 - 432p^2x'^2 - 12.22x'^2$$

$$+ 31.125px'^4)x'^2 + (408p^2 - 192x'^2)x'\xi' + 1302\xi'^2 \geq 0$$

for $1 \leq p \leq 2$, $x'^2/p^2 \leq 1/40$, which imply the desired result.

Thus we have $\Re a_8 \leq 8$ in the present case with equality holding only for $p = 2$.

4. Case $1 \leq p$, $1/40 \leq x'^2/p^2 \leq 1/20$ and $y \geq 0$. In this case we put $A = 1$, $B = 5/4$, $C = 3/5$ in (A). Then we have

$$\Re a_8 \leq U + \left(-\frac{1}{384}p^5 + \frac{15}{16}p^4x + \frac{5}{6}p^2 - \frac{673}{192}p^3x'^2 - \frac{15}{4}p^2xx'^2 - \frac{5}{6}x'^2 + \frac{129}{128}px'^4\right)y$$

$$+ \left(\frac{3}{64}p^4 + \frac{9}{20}p^3x - \frac{27}{160}p^2x'^2 - \frac{18}{5}px'^2 + \frac{11}{64}x'^4\right)\eta$$

$$+ \left(-\frac{97}{48}p^3 + \frac{25}{8}p^2 + \frac{2}{3} - \frac{7}{8}px'^2\right)y^2 + \left(-\frac{7}{8}p^2 + 3p - \frac{29}{8}x'^2\right)y\eta$$

$$+ \left(\frac{69}{100}p + \frac{18}{25}\right)\eta^2 + py\xi + \frac{14}{5}\eta\xi + y\varphi - \frac{3}{8}py^3 - \frac{1}{5}y^2\eta$$

$$+ \left(\frac{113}{64\cdot12}p^5 - \frac{9}{4}p^4 - \frac{25}{48}p^2 - \frac{331}{64\cdot12}p^3x'^2 + \frac{9}{2}p^2x'^2 + \frac{25}{96}x'^2 - \frac{83}{256}px'^4\right)x'^2$$

$$+ \left(-\frac{17}{4}p^2 + 2x'^2\right)x'\xi' - \frac{7}{2}px'\varphi' - 2x'\tau' - \frac{5}{2}x'y\xi'.$$

Let $\mathfrak{A}$ and $\mathfrak{B}$ be constants, to be fixed later, such that $3p^2 - \mathfrak{A}x'^2 \geq 0$ and $9p^2 - \mathfrak{B}x'^2 \geq 0$. Applying Lemma 3 to $(3p^4/64 - \mathfrak{A}p^2x'^2/64)(\eta + 13py/6)$ and Lemma 2 to $(9p^3x/20 - \mathfrak{B}pxx'^2/20)(\eta + 3py/2)$ we have

$$\Re a_8 \leqq \frac{2}{7} + \frac{1}{8}p + \frac{2}{5}p^2 + \frac{19}{15}p^3 - \frac{19}{480}p^4 + \frac{49}{30}p^5 - \frac{51}{32}p^6 + \left(\frac{19}{40} + \frac{27}{448} - \frac{269}{96 \cdot 16}\right)p^7$$

$$+ \left\{ -\frac{73}{240}p^5 + \frac{21}{40}p^4 - \frac{1}{4}p^3 + \frac{5}{6}p^2 + \left(\frac{5}{384}\mathfrak{A} - \frac{3}{40}\mathfrak{B} + \frac{83}{192}\right)p^3 x'^2 + \left(\frac{3}{20}\mathfrak{B}\right.\right.$$

$$\left. -\frac{15}{2}\right)p^2 x'^2 + \frac{1}{12}\mathfrak{A}px'^2 - \frac{5}{6}x'^2 + \left(\frac{63}{64} - \frac{1}{16}\mathfrak{A}\right)px'^4 + \frac{\mathfrak{A}}{128}\frac{x'^6}{p}\right\} y$$

$$+ \left\{ \left(\frac{1}{64}\mathfrak{A} - \frac{1}{20}\mathfrak{B} - \frac{9}{320}\right)p^2 + \left(\frac{1}{10}\mathfrak{B} - \frac{18}{5}\right)p + \left(\frac{11}{64} - \frac{3}{64}\mathfrak{A}\right)x'^2\right\} x'^2 \eta$$

$$+ \left\{ -\frac{851}{384}p^3 + \frac{25}{8}p^2 - \frac{3}{32}p + \frac{2}{3} + \left(\frac{25}{384}\mathfrak{A} - \frac{363}{384}\right)px'^2 + \frac{\mathfrak{A}}{32}\frac{x'^2}{p} + \frac{3\mathfrak{A}}{128}\frac{x'^4}{p}\right\} y^2$$

(D) $$+ \left\{ -\frac{43}{32}p^2 + 3p + \left(\frac{5}{32}\mathfrak{A} - \frac{29}{8}\right)x'^2\right\} y\eta + \left(\frac{327}{800}p + \frac{18}{25} + \frac{3\mathfrak{A}}{32}\frac{x'^2}{p}\right)\eta^2$$

$$+ py\xi + \frac{14}{5}\eta\xi + y\varphi - \frac{3}{8}py^3 - \frac{1}{5}y^2\eta$$

$$+ \left\{ \left(\frac{1}{32 \cdot 48}\mathfrak{A} - \frac{13}{5 \cdot 48}\mathfrak{B} + \frac{221}{96 \cdot 80}\right)p^5 + \left(\frac{5}{24}\mathfrak{B} - \frac{81}{40}\right)p^4 + \left(\frac{1}{18}\mathfrak{A} - \frac{1}{5}\mathfrak{B} - \frac{1}{6}\right)p^3\right.$$

$$+ \left(\frac{1}{30}\mathfrak{B} - \frac{25}{48}\right)p^2 - \left(\frac{2}{9}\mathfrak{A} + \frac{1}{15}\mathfrak{B}\right)p - \frac{\mathfrak{A}}{24}\frac{1}{p} + \left(\frac{1}{32 \cdot 16}\mathfrak{A} + \frac{1}{80}\mathfrak{B} - \frac{671}{96 \cdot 16}\right)p^3 x'^2$$

$$+ \left(\frac{9}{2} - \frac{1}{40}\mathfrak{B}\right)p^2 x'^2 + \frac{1}{18}\mathfrak{A}px'^2 + \frac{25}{96}x'^2 + \left(\frac{1}{32 \cdot 16}\mathfrak{A} - \frac{167}{32 \cdot 16}\right)px'^4$$

$$+ \frac{\mathfrak{A}}{96 \cdot 16}\frac{x'^6}{p}\right\} x'^2$$

$$+ \left(-\frac{17}{4}p^2 + 2x'^2\right)x'\xi' - \frac{7}{2}px'\varphi' - 2x'\tau' - \frac{5}{2}x'y\xi'.$$

Case i) $1.9 \leqq p$.

Since $|a_2| \leqq 2$ we have $p < 1.98$ in this case. Applying the trivial inequality $2AXY \leqq |A|\alpha X^2 + |A|\alpha' Y^2$, $\alpha' = 1/\alpha$, $\alpha > 0$ we have

$$\left(-\frac{73}{240}p^5 + \frac{21}{40}p^4 + \frac{17}{12}p^3 - \frac{5}{2}p^2\right)y = \frac{1}{240}p^2(73p^2 + 20p - 300)xy$$

$$\leqq \frac{1}{480}\alpha p^2(73p^2 + 20p - 300)x^2 + \frac{1}{480\alpha}p^2(73p^2 + 20p - 300)y^2.$$

Hence from (D) we have

$$\Re a_8 \leqq 8 - \frac{x}{96}\{160 + (266 - 25.6\alpha)x + (1244.2 + 275.2\alpha)x^2 - (2064.7$$

$$+ 314.4\alpha)x^3 + (1224.95 + 120.8\alpha)x^4 - (331.025 + 14.6\alpha)x^5 + 34x^6\}$$

$$+ \left\{ -\frac{5}{3}p^3 + \frac{10}{3}p^2 + \left(\frac{5}{384}\mathfrak{A} - \frac{3}{40}\mathfrak{B} + \frac{83}{192}\right)p^3 x'^2 + \left(\frac{3}{20}\mathfrak{B} - \frac{15}{2}\right)p^2 x'^2 \right.$$

$$\left. + \frac{1}{12}\mathfrak{A}px'^2 - \frac{5}{6}x'^2 + \left(\frac{63}{64} - \frac{1}{16}\mathfrak{A}\right)px'^4 + \frac{\mathfrak{A}}{128}\frac{x'^6}{p} \right\}y$$

$$+ \left\{ \left(\frac{1}{64}\mathfrak{A} - \frac{1}{20}\mathfrak{B} - \frac{9}{320}\right)p^2 + \left(\frac{1}{10}\mathfrak{B} - \frac{18}{5}\right)p + \left(\frac{11}{64} - \frac{3}{64}\mathfrak{A}\right)x'^2 \right\}x'^2\eta$$

$$- \frac{1}{96}\left[\left\{ -\frac{73}{5\alpha}p^4 + \left(212.75 - \frac{4}{\alpha}\right)p^3 - \left(300 - \frac{60}{\alpha}\right)p^2 + 9p - 64 \right. \right.$$

$$(E) \qquad \left. + (90.75 - 6.25\mathfrak{A})px'^2 - 3\mathfrak{A}\frac{x'^2}{p} - 2.25\mathfrak{A}\frac{x'^4}{p} \right\}y^2$$

$$+ \{129p^2 - 288p + (348 - 15\mathfrak{A})x'^2\}y\eta + \left(-39.24p - 69.12 - 9\mathfrak{A}\frac{x'^2}{p} \right)\eta^2$$

$$\left. - 96py\xi - 268.8\eta\xi - 96y\varphi \right]$$

$$- \frac{1}{96}\left[\left\{ (-0.0625\mathfrak{A} + 5.2\mathfrak{B} - 2.7625)p^5 + (194.4 - 20\mathfrak{B})p^4 + (16 - 5.3333\mathfrak{A} + 19.2\mathfrak{B})p^3 \right. \right.$$

$$+ (50 - 3.2\mathfrak{B})p^2 + (21.3334\mathfrak{A} + 6.4\mathfrak{B})p + 4\mathfrak{A}\frac{1}{p}$$

$$+ (41.9375 - 0.1875\mathfrak{A} - 1.2\mathfrak{B})p^3 x'^2 + (2.4\mathfrak{B} - 432)p^2 x'^2 - 5.3333\mathfrak{A}px'^2$$

$$\left. - 25x'^2 + (31.3125 - 0.1875\mathfrak{A})px'^4 - 0.0625\mathfrak{A}\frac{x'^6}{p} \right\}x'^2$$

$$\left. + (408p^2 - 192x'^2)x'\xi' + 336px'\varphi' + 192x'\tau' \right] - \frac{3}{8}py^3 - \frac{1}{5}y^2\eta - \frac{5}{2}x'y\xi'.$$

Here we put $\mathfrak{A} = -70$, $\mathfrak{B} = 180$, $\alpha = 16$. We remark the following facts:

$$\left(-\frac{5}{3}p^3 + \frac{10}{3}p^2 - \frac{671}{48}p^3 x'^2 + \frac{39}{2}p^2 x'^2 - \frac{35}{6}px'^2 - \frac{5}{6}x'^2 + \frac{343}{64}px'^4 - \frac{35}{64}\frac{x'^6}{p} \right)y$$

$$+ \frac{23}{5}p^3 x'^2 y \leqq 0,$$

$$\left(-\frac{3239}{320}p^2 + \frac{72}{5}p + \frac{221}{64}x'^2 \right)x'^2\eta \leqq \left(\frac{3239}{320}p^2 - \frac{72}{5}p - \frac{221}{64}x'^2 \right)\sqrt{\frac{4 - p^2}{5}}\, x'^2$$

$$\leqq \frac{1}{96}(271.39581 p^2 - 386.10432 p - 92.58795 x'^2)x'^2,$$

$$-\frac{23}{5}p^3 x'^2 y - \frac{5}{2}x'y\xi' \leqq \frac{125}{368 p^3}y\xi'^2 \leqq \frac{125}{368 p^3}\sqrt{\frac{4-p^2}{3}}\,\xi'^2 \leqq \frac{1.72}{96}\xi'^2,$$

$$-\frac{3}{8}py^3 - \frac{1}{5}y^2\eta \leqq \frac{2}{75 p}y\eta^2 \leqq \frac{2}{75 p}\sqrt{\frac{4-p^2}{3}}\,\eta^2 \leqq \frac{0.49}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96$ we have

$$\Re a_8 \leqq 8 - \frac{x^2}{96}P(x) - \frac{1}{96}Q - \frac{1}{96}R,$$

$$P(x) = -103.6 + 5647.4x - 7095.1x^2 + 3157.75x^3 - 564.625x^4 + 34x^5,$$

$$Q = \left(-0.9125 p^4 + 212.5 p^3 - 296.25 p^2 + 9p + 56 + 528.25 px'^2 + 210\frac{x'^2}{p} + 157.5\frac{x'^4}{p}\right)y^2$$

$$+ (129 p^2 - 288 p + 1398 x'^2)y\eta + \left(-39.24 p + 130.39 + 630\frac{x'^2}{p}\right)\eta^2$$

$$- 96 py\xi - 268.8\eta\xi + 280\xi^2 - 96 y\varphi + 360\varphi^2,$$

$$R = \left(937.6125 p^5 - 3405.6 p^4 + 3845.331 p^3 - 797.396 p^2 + 44.766 p + 40 - 280\frac{1}{p}\right.$$

$$\left. - 160.938 p^3 x'^2 + 373.331 px'^2 + 67.587 x'^2 + 44.437 px'^4 + 4.375\frac{x'^6}{p}\right)x'^2$$

$$+ (408 p^2 - 192 x'^2)x'\xi' + 278.28\xi'^2 + 336 px'\varphi' + 360\varphi'^2 + 192 x'\tau' + 440\tau'^2.$$

$P(x)$ is monotone increasing for $0.02 \leqq x \leqq 0.1$ and $P(0.02) > 0$. Hence $P(x) > 0$ for $0.02 \leqq x \leqq 0.1$.

Since $6.4 y^2 - 96 y\varphi + 360\varphi^2 \geqq 0$ and $20.95 x'^2 + 192 x'\tau' + 440\tau'^2 \geqq 0$ we consider $Q^* = Q - 6.4 y^2 + 96 y\varphi - 360\varphi^2$ and $R^* = R - 20.95 x'^2 - 192 x'\tau' - 440\tau'^2$. We can prove the positive definiteness of the symmetric matrices of Q^* and R^* for $1.9 \leqq p \leqq 1.98$, $1/40 \leqq x'^2/p^2 \leqq 1/20$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8 < 8$ for $1.9 \leqq p \leqq 1.98$, $1/40 \leqq x'^2/p^2 \leqq 1/20$, $y \geqq 0$.

Case ii) $1.7 \leqq p \leqq 1.9$.

Applying the trivial inequality we have

$$\left(-\frac{73}{240}p^5 + \frac{21}{40}p^4 + \frac{71}{60}p^3 - \frac{61}{30}p^2\right)y = \frac{1}{240}p^2(73 p^2 + 20 p - 244)xy$$

$$\leqq \frac{1}{480}\alpha p^2(73 p^2 + 20 p - 244)x^2 + \frac{1}{480\alpha}p^2(73 p^2 + 20 p - 244)y^2.$$

Hence from (D) we have

$$\Re a_8 \leqq 8 - \frac{x}{96}\{160 + (266 - 70.4\alpha)x + (1244.2 + 320\alpha)x^2 - (2064.7$$

$$+ 325.6\alpha)x^3 + (1224.95 + 120.8\alpha)x^4 - (331.025 + 14.6\alpha)x^5 + 34x^6\}$$

$$+ \left[-\frac{43}{30}p^3 + \frac{43}{15}p^2 + \left(\frac{5}{384}\mathfrak{A} - \frac{3}{40}\mathfrak{B} + \frac{83}{192}\right)p^3 x'^2 + \left(\frac{3}{20}\mathfrak{B} - \frac{15}{2}\right)p^2 x'^2 \right.$$

$$\left. + \frac{1}{12}\mathfrak{A}px'^2 - \frac{5}{6}x'^2 + \left(\frac{63}{64} - \frac{1}{16}\mathfrak{A}\right)px'^4 + \frac{\mathfrak{A}}{128}\frac{x'^6}{p} \right]y$$

$$+ \left[\left(\frac{1}{64}\mathfrak{A} - \frac{1}{20}\mathfrak{B} - \frac{9}{320}\right)p^2 + \left(\frac{1}{10}\mathfrak{B} - \frac{18}{5}\right)p + \left(\frac{11}{64} - \frac{3}{64}\mathfrak{A}\right)x'^2 \right]x'^2\eta$$

$$- \frac{1}{96}\left[\left\{ -\frac{73}{5\alpha}p^4 + \left(212.75 - \frac{4}{\alpha}\right)p^3 - \left(300 - \frac{244}{5\alpha}\right)p^2 + 9p - 64 \right.\right.$$

$$\left. + (90.75 - 6.25\mathfrak{A})px'^2 - 3\mathfrak{A}\frac{x'^2}{p} - 2.25\mathfrak{A}\frac{x'^4}{p} \right\}y^2$$

(F) $$\quad + \{129p^2 - 288p + (348 - 15\mathfrak{A})x'^2\}y\eta + \left(-39.24p - 69.12 - 9\mathfrak{A}\frac{x'^2}{p}\right)\eta^2$$

$$- 96py\xi - 268.8\eta\xi - 96y\varphi \Bigg]$$

$$- \frac{1}{96}\left[\left\{ (-0.0625\mathfrak{A} + 5.2\mathfrak{B} - 2.7625)p^5 + (194.4 - 20\mathfrak{B})p^4 + (16 - 5.3333\mathfrak{A} + 19.2\mathfrak{B})p^3 \right.\right.$$

$$+ (50 - 3.2\mathfrak{B})p^2 + (21.3334\mathfrak{A} + 6.4\mathfrak{B})p + 4\mathfrak{A}\frac{1}{p}$$

$$+ (41.9375 - 0.1875\mathfrak{A} - 1.2\mathfrak{B})p^3 x'^2 + (2.4\mathfrak{B} - 432)p^2 x'^2 - 5.3333\mathfrak{A}px'^2$$

$$- 25x'^2 + (31.3125 - 0.1875\mathfrak{A})px'^4 - 0.0625\mathfrak{A}\frac{x'^6}{p} \Bigg\}x'^2$$

$$+ (408p^2 - 192x'^2)x'\xi' + 336px'\varphi' + 192x'\tau' \Bigg]$$

$$- \frac{3}{8}py^3 - \frac{1}{5}y^2\eta - \frac{5}{2}x'y\xi'.$$

Here we put $\mathfrak{A} = -30$, $\mathfrak{B} = 180$, $\alpha = 4$. We remark the following facts:

$$\left(-\frac{43}{30}p^3 + \frac{43}{15}p^2 - \frac{323}{24}p^3 x'^2 + \frac{39}{2}p^2 x'^2 - \frac{5}{2}px'^2 - \frac{5}{6}x'^2 + \frac{183}{64}px'^4 \right.$$

$$-\frac{15}{64}\frac{x'^6}{p}\Big)y+3p^3x'^2y\leqq 0,$$

$$\Big(-\frac{3039}{320}p^2+\frac{72}{5}p+\frac{101}{64}x'^2\Big)x'^2\eta\leqq\Big(\frac{3039}{320}p^2-\frac{72}{5}p-\frac{101}{64}x'^2\Big)\sqrt{\frac{4-p^2}{5}}\,x'^2$$

$$\leqq\frac{1}{96}(429.59304p^2-651.38688p-71.3868x'^2)x'^2,$$

$$-3p^3x'^2y-\frac{5}{2}x'y\xi'\leqq\frac{25}{48p^3}y\xi'^2\leqq\frac{25}{48p^3}\sqrt{\frac{4-p^2}{3}}\,\xi'^2\leqq\frac{6.2}{96}\xi'^2,$$

$$-\frac{3}{8}py^3-\frac{1}{5}y^2\eta\leqq\frac{2}{75p}y\eta^2\leqq\frac{0.92}{96}\eta^2.$$

Using these remarks and applying Lemma 1 to the term $-160x/96-248x^2/96$ we have

$$\Re a_8\leqq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=-223.6+2586.2x-3367.1x^2+1708.15x^3-389.425x^4+34x^5,$$

$$Q=\Big(-3.65p^4+211.75p^3-287.8p^2-177p+428+278.25px'^2+90\frac{x'^2}{p}+67.5\frac{x'^4}{p}\Big)y^2$$

$$+(129p^2-288p+798x'^2)y\eta+\Big(-349.24p+749.96+270\frac{x'^2}{p}\Big)\eta^2$$

$$-96py\xi-268.8\eta\xi+(1148-434p)\xi^2-96y\varphi+(1476-558p)\varphi^2,$$

$$R=\Big(935.1125p^5-3405.6p^4+3631.999p^3-955.594p^2+1101.384p+164-120\frac{1}{p}$$

$$-168.4375p^3x'^2+159.999px'^2+46.386x'^2+36.937px'^4+1.875\frac{x'^6}{p}\Big)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+(1141.8-434p)\xi'^2+336px'\varphi'+(1476-558p)\varphi'^2$$

$$+192x'\tau'+(1804-682p)\tau'^2.$$

$P(x)$ is monotone increasing for $0.1\leqq x\leqq 0.3$ and $P(0.1)>0$. Hence $P(x)>0$ for $0.1\leqq x\leqq 0.3$.

Since $(6.7p-7)y^2-96y\varphi+(1476-558p)\varphi^2\geqq 0$ and $(22.2p-23.4)x'^2+192x'\tau'$ $+(1804-682p)\tau'^2\geqq 0$ for $1.7\leqq p\leqq 1.9$ we consider $Q^*=Q-(6.7p-7)y^2+96y\varphi-(1476-558p)\varphi^2$ and $R^*=R-(22.2p-23.4)x'^2-192x'\tau'-(1804-682p)\tau'^2$. We can prove the positive definiteness of the symmetric matrices of Q^* and R^* for $1.7\leqq p\leqq 1.9$, $1/40\leqq x'^2/p^2\leqq 1/20$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8 < 8$ for $1.7 \leq p \leq 1.9$, $1/40 \leq x'^2/p^2 \leq 1/20$, $y \geq 0$.

Case iii) $1.4 \leq p \leq 1.7$.

Applying the trivial inequality we have

$$\left(-\frac{73}{240}p^5 + \frac{21}{40}p^4 - \frac{1}{4}p^3 + \frac{5}{6}p^2\right)y = \frac{1}{240}p^2(73p^2 + 20p + 100)xy$$

$$\leq \frac{1}{480}\alpha p^2(73p^2 + 20p + 100)x^2 + \frac{1}{480\alpha}p^2(73p^2 + 20p + 100)y^2.$$

Hence from (D) we have

$$\Re a_8 \leq 8 - \frac{x}{96}\{160 + (266 - 345.6\alpha)x + (1244.2 + 595.2\alpha)x^2 - (2064.7 + 394.4\alpha)x^3$$

$$+ (1224.95 + 120.8\alpha)x^4 - (331.025 + 14.6\alpha)x^5 + 34x^6\}$$

$$+ \left[\left(\frac{5}{384}\mathfrak{A} - \frac{3}{40}\mathfrak{B} + \frac{83}{192}\right)p^3 + \left(\frac{3}{20}\mathfrak{B} - \frac{15}{2}\right)p^2 + \frac{1}{12}\mathfrak{A}p - \frac{5}{6}\right.$$

$$\left.+ \left(\frac{63}{64} - \frac{1}{16}\mathfrak{A}\right)px'^2 + \frac{\mathfrak{A}}{128}\frac{x'^4}{p}\right]x'^2 y$$

$$+ \left[\left(\frac{1}{64}\mathfrak{A} - \frac{1}{20}\mathfrak{B} - \frac{9}{320}\right)p^2 + \left(\frac{1}{10}\mathfrak{B} - \frac{18}{5}\right)p + \left(\frac{11}{64} - \frac{3}{64}\mathfrak{A}\right)x'^2\right]x'^2\eta$$

$$- \frac{1}{96}\left[\left\{-\frac{73}{5\alpha}p^4 + \left(212.75 - \frac{4}{\alpha}\right)p^3 - \left(300 + \frac{20}{\alpha}\right)p^2 + 9p - 64\right.\right.$$

$$\left.+ (90.75 - 6.25\mathfrak{A})px'^2 - 3\mathfrak{A}\frac{x'^2}{p} - 2.25\mathfrak{A}\frac{x'^4}{p}\right\}y^2$$

(G) $$+ \{129p^2 - 288p + (348 - 15\mathfrak{A})x'^2\}y\eta + \left(-39.24p - 69.12 - 9\mathfrak{A}\frac{x'^2}{p}\right)\eta^2$$

$$\left.- 96py\xi - 268.8\eta\xi - 96y\varphi\right]$$

$$- \frac{1}{96}\left[\left\{(-0.0625\mathfrak{A} + 5.2\mathfrak{B} - 2.7625)p^5 + (194.4 - 20\mathfrak{B})p^4 + (16 - 5.3333\mathfrak{A} + 19.2\mathfrak{B})p^3\right.\right.$$

$$+ (50 - 3.2\mathfrak{B})p^2 + (21.3334\mathfrak{A} + 6.4\mathfrak{B})p + 4\mathfrak{A}\frac{1}{p}$$

$$+ (41.9375 - 0.1875\mathfrak{A} - 1.2\mathfrak{B})p^3x'^2 + (2.4\mathfrak{B} - 432)p^2x'^2 - 5.3333\mathfrak{A}px'^2$$

$$\left.- 25x'^2 - (0.1875\mathfrak{A} - 31.3125)px'^4 - 0.0625\mathfrak{A}\frac{x'^6}{p}\right\}x'^2$$

$$+(408p^2-192x'^2)x'\xi'+336px'\varphi'+192x'\tau'\Big]$$

$$-\frac{3}{8}py^3-\frac{1}{5}y^2\eta-\frac{5}{2}x'y\xi'.$$

Here we put $\mathfrak{A}=-25$, $\mathfrak{B}=40$, $\alpha=1.1$. We remark the following facts:

$$\left(-\frac{1111}{384}p^3-\frac{3}{2}p^2-\frac{25}{12}p-\frac{5}{6}+\frac{163}{64}px'^2-\frac{25}{128}\frac{x'^4}{p}\right)x'^2y+\frac{7}{10}p^3x'^2y\leqq0,$$

$$\left(-\frac{387}{160}p^2+\frac{2}{5}p+\frac{43}{32}x'^2\right)x'^2\eta\leqq\left(\frac{387}{160}p^2-\frac{2}{5}p-\frac{43}{32}x'^2\right)\sqrt{\frac{4-p^2}{5}}x'^2$$

$$\leqq\frac{1}{96}(148.32936p^2-24.52992p-82.4052x'^2)x'^2,$$

$$-\frac{7}{10}p^3x'^2y-\frac{5}{2}x'y\xi'\leqq\frac{125}{56p^3}y\xi'^2\leqq\frac{125}{56p^3}\sqrt{\frac{4-p^2}{3}}\xi'^2\leqq\frac{64.4}{96}\xi'^2,$$

$$-\frac{3}{8}py^3-\frac{1}{5}y^2\eta\leqq\frac{2}{75p}y\eta^2\leqq\frac{2}{75p}\sqrt{\frac{4-p^2}{3}}\eta^2\leqq\frac{1.51}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96$ $-329.2x^2/96$ we have

$$\Re a_8\leqq8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=-403.36+1981.22x-2498.54x^2+1357.83x^3-347.085x^4+34x^5,$$

$$Q=\left(-13.28p^4+209.11p^3-318.19p^2-237.9p+549.8+247px'^2+75\frac{x'^2}{p}+56.25\frac{x'^4}{p}\right)y^2$$

$$+(129p^2-288p+723x'^2)y\eta+\left(-450.74p+952.37+225\frac{x'^2}{p}\right)\eta^2-96py\xi$$

$$-268.8\eta\xi+(1432.2-576.1p)\xi^2-96y\varphi+(1841.4-740.7p)\varphi^2,$$

$$R=\left(206.8\,p^5-605.6p^4+917.3325p^3-226.33p^2-335.106p+204.6-100\frac{1}{p}\right.$$

$$\left.-1.375p^3x'^2-336p^2x'^2+133.3325px'^2+57.4052x'^2+36px'^4+1.5625\frac{x'^6}{p}\right)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+(1367.8-576.1p)\xi'^2+336px'\varphi'$$

$$+(1841.4-740.7p)\varphi'^2+192x'\tau'+(2250.6-905.3p)\tau'^2.$$

$P(x)$ is monotone increasing for $0.3\leqq x\leqq0.6$ and $P(0.3)>0$. Hence $P(x)>0$ for

$0.3 \leq x \leq 0.6$.

Since $(5.8p-5.2)y^2-96y\varphi+(1841.4-740.7p)\varphi^2 \geq 0$ and $(19.3p-17.6)x'^2+192x'\tau'$ $+(2250.6-905.3p)\tau'^2 \geq 0$ for $1.4 \leq p \leq 1.7$ we consider $Q^*=Q-(5.8p-5.2)y^2+96y\varphi$ $-(1841.4-740.7p)\varphi^2$ and $R^*=R-(19.3p-17.6)x'^2-192x'\tau'-(2250.6-905.3p)\tau'^2$. We can prove the positive definiteness of the symmetric matrices of Q^* and R^* for $1.4 \leq p \leq 1.7$, $1/40 \leq x'^2/p^2 \leq 1/20$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8 < 8$ for $1.4 \leq p \leq 1.7$, $1/40 \leq x'^2/p^2 \leq 1/20$, $y \geq 0$.

Case iv) $1 \leq p \leq 1.4$.

In this case we start from (G) with $\mathfrak{A}=0$, $\mathfrak{B}=60$, $\alpha=1.1$. We remark the following facts:

$$\left(-\frac{781}{192}p^3+\frac{3}{2}p^2-\frac{5}{6}+\frac{63}{64}px'^2\right)x'^2y+\frac{29}{10}p^3x'^2y \leq 0,$$

$$\left(-\frac{969}{320}p^2+\frac{12}{5}p+\frac{11}{64}x'^2\right)x'^2\eta \leq \left(\frac{969}{320}p^2-\frac{12}{5}p-\frac{11}{64}x'^2\right)\sqrt{\frac{4-p^2}{5}}x'^2$$

$$\leq \frac{1}{96}(225.17622p^2-178.46784p-12.7809x'^2)x'^2,$$

$$-\frac{29}{10}p^3x'^2y-\frac{5}{2}x'y\xi' \leq \frac{125}{232p^3}y\xi'^2 \leq \frac{125}{232p^3}\sqrt{\frac{4-p^2}{3}}\xi'^2 \leq \frac{51.8}{96}\xi'^2,$$

$$-\frac{3}{8}py^3-\frac{1}{5}y^2\eta \leq \frac{2}{75p}y\eta^2 \leq \frac{2}{75p}\sqrt{\frac{4-p^2}{3}}\eta^2 \leq \frac{2.56}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96$ $-490x^2/96$ we have

$$\Re a_8 \leq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=-564.16+2021.42x-2498.54x^2+1357.83x^3-347.085x^4+34x^5,$$

$$Q=(-13.28p^4+209.11p^3-318.19p^2-358.5p+791+90.75x'^2)y^2$$

$$+(129p^2-288p+348x'^2)y\eta+(-651.74p+1353.32)\eta^2-96py\xi$$

$$-268.8\eta\xi+(1995-857.5p)\xi^2-96y\varphi+(2565-1102.5p)\varphi^2,$$

$$R=(309.2375p^5-1005.6p^4+1168p^3-367.177p^2+439.967p+285$$

$$-30.063p^3x'^2-288p^2x'^2-12.22x'^2+31.3125px'^4)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+(1943.2-857.5p)\xi'^2+336px'\varphi'$$

$$+(2565-1102.5p)\varphi'^2+192x'\tau'+(3135-1347.5p)\tau'^2.$$

$P'(x)$ is monotone decreasing for $0.6 \leq x \leq 1$ and $P'(0.6) > 0$, $P'(1) < 0$. Further $P(0.6) > 0$, $P(1) > 0$. Hence $P(x) > 0$ for $0.6 \leq x \leq 1$.

Since $(4.8p - 3.2)y^2 - 96y\varphi + (2565 - 1102.5p)\varphi^2 \geq 0$ and $(15.8p - 10.6)x'^2 + 192x'\tau'$ $+ (3135 - 1347.5p)\tau'^2 \geq 0$ for $1 \leq p \leq 1.4$ we consider $Q^* = Q - (4.8p - 3.2)y^2 + 96y\varphi$ $- (2565 - 1102.5p)\varphi^2$ and $R^* = R - (15.8p - 10.6)x'^2 - 192x'\tau' - (3135 - 1347.5p)\tau'^2$. We can prove the positive definiteness of the symmetric matrices of Q^* and R^* for $1 \leq p \leq 1.4$, $1/40 \leq x'^2/p^2 \leq 1/20$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8 < 8$ for $1 \leq p \leq 1.4$, $1/40 \leq x'^2/p^2 \leq 1/20$, $y \geq 0$.

5. Case $1 \leq p$, $1/40 \leq x'^2/p^2 \leq 1/20$ and $y \leq 0$. In this case we put $A = 1/2$, $B = 3/2$, $C = 3/5$ in (A).

Applying the trivial inequality $2AXY \leq |A|X^2 + |A|Y^2$ and Lemma 1 we have

$$\left(-\frac{413}{384}p^5 + \frac{934}{384}p^4 + \frac{5}{12}p^2 + \frac{151}{192}p^3x'^2 - 9p^2x'^2 - \frac{5}{12}x'^2 + \frac{149}{128}px'^4 \right)y - 2x'y\xi'$$

$$\leq -\frac{y}{384}(413p^5 - 934p^4 - 160p^2 - 302p^3x'^2 + 3456p^2x'^2 + 160x'^2 + 384x'^2 - 447px'^4 + 384\xi'^2)$$

$$\leq -\frac{y}{384}(413p^5 - 934p^4 - 214.86p^2 + 219.44 - 302p^3x'^2 + 3456p^2x'^2 + 489.14x'^2$$

$$-447px'^4) \leq 0$$

for $1 \leq p \leq 1.98$, $1/40 \leq x'^2/p^2 \leq 1/20$, $y \leq 0$. Then we have

$$\Re a_8 \leq U - \frac{35}{192}p^4y + \left(-\frac{129}{320}p^4 + \frac{9}{10}p^3 - \frac{27}{160}p^2x'^2 - \frac{18}{5}px'^2 + \frac{11}{64}x'^4 \right)\eta$$

$$+ \left(-\frac{127}{48}p^3 + \frac{9}{2}p^2 + \frac{1}{6} - \frac{35}{16}px'^2 \right)y^2 + \left(-\frac{13}{10}p^2 + \frac{18}{5}p - \frac{9}{2}x'^2 \right)y\eta$$

(H) $$+ \left(\frac{69}{100}p + \frac{18}{25} \right)\eta^2 + py\xi + \frac{14}{5}\eta\xi + 2y\varphi + \frac{3}{20}y^2\eta$$

$$+ \left(\frac{113}{64 \cdot 12}p^5 - \frac{9}{4}p^4 - \frac{25}{48}p^2 - \frac{331}{64 \cdot 12}p^3x'^2 + \frac{9}{2}p^2x'^2 + \frac{25}{96}x'^2 - \frac{83}{256}px'^4 \right)x'^2$$

$$+ \left(-\frac{17}{4}p^2 + 2x'^2 \right)x'\xi' - \frac{7}{2}px'\varphi' - 2x'\tau' - 2x'y\xi'.$$

Case i) $\eta \geq 0$. By Lemma 1,

$$\left(-\frac{129}{320}p^4 + \frac{111}{320}p^3 - \frac{27}{160}p^2x'^2 - \frac{18}{5}px'^2 + \frac{11}{64}x'^4 \right)\eta + \frac{3}{20}y^2\eta$$

$$\leq \frac{\eta}{320}(-129p^4 + 111p^3 - 54p^2x'^2 - 1152px'^2 + 55x'^4 + 64 - 16p^2 - 16x'^2) \leq 0.$$

By Lemma 2

$$\frac{177}{320}p^3\eta - \frac{35}{192}p^4y \leqq \frac{9}{16}p^3\left(\eta - \frac{1}{3}py\right) \leqq -\frac{13}{256}p^6 + \frac{1}{128}p^5 + \frac{3}{8}p^3 + \frac{9}{64}p^4x'^2.$$

Further by Lemma 1

$$-\frac{300}{96}x \leqq -\frac{75}{96}(x^2+x'^2+3y^2+5\eta^2+7\xi^2+7\xi'^2+9\varphi^2+9\varphi'^2+11\tau'^2),$$

$$-\frac{538.8}{96}x^2 \leqq -\frac{134.7}{96}x(x^2+x'^2+3y^2+5\eta^2+7\xi^2+7\xi'^2+9\varphi^2+9\varphi'^2+11\tau'^2).$$

Making use of these inequalities and (H) we have

$$\Re a_8 \leqq 8 - \frac{x^2}{96}P_0(x) - \frac{1}{96}Q_0 - \frac{1}{96}R_0,$$

$$P_0(x) = 1402.2 - 2801.3x + 2011x^2 - 745.05x^3 + 146.425x^4 - 13x^5,$$

$$Q_0 = (254p^3 - 432p^2 - 404.1p + 1017.2 + 210px'^2)y^2 + (124.8p^2 - 345.6p + 432x'^2)y\eta$$

$$+ (-739.74p + 1652.88)\eta^2 - 96py\xi - 268.8\eta\xi + (2410.8 - 942.9p)\xi^2$$

$$- 192y\varphi + (3099.6 - 1212.3p)\varphi^2,$$

$$R_0 = (-14.125p^5 + 202.5p^4 + 50p^2 - 134.7p + 344.4 + 41.375p^3x'^2$$

$$- 432p^2x'^2 - 25x'^2 + 31.125px'^4)x'^2$$

$$+ (408p^2 - 192x'^2)x'\xi' + (2410.8 - 942.9p)\xi'^2 + 336px'\varphi'$$

$$+ (3099.6 - 1212.3p)\varphi'^2 + 192x'\tau' + (3788.4 - 1481.7p)\tau'^2.$$

$P_0(x)$ is monotone decreasing for $0.02 \leqq x \leqq 1$ and $P_0(1) > 0$. Hence $P_0(x) > 0$ for $0.02 \leqq x \leqq 1$.

Q_0 is positive definite for $1 \leqq p \leqq 1.98$, $1/40 \leqq x'^2/p^2 \leqq 1/20$. Indeed we have

$$(8.77p - 3.88)y^2 - 192y\varphi + (3099.6 - 1212.3p)\varphi^2 \geqq 0,$$

$$51.61\eta^2 - 268.8\eta\xi + 350\xi^2 \geqq 0,$$

$$(-2.45p + 57.71)y^2 - 96py\xi + (2060.8 - 942.9p)\xi^2 \geqq 0$$

and

$$(254p^3 - 432p^2 - 410.42p + 963.37 + 210px'^2)y^2$$

$$+ (124.8p^2 - 345.6p + 432x'^2)y\eta + (-739.74p + 1601.27)\eta^2 \geqq 0$$

for $1 \leqq p \leqq 1.98$, $1/40 \leqq x'^2/p^2 \leqq 1/20$. This implies the positive definiteness of Q_0 for

$1 \le p \le 1.98$, $1/40 \le x'^2/p^2 \le 1/20$.

R_0 is positive definite for $1 \le p \le 1.98$, $1/40 \le x'^2/p^2 \le 1/20$. Indeed we have

$$(7.18p - 3.18)x'^2 + 192x'\tau' + (3788.4 - 1481.7p)\tau'^2 \ge 0,$$

$$(-23.29p + 213.9)x'^2 + 336px'\varphi' + (3099.6 - 1212.3p)\varphi'^2 \ge 0$$

and

$$(-14.125p^5 + 202.5p^4 + 50p^2 - 118.59p + 133.68 + 41.375p^3x'^2 - 432p^2x'^2$$

$$-25x'^2 + 31.125px'^4)x'^2 + (408p^2 - 192x'^2)x'\xi' + (2410.8 - 942.9p)\xi'^2 \ge 0$$

for $1 \le p \le 1.98$, $1/40 \le x'^2/p^2 \le 1/20$, which imply the desired result.

Thus we have $\Re a_8 < 8$ in the present case.

Case ii) $\eta \le 0$. By Lemma 1

$$\left(-\frac{129}{320}p^4 + \frac{129}{160}p^3 - \frac{27}{160}p^2x'^2 - \frac{18}{5}px'^2 + \frac{11}{64}x'^4\right)\eta$$

$$\le -\frac{\eta}{320}(54p^2 + 1152p - 55x'^2)$$

$$\le \frac{1}{96}(12.549p^2 + 267.702p - 12.78x'^2)x'^2.$$

By Lemma 2

$$\frac{3}{32}p^3\eta - \frac{70}{384}p^4y \le \frac{3}{32}p^3(\eta - 2py)$$

$$\le -\frac{31}{512}p^6 + \frac{27}{256}p^5 + \frac{1}{16}p^3 + \frac{3}{128}p^4x'^2.$$

Further by Lemma 1

$$-\frac{510}{96}x \le -\frac{127.5}{96}(x^2 + x'^2 + 3y^2 + 5\eta^2 + 7\xi^2 + 7\xi'^2 + 9\varphi^2 + 9\varphi'^2 + 11\tau'^2),$$

$$-\frac{357.6}{96}x^2 \le -\frac{89.4}{96}x(x^2 + x'^2 + 3y^2 + 5\eta^2 + 7\xi^2 + 7\xi'^2 + 9\varphi^2 + 9\varphi'^2 + 11\tau'^2).$$

Making use of these remarks and (H) we have

$$\Re a_8 \le 8 - \frac{x^2}{96}P_1(x) - \frac{1}{96}Q_1 - \frac{1}{96}R_1,$$

$$P_1(x) = 1290.9 - 2651.6x + 1973.5x^2 - 746.925x^3 + 147.3625x^4 - 13x^5,$$

$$Q_1 = (254p^3 - 432p^2 - 268.2p + 902.9 + 210px'^2)y^2 + (124.8p^2 - 345.6p + 432x'^2)y\eta$$

$$+ (-513.24p + 1462.38)\eta^2 - 96py\xi - 268.8\eta\xi + (2144.1 - 625.8p)\xi^2$$

$$- 192y\varphi + (2756.7 - 804.6p)\varphi^2,$$

$$R_1 = (-14.125p^5 + 213.75p^4 + 37.451p^2 - 357.102p + 306.3$$

$$+ 41.375p^3x'^2 - 432p^2x'^2 - 12.22x'^2 + 31.125px'^4)x'^2$$

$$+ (408p^2 - 192x'^2)x'\xi' + (2144.1 - 625.8p)\xi'^2 + 336px'\varphi'$$

$$+ (2756.7 - 804.6p)\varphi'^2 + 192x'\tau' + (3369.3 - 983.4p)\tau'^2.$$

$P_1(x)$ is monotone decreasing for $0.02 \leq x \leq 1$ and $P_1(1) > 0$. Hence $P_1(x) > 0$ for $0.02 \leq x \leq 1$.

Since $Q_1 \geq Q_0$ for $1 \leq p \leq 1.98$, $1/40 \leq x'^2/p^2 \leq 1/20$, Q_1 is positive definite there. Here Q_0 was defined in Case i) and was positive definite for $1 \leq p \leq 1.98$, $1/40 \leq x'^2/p^2 \leq 1/20$.

R_1 is positive definite for $1 \leq p \leq 1.98$, $1/40 \leq x'^2/p^2 \leq 1/20$. Indeed we have

$$(2.71p + 1.16)x'^2 + 192x'\tau' + (3369.3 - 983.4p)\tau'^2 \geq 0,$$

$$(-35.08p + 168.55)x'^2 + 336px'\varphi' + (2756.7 - 804.6p)\varphi'^2 \geq 0$$

and

$$(-14.125p^5 + 213.75p^4 + 37.451p^2 - 324.732p + 136.59 + 41.375p^3x'^2 - 432p^2x'^2$$

$$- 12.22x'^2 + 31.125px'^4)x'^2 + (408p^2 - 192x'^2)x'\xi' + (2144.1 - 625.8p)\xi'^2 \geq 0$$

for $1 \leq p \leq 1.98$, $1/40 \leq x'^2/p^2 \leq 1/20$, which imply the desired result.

Thus we have $\Re a_8 < 8$ in the present case.

6. Case $1 \leq p$, $1/20 \leq x'^2/p^2 \leq 1/10$ and $y \geq 0$.

Case i) $1.9 \leq p$.

Since $|a_2| \leq 2$ we have $p < 1.952$ in this case. We start from (E) with $\mathfrak{A} = -21$, $\mathfrak{B} = 90$, $\alpha = 15$. We remark the following facts:

$$\left(-\frac{5}{3}p^3 + \frac{10}{3}p^2 - \frac{1243}{192}p^3x'^2 + 6p^2x'^2 - px'^2 - \frac{5}{6}x'^2 + \frac{111}{64}px'^4\right.$$

$$\left. - \frac{3}{32}\frac{x'^6}{p}\right)y + \frac{61}{20}p^3x'^2y \leq 0,$$

$$\left(-\frac{1509}{320}p^2 + \frac{27}{5}p + \frac{47}{64}x'^2\right)x'^2\eta \leq \left(\frac{1509}{320}p^2 - \frac{27}{5}p - \frac{47}{64}x'^2\right)\sqrt{\frac{4-p^2}{5}}x'^2$$

$$\leq \frac{1}{96}(126.43911p^2 - 144.78912p - 19.69065x'^2)x'^2,$$

$$-\frac{61}{20}p^3x'^2y-\frac{5}{2}x'y\xi'\leq\frac{125}{244p^3}y\xi'^2\leq\frac{125}{244p^3}\sqrt{\frac{4-p^2}{3}}\xi'^2\leq\frac{7.8}{96}\xi'^2,$$

$$-\frac{3}{8}py^3-\frac{1}{5}y^2\eta\leq\frac{2}{75p}y\eta^2\leq\frac{0.49}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96$ $-164x^2/96$ we have

$$\Re a_8\leq8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=-242+5413.2x-6780.7x^2+3036.95x^3-550.025x^4+34x^5,$$

$$Q=\left(-0.98p^4+212.48p^3-296p^2-114p+302+165.75px'^2+36\frac{x'^2}{p}+27\frac{x'^4}{p}\right)y^2$$

$$+(129p^2-288p+528x'^2)y\eta+\left(-244.24p+540.39+108\frac{x'^2}{p}\right)\eta^2$$

$$-96py\xi-268.8\eta\xi+(854-287p)\xi^2-96y\varphi+(1098-369p)\varphi^2,$$

$$R=\left(465.9875p^5-1605.6p^4+1807.9996p^3-364.4392p^2+423.7883p+122-48\frac{1}{p}\right.$$

$$\left.-63.8125p^3x'^2-216p^2x'^2+63.9996px'^2-5.3094x'^2+33.5625px'^4+0.75\frac{x'^6}{p}\right)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+(846.2-287p)\xi'^2+336px'\varphi'$$

$$+(1098-369p)\varphi'^2+192x'\tau'+(1342-451p)\tau'^2.$$

$P(x)$ is monotone increasing for $0.048\leq x\leq0.1$ and $P(0.048)>0$. Hence $P(x)>0$ for $0.048\leq x\leq0.1$.

Since $(5.9p-5.4)y^2-96y\varphi+(1098-369p)\varphi^2\geq0$ and $(19.5p-18)x'^2+192x'\tau'+(1342-451p)\tau'^2\geq0$ for $1.9\leq p<1.952$ we consider $Q^*=Q-(5.9p-5.4)y^2+96y\varphi-(1098-369p)\varphi^2$ and $R^*=R-(19.5p-18)x'^2-192x'\tau'-(1342-451p)\tau'^2$. We can prove the positive definiteness of the symmetric matrices of Q^* and R^* for $1.9\leq p<1.952$, $1/20\leq x'^2/p^2$ $\leq1/10$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8<8$ for $1.9\leq p<1.952$, $1/20\leq x'^2/p^2\leq1/10$, $y\geq0$.

Case ii) $1.7\leq p\leq1.9$

We start from (F) with $\mathfrak{A}=0$, $\mathfrak{B}=90$, $\alpha=4$. We remark the following facts:

$$\left(-\frac{43}{30}p^3+\frac{43}{15}p^2-\frac{1213}{192}p^3x'^2+6p^2x'^2-\frac{5}{6}x'^2+\frac{63}{64}px'^4\right)y+\frac{109}{100}p^3x'^2y\leq0,$$

$$\left(-\frac{1449}{320}p^2+\frac{27}{5}p+\frac{11}{64}x'^2\right)x'^2\eta\leq\left(\frac{1449}{320}p^2-\frac{27}{5}p-\frac{11}{64}x'^2\right)\sqrt{\frac{4-p^2}{5}}\,x'^2$$

$$\leq\frac{1}{96}(204.83064p^2-244.27008p-7.7748x'^2)x'^2,$$

$$-\frac{109}{100}p^3x'^2y-\frac{5}{2}x'y\xi'\leq\frac{625}{436p^3}y\xi'^2\leq\frac{625}{436p^3}\sqrt{\frac{4-p^3}{3}}\,\xi'^2\leq\frac{17.04}{96}\xi'^2,$$

$$-\frac{3}{8}py^3-\frac{1}{5}y^2\eta\leq\frac{2}{75p}y\eta^2\leq\frac{0.92}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96$ $-248x^2/96$ we have

$$\Re a_8\leq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=-223.6+2586.2x-3367.1x^2+1708.15x^3-389.425x^4+34x^5,$$

$$Q=(-3.65p^4+211.75p^3-287.8p^2-177p+428+90.75x'^2)y^2$$

$$+(129p^2-288p+348x'^2)y\eta+(-349.24p+749.96)\eta^2-96py\xi$$

$$-268.8\eta\xi+(1148-434p)\xi^2-96y\varphi+(1476-558p)\varphi^2,$$

$$R=(465.2375p^5-1605.6p^4+1744p^3-442.83064p^2+758.27008p+164$$

$$-66.0625p^3x'^2-216p^2x'^2-17.2252x'^2+31.3125px'^4)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+(1130.96-434p)\xi'^2+336px'\varphi'$$

$$+(1476-558p)\varphi'^2+192x'\tau'+(1804-682p)\tau'^2.$$

$P(x)$ is monotone increasing for $0.1\leq x\leq 0.3$ and $P(0.1)>0$. Hence $P(x)>0$ for $0.1\leq x\leq 0.3$.

Since $(6.7p-7)y^2-96y\varphi+(1476-558p)\varphi^2\geq 0$ and $(22.2p-23.4)x'^2+192x'\tau'+(1804-682p)\tau'^2\geq 0$ for $1.7\leq p\leq 1.9$ we consider $Q^*=Q-(6.7p-7)y^2+96y\varphi-(1476-558p)\varphi^2$ and $R^*=R-(22.2p-23.4)x'^2-192x'\tau'-(1804-682p)\tau'^2$. We can prove the positive definiteness of the symmetric matrices of Q^* and R^* for $1.7\leq p\leq 1.9$, $1/20\leq x'^2/p^2\leq 1/10$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8<8$ for $1.7\leq p\leq 1.9$, $1/20\leq x'^2/p^2\leq 1/10$, $y\geq 0$.

Case iii) $1.4\leq p\leq 1.7$

In this case we apply the same process as in the case $1.4\leq p\leq 1.7$, $1/40\leq x'^2/p^2$ $\leq 1/20$, $y\geq 0$. Then we have

$$\Re a_8 \leq 8 - \frac{x^2}{96} P(x) - \frac{1}{96} Q - \frac{1}{96} R,$$

where $P(x)$, Q and R are the same as them in the case $1.4 \leq p \leq 1.7$, $1/40 \leq x'^2/p^2 \leq 1/20$, $y \geq 0$. It is not so difficult to prove the positive definiteness of Q and R for $1.4 \leq p \leq 1.7$, $1/20 \leq x'^2/p^2 \leq 1/10$. Thus we have $\Re a_8 < 8$ in the present case.

Case iv) $1 \leq p \leq 1.4$.

We apply the same process as in the case $1 \leq p \leq 1.4$, $1/40 \leq x'^2/p^2 \leq 1/20$, $y \geq 0$ and we have

$$\Re a_8 \leq 8 - \frac{x^2}{96} P(x) - \frac{1}{96} Q - \frac{1}{96} R,$$

Where $P(x)$, Q and R are the same as them in the case $1 \leq p \leq 1.4$, $1/40 \leq x'^2/p^2 \leq 1/20$, $y \geq 0$. It is not so difficult to prove the positive definiteness of Q and R for $1 \leq p \leq 1.4$, $1/20 \leq x'^2/p^2 \leq 1/10$. Thus we have $\Re a_8 < 8$ in the present case.

7.   Case $1 \leq p$, $1/20 \leq x'^2/p^2 \leq 1/10$ and $y \leq 0$. In this case we put $A = 1/2$, $B = 3/2$, $C = 3/5$ in (A). Then we have

(I)
$$
\begin{aligned}
\Re a_8 \leq U + &\left(-\frac{413}{384} p^5 + \frac{9}{4} p^4 + \frac{5}{12} p^2 + \frac{151}{192} p^3 x'^2 - 9 p^2 x'^2 - \frac{5}{12} x'^2 + \frac{149}{128} p x'^4 \right) y \\
&+ \left(-\frac{129}{320} p^4 + \frac{9}{10} p^3 - \frac{27}{160} p^2 x'^2 - \frac{18}{5} p x'^2 + \frac{11}{64} x'^4 \right) \eta \\
&+ \left(-\frac{127}{48} p^3 + \frac{9}{2} p^2 + \frac{1}{6} - \frac{35}{16} p x'^2 \right) y^2 + \left(-\frac{13}{10} p^2 + \frac{18}{5} p - \frac{9}{2} x'^2 \right) y \eta \\
&+ \left(\frac{69}{100} p + \frac{18}{25} \right) \eta^2 + p y \xi + \frac{14}{5} \eta \xi + 2 y \varphi + \frac{3}{20} y^2 \eta \\
&+ \left(\frac{113}{64 \cdot 12} p^5 - \frac{9}{4} p^4 - \frac{25}{48} p^2 - \frac{331}{64 \cdot 12} p^3 x'^2 + \frac{9}{2} p^2 x'^2 + \frac{25}{96} x'^2 - \frac{83}{256} p x'^4 \right) x'^2 \\
&+ \left(-\frac{17}{4} p^2 + 2 x'^2 \right) x' \xi' - \frac{7}{2} p x' \varphi' - 2 x' \tau' - 2 x' y \xi'.
\end{aligned}
$$

Case i) $\eta \geq 0$. We remark the following facts:

$$\left(-\frac{413}{384} p^5 + \frac{1061}{384} p^4 + \frac{5}{12} p^2 + \frac{151}{192} p^3 x'^2 - 9 p^2 x'^2 - \frac{5}{12} x'^2 + \frac{149}{128} p x'^4 \right) y - 2 x' y \xi' \leq 0,$$

$$\left(-\frac{129}{320} p^4 + \frac{141}{320} p^3 - \frac{27}{160} p^2 x'^2 - \frac{18}{5} p x'^2 + \frac{11}{64} x'^4 \right) \eta + \frac{3}{20} y^2 \eta \leq 0.$$

Further by Lemma 2

$$\frac{147}{320}p^3\eta-\frac{197}{384}p^4y\leqq\frac{15}{32}p^3\left(\eta-\frac{10}{9}py\right)$$

$$\leqq-\frac{1145}{13824}p^6+\frac{605}{6912}p^5+\frac{5}{16}p^3+\frac{15}{128}p^4x'^2.$$

Making use of these remarks and applying Lemma 1 to the term $-249.2x/96$ $-591.2x^2/96$ we have, from (I)

$$\Re a_8\leqq 8-\frac{x^2}{96}P_0(x)-\frac{1}{96}Q_0-\frac{1}{96}R_0,$$

$$P_0(x)=1499.2-2980.4x+2119.05x^2-774.32x^3+149.5x^4-12.04x^5,$$

$$Q_0=(254p^3-432p^2-443.4p+1057.7+210px'^2)y^2+(124.8p^2$$

$$-345.6p+432x'^2)y\eta+(-805.24p+1720.38)\eta^2-96py\xi-268.8\eta\xi$$

$$+(2505.3-1034.6p)\xi^2-192y\varphi+(3221.1-1330.2p)\varphi^2,$$

$$R_0=(-14.125p^5+204.75p^4+50p^2-147.8p+357.9+41.375p^3x'^2$$

$$-432p^2x'^2-25x'^2+31.125px'^4)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+(2505.3-1034.6p)\xi'^2+336px'\varphi'$$

$$+(3221.1-1330.2p)\varphi'^2+192x'\tau'+(3936.9-1625.8p)\tau'^2.$$

$P_0(x)$ is monotone decreasing for $0.048\leqq x\leqq 1$ and $P_0(1)>0$. Hence $P_0(x)>0$ for $0.048\leqq x\leqq 1$.

Q_0 is positive definite for $1\leqq p\leqq 1.952$, $1/20\leqq x'^2/p^2\leqq 1/10$. Indeed we have

$$(11.56p-6.68)y^2-192y\varphi+(3221.1-1330.2p)\varphi^2\geqq 0,$$

$$60.22\eta^2-268.8\eta\xi+300\xi^2\geqq 0,$$

$$(-2.23p+51.68)y^2-96py\xi+(2205.3-1034.6p)\xi^2\geqq 0$$

and

$$(254p^3-432p^2-452.73p+1012.7+210px'^2)y^2$$

$$+(124.8p^2-345.6p+432x'^2)y\eta+(-805.24p+1660.16)\eta^2\geqq 0$$

for $1\leqq p\leqq 1.952$, $1/20\leqq x'^2/p^2\leqq 1/10$. This implies the positive definiteness of Q_0 for $1\leqq p\leqq 1.952$, $1/20\leqq x'^2/p^2\leqq 1/10$.

R_0 is positive definite for $1\leqq p\leqq 1.952$, $1/20\leqq x'^2/p^2\leqq 1/10$. Indeed we have

$$(9.46p-5.47)x'^2+192x'\tau'+(3936.9-1625.8p)\tau'^2\geqq 0,$$

$$(-21.22p+213.64)x'^2+336px'\varphi'+(3221.1-1330.2p)\varphi'^2\geqq 0,$$

$$(-14.125p^5+204.75p^4+50p^2-136.04p+149.73+41.375p^3x'^2$$

$$-432p^2x'^2-25x'^2+31.125px'^4)x'^2+(408p^2-192x'^2)x'\xi'$$

$$+(2505.3-1034.6p)\xi'^2\geqq 0$$

for $1\leqq p\leqq 1.952$, $1/20\leqq x'^2/p^2\leqq 1/10$, which imply the desired result. Thus we have $\Re a_8<8$ in the present case.

Case ii) $\eta\leqq 0$, $1.6\leqq p\leqq 1.952$. We remark the following facts:

$$\left(-\frac{413}{384}p^5+\frac{9}{4}p^4+\frac{5}{12}p^2+\frac{151}{192}p^3x'^2-9p^2x'^2-\frac{5}{12}x'^2+\frac{149}{128}px'^4\right)y-2x'y\xi'$$

$$\leqq-\frac{y}{384}\left(3828p^3-5184p^2-1659.46+2194.4\frac{1}{p^2}-447px'^2\right)x'^2$$

$$\leqq\frac{1}{96}\left(663.106p^3-897.998p^2-287.459+380.125\frac{1}{p^2}-77.431px'^2\right)x'^2,$$

$$\left(-\frac{129}{320}p^4+\frac{9}{10}p^3-\frac{27}{160}p^2x'^2-\frac{18}{5}px'^2+\frac{11}{64}x'^4\right)\eta$$

$$\leqq-\frac{\eta}{320}(1344p^2-1728p-55x'^2)x'^2$$

$$\leqq\frac{1}{96}(216.398p^2-278.225p-8.855x'^2)x'^2.$$

Making use of these remarks and applying Lemma 1 to the term $-744x/96-560x^2/96$ we have, from (I)

$$\Re a_8\leqq 8-\frac{x^2}{96}P_1(x)-\frac{1}{96}Q_1-\frac{1}{96}R_1,$$

$$P_1(x)=598-2082x+1726x^2-687.3x^3+141.55x^4-13x^5,$$

$$Q_1=(254p^3-432p^2-420p+1382+210px'^2)y^2+(124.8p^2-345.6p$$

$$+432x'^2)y\eta+(-766.24p+2260.88)\eta^2-96py\xi-268.8\eta\xi$$

$$+(3262-980p)\xi^2-192y\varphi+(4194-1260p)\varphi^2,$$

$$R_1=\left(-14.125p^5+216p^4-663.106p^3+731.6p^2+138.225p+753.459\right.$$

$$\left.-380.125\frac{1}{p^2}+41.375p^3x'^2-432p^2x'^2+77.431px'^2-33.855x'^2\right.$$

$$+31.125 p x'^4 \Big) x'^2 + (408 p^2 - 192 x'^2) x' \xi' + (3262 - 980 p) \xi'^2$$

$$+336 p x' \varphi' + (4194 - 1260 p) \varphi'^2 + 192 x' \tau' + (5126 - 1540 p) \tau'^2.$$

$P_1(x)$ is monotone decreasing for $0.048 \leq x \leq 0.4$ and $P_1(0.4) > 0$. Hence $P_1(x) > 0$ for $0.048 \leq x \leq 0.4$.

Since $Q_1 \geq Q_0$ for $1.6 \leq p \leq 1.952$, $1/20 \leq x'^2/p^2 \leq 1/10$, Q_1 is positive definite there. Here Q_0 was defined in Case i) and was positive definite for $1 \leq p \leq 1.952$, $1/20 \leq x'^2/p^2 \leq 1/10$.

R_1 is positive definite for $1.6 \leq p \leq 1.952$, $1/20 \leq x'^2/p^2 \leq 1/10$. Indeed we have

$$(2.6 p - 0.69) x'^2 + 192 x' \tau' + (5126 - 1540 p) \tau'^2 \geq 0;$$

$$(-22.4 p + 105.73) x'^2 + 336 p x' \varphi' + (4194 - 1260 p) \varphi'^2 \geq 0$$

and

$$x'^2 \Big(-14.125 p^5 + 216 p^4 - 663.106 p^3 + 731.6 p^2 + 158.025 p + 648.419$$

$$-380.125 \frac{1}{p^2} + 41.375 p^3 x'^2 - 432 p^2 x'^2 + 77.431 p x'^2 - 33.855 x'^2$$

$$+31.125 p x'^4 \Big) + (408 p^2 - 192 x'^2) x' \xi' + (3262 - 980 p) \xi'^2 \geq 0$$

for $1.6 \leq p \leq 1.952$, $1/20 \leq x'^2/p^2 \leq 1/10$. This implies the desired result. Thus we have $\Re a_8 < 8$ in the present case.

Case iii) $\eta \leq 0$, $1 \leq p \leq 1.6$. We remark the following facts:

$$\Big(-\frac{413}{384} p^5 + \frac{924}{384} p^4 + \frac{5}{12} p^2 + \frac{151}{192} p^3 x'^2 - 9 p^2 x'^2 - \frac{5}{12} x'^2$$

$$+\frac{149}{128} p x'^4 \Big) y - 2 x' y \xi' \leq 0,$$

$$\Big(-\frac{129}{320} p^4 + \frac{207}{320} p^3 - \frac{27}{160} p^2 x'^2 - \frac{18}{5} p x'^2 + \frac{11}{64} x'^4 \Big) \eta$$

$$\leq -\frac{\eta}{160} (27 p^2 + 576 p - 27.5 x'^2) x'^2$$

$$\leq \frac{1}{96} (12.549 p^2 + 267.702 p - 12.78 x'^2) x'^2.$$

Further by Lemma 2

$$\frac{27}{128}p^3\eta - \frac{5}{32}p^4y = \frac{27}{128}p^3\left(\eta - \frac{20}{27}py\right)$$

$$\leqq -\frac{1141}{55296}p^6 + \frac{9}{64}p^3 + \frac{169}{27648}p^5 + \frac{27}{512}p^4x'^2.$$

Making use of these remarks and applying Lemma 1 to the term $-572.6x/96$ $-292.88x^2/96$, from (I) we have

$$\Re a_8 \leqq 8 - \frac{x^2}{96}P_2(x) - \frac{1}{96}Q_2 - \frac{1}{96}R_2,$$

$$P_2(x) = 1169.74 - 2428.76x + 1838.98x^2 - 710.49x^3 + 143.53x^4 - 13x^5,$$

$$Q_2 = (254p^3 - 432p^2 - 219.66p + 852.77 + 210px'^2)y^2$$

$$+ (124.8p^2 - 345.6p + 432x'^2)y\eta + (-432.34p + 1378.83)\eta^2$$

$$- 96py\xi - 268.8\eta\xi + (2027.13 - 512.54p)\xi^2 - 192y\varphi$$

$$+ (2606.31 - 658.98p)\varphi^2,$$

$$R_2 = (-14.125p^5 + 210.9375p^4 + 37.451p^2 - 340.922p + 289.59$$

$$+ 41.375p^3x'^2 - 432p^2x'^2 - 12.22x'^2 + 31.125px'^4)x'^2$$

$$+ (408p^2 - 192x'^2)x'\xi' + (2027.13 - 512.54p)\xi'^2 + 336px'\varphi'$$

$$+ (2606.31 - 658.98p)\varphi'^2 + 192x'\tau' + (3185.49 - 805.42p)\tau'^2.$$

$P_2(x)$ is monotone decreasing for $0.4 \leqq x \leqq 1$ and $P_2(1) = 0$. Hence $P_2(x) \geqq 0$ for $0.4 \leqq x \leqq 1$.

Since $Q_2 \geqq Q_0$ for $1 \leqq p \leqq 1.6$, $1/20 \leqq x'^2/p^2 \leqq 1/10$, Q_2 is positive definite there. Here Q_0 was defined in Case i) and was positive definite for $1 \leqq p \leqq 1.952$, $1/20 \leqq x'^2/p^2 \leqq 1/10$.

R_2 is positive definite for $1 \leqq p \leqq 1.6$, $1/20 \leqq x'^2/p^2 \leqq 1/10$. Indeed we have

$$(1.98p + 1.9)x'^2 + 192x'\tau' + (3185.49 - 805.42p)\tau'^2 \geqq 0,$$

$$(-42.83p + 115.09)x'^2 + 336px'\varphi' + (2606.31 - 658.98p)\varphi'^2 \geqq 0$$

and

$$(-14.125p^5 + 210.9375p^4 + 37.451p^2 - 300.072p + 172.6$$

$$+ 41.375p^3x'^2 - 432p^2x'^2 - 12.22x'^2 + 31.125px'^4)x'^2$$

$$+ (408p^2 - 192x'^2)x'\xi' + (2027.13 - 512.54p)\xi'^2 \geqq 0$$

for $1 \leqq p \leqq 1.6$, $1/20 \leqq x'^2/p^2 \leqq 1/10$, which imply the desired result.

Thus we have $\Re a_8 < 8$ in the present case.

8. Case $1 \leqq p$, $1/10 \leqq x'^2/p^2 \leqq \tan^2(\pi/7)$ and $y \geqq 0$. It is evident that $p < 1.907$ and $x'^2/p^2 < 0.2379$ in this case.

Case i)   Case $1.8 \leqq p < 1.907$.

We start from (D) with $\mathfrak{A} = -21.2$, $\mathfrak{B} = 37.8$, and use the following trivial inequality

$$\left(-\frac{73}{240}p^5 + \frac{21}{40}p^4 + \frac{13}{10}p^3 - \frac{34}{15}p^2\right)y$$

$$= \frac{1}{240}p^2(73p^2 + 20p - 272)xy$$

$$\leqq \frac{1}{80}p^2(73p^2 + 20p - 272)x^2 + \frac{1}{2880}p^2(73p^2 + 20p - 272)y^2.$$

Then we have

$$\mathfrak{R}a_8 \leqq 8 - \frac{x}{96}(160 - 22x + 3029.8x^2 - 3984.7x^3 + 1949.75x^4 - 418.625x^5 + 34x^6)$$

$$+ \left(-\frac{31}{20}p^3 + \frac{31}{10}p^2 - \frac{257.16}{96}p^3x'^2 - 1.83p^2x'^2 - \frac{10.6}{6}px'^2 - \frac{5}{6}x'^2\right.$$

$$\left. + \frac{73.9}{32}px'^4 - \frac{5.3}{32}\frac{x'^6}{p}\right)y$$

$$+ \left(-\frac{359.9}{160}p^2 + 0.18p + \frac{37.3}{32}x'^2\right)x'^2\eta$$

$$- \frac{1}{96}\left\{\left(-2.434p^4 + 212.08p^3 - 290.94p^2 + 9p - 64 + 223.25px'^2 + 63.6\frac{x'^2}{p}\right.\right.$$

$$\left. + 47.7\frac{x'^4}{p}\right)y^2 + (129p^2 - 288p + 666x'^2)y\eta + \left(-39.24p - 69.12\right.$$

$$\left.\left. + 190.8\frac{x'^2}{p}\right)\eta^2 - 96py\xi - 268.8\eta\xi - 96y\varphi\right\}$$

$$- \frac{1}{96}\left\{\left(195.1225p^5 - 561.6p^4 + 854.8259p^3 - 70.96p^2 - 210.349p\right.\right.$$

$$\left. - 84.8\frac{1}{p} + 0.5525p^3x'^2 - 341.28p^2x'^2 + 113.065px'^2 - 25x'^2\right.$$

$$\left.\left. + 35.287px'^4 + 1.325\frac{x'^6}{p}\right)x'^2 + (408p^2 - 192x'^2)x'\xi' + 336px'\varphi' + 192x'\tau'\right\}$$

$$- \frac{3}{8}py^3 - \frac{1}{5}y^2\eta - \frac{5}{2}x'y\xi'.$$

We remark the following facts:

$$\left(-\frac{31}{20}p^3+\frac{31}{10}p^2-\frac{257.16}{96}p^3x'^2-1.83p^2x'^2-\frac{10.6}{6}px'^2-\frac{5}{6}x'^2\right.$$

$$\left.+\frac{73.9}{32}px'^4-\frac{5.3}{32}\frac{x'^6}{p}\right)y+\frac{33}{10}p^3x'^2y\leqq 0,$$

$$\left(-\frac{359.9}{160}p^2+0.18p+\frac{37.3}{32}x'^2\right)x'^2\eta\leqq\left(\frac{359.9}{160}p^2-0.18p-\frac{37.3}{32}x'^2\right)\sqrt{\frac{4-p^2}{5}}x'^2$$

$$\leqq\frac{1}{96}(84.1951p^2-6.7374p-43.6298x'^2)x'^2,$$

$$-\frac{33}{10}p^3x'^2y-\frac{5}{2}x'y\xi'\leqq\frac{125}{264p^3}y\xi'^2\leqq\frac{125}{264p^3}\sqrt{\frac{4-p^2}{3}}\xi'^2\leqq\frac{3.93}{96}\xi'^2,$$

$$-\frac{3}{8}py^3-\frac{1}{5}y^2\eta\leqq\frac{2}{75p}y\eta^2\leqq\frac{0.72}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96$ $-272.8x^2/96$ we have

$$\Re a_8\leqq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=-254.8+3098x-3984.7x^2+1949.75x^3-418.625x^4+34x^5,$$

$$Q=\left(-2.434p^4+212.08p^3-290.94p^2-195.6p+465.2+223.25px'^2\right.$$

$$+63.6\frac{x'^2}{p}+47.7\frac{x'^4}{p}\right)y^2+(129p^2-288p+666x'^2)y\eta$$

$$+\left(-380.24p+812.16+190.8\frac{x'^2}{p}\right)\eta^2-96py\xi$$

$$-268.8\eta\xi+(1234.8-477.4p)\xi^2-96y\varphi+(1587.6-613.8p)\varphi^2,$$

$$R=\left(195.1225p^5-561.6p^4+854.8259p^3-155.1551p^2-271.8116p\right.$$

$$+176.4-84.8\frac{1}{p}+0.5525p^3x'^2-341.28p^2x'^2$$

$$+113.065px'^2+18.6298x'^2+35.287px'^4+1.325\frac{x'^6}{p}\right)x'^2$$

$$+(408p^2-192x^2)x'\xi'+(1230.87-477.4p)\xi'^2+336px'\varphi'$$

$$+(1587.6-613.8p)\varphi'^2+192x'\tau'+(1940.4-750.2p)\tau'^2.$$

$P(x)$ is monotone increasing for $0.093 \leqq x \leqq 0.2$ and $P(0.093) > 0$. Hence $P(x) > 0$ for $0.093 \leqq x \leqq 0.2$.

Since $(8.1p - 9.8)y^2 - 96y\varphi + (1587.6 - 613.8p)\varphi^2 \geqq 0$ and $(26.7p - 32.4)x'^2 + 192x'\tau' + (1940.4 - 750.2p)\tau'^2 \geqq 0$ for $1.8 \leqq p < 1.907$ we consider $Q^* = Q - (8.1p - 9.8)y^2 + 96y\varphi - (1587.6 - 613.8p)\varphi^2$ and $R^* = R - (26.7p - 32.4)x'^2 - 192x'\tau' - (1940.4 - 750.2p)\tau'^2$. We can prove the positive definiteness of the symmetric matrices of Q^* and R^* for $1.8 \leqq p < 1.907$, $1/10 \leqq x'^2/p^2 < 0.2379$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8 < 8$ for $1.8 \leqq p < 1.907$, $1/10 \leqq x'^2/p^2 \leqq \tan^2(\pi/7)$, $y \geqq 0$.

Case ii) $1.7 \leqq p \leqq 1.8$.

In this case we start from (F) with $\mathfrak{A} = 0$, $\mathfrak{B} = 37.8$, $\alpha = 1$. We remark the following facts:

$$\left(-\frac{43}{30}p^3 + \frac{43}{15}p^2 - \frac{115.33}{48}p^3x'^2 - 1.83p^2x'^2 - \frac{5}{6}x'^2 + \frac{63}{64}px'^4 \right)y + \frac{5}{2}p^3x'^2y \leqq 0,$$

$$\left(-\frac{306.9}{160}p^2 + 0.18p + \frac{11}{64}x'^2 \right)x'^2\eta \leqq \left(\frac{306.9}{160}p^2 - 0.18p - \frac{11}{64}x'^2 \right)\sqrt{\frac{4-p^2}{5}}\,x'^2$$

$$\leqq \frac{1}{96}(86.766768p^2 - 8.142336p - 7.7748x'^2)x'^2,$$

$$-\frac{5}{2}p^3x'^2y - \frac{5}{2}x'y\xi' \leqq \frac{5}{8p^3}y\xi'^2 \leqq \frac{5}{8p^3}\sqrt{\frac{4-p^2}{3}}\,\xi'^2 \leqq \frac{7.5}{96}\xi'^2,$$

$$-\frac{3}{8}py^3 - \frac{1}{5}y^2\eta \leqq \frac{2}{75p}y\eta^2 \leqq \frac{0.92}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96 - 484x^2/96$ we have

$$\Re a_8 \leqq 8 - \frac{x^2}{96}P(x) - \frac{1}{96}Q - \frac{1}{96}R,$$

$$P(x) = -248.4 + 1685.2x - 2390.3x^2 + 1345.75x^3 - 345.625x^4 + 34x^5,$$

$$Q = (-14.6p^4 + 208.75p^3 - 251.2p^2 - 354p + 782 + 90.75px'^2)y^2$$

$$+ (129p^2 - 288p + 348x'^2)y\eta + (-644.24p + 1339.96)\eta^2 - 268.8\eta\xi - 96py\xi$$

$$+ (1974 - 847p)\xi^2 - 96y\varphi + (2538 - 1089p)\varphi^2,$$

$$R = (193.7975p^5 - 561.6p^4 + 741.76p^3 - 157.7268p^2 + 129.0623p + 282$$

$$- 3.4225p^3x'^2 - 341.28p^2x'^2 - 17.2252x'^2 + 31.3125px'^4)x'^2$$

$$+ (408p^2 - 192x'^2)x'\xi' + (1966.5 - 847p)\xi'^2 + 336px'\varphi'$$

$$+ (2538 - 1089p)\varphi'^2 + 192x'\tau' + (3102 - 1331p)\tau'^2.$$

$P(x)$ is monotone increasing for $0.2 \leq x \leq 0.3$ and $P(0.2) > 0$. Hence $P(x) > 0$ for $0.2 \leq x \leq 0.3$.

Since $(10.14p - 13.88)y^2 - 96y\varphi + (2538 - 1089p)\varphi^2 \geq 0$ and $(33.3p - 45.6)x'^2 + 192x'\tau' + (3102 - 1331p)\tau'^2 \geq 0$ for $1.7 \leq p \leq 1.8$ we consider $Q^* = Q - (10.14p - 13.88)y^2 + 96y\varphi - (2538 - 1089p)\varphi^2$ and $R^* = R - (33.3p - 45.6)x'^2 - 192x'\tau' - (3102 - 1331p)\tau'^2$. We can prove the positive definiteness of the symmetric matrices of Q^* and R^* for $1.7 \leq p \leq 1.8$, $1/10 \leq x'^2/p^2 \leq 0.2379$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8 < 8$ for $1.7 \leq p \leq 1.8$, $1/10 \leq x'^2/p^2 \leq \tan^2(\pi/7)$, $y \geq 0$.

Case iii) $1.4 \leq p \leq 1.7$.

In this case we start from (G) with $\mathfrak{A} = 0$, $\mathfrak{B} = 37.8$, $\alpha = 1.1$. We remark the following facts:

$$\left(-\frac{115.33}{48}p^3 - 1.83p^2 - \frac{5}{6} + \frac{63}{64}px'^2\right)x'^2y + \frac{17}{5}p^3x'^2y \leq 0,$$

$$\left(-\frac{613.8}{320}p^2 + 0.18p + \frac{11}{64}x'^2\right)x'^2\eta \leq \left(\frac{613.8}{320}p^2 - 0.18p - \frac{11}{64}x'^2\right)\sqrt{\frac{4-p^2}{5}}x'^2$$

$$\leq \frac{1}{96}(117.628632p^2 - 11.038464p - 10.5402x'^2)x'^2,$$

$$-\frac{17}{5}p^3x'^2y - \frac{5}{2}x'y\xi' \leq \frac{125}{272p^3}y\xi'^2 \leq \frac{125}{272p^3}\sqrt{\frac{4-p^2}{3}}\xi'^2 \leq \frac{13.3}{96}\xi'^2,$$

$$-\frac{3}{8}py^3 - \frac{1}{5}y^2\eta \leq \frac{2}{75p}y\eta^2 \leq \frac{1.15}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96 -329.2x^2/96$ we have

$$\Re a_8 \leq 8 - \frac{x^2}{96}P(x) - \frac{1}{96}Q - \frac{1}{96}R,$$

$$P(x) = -403.36 + 1981.22x - 2498.54x^2 + 1357.83x^3 - 347.085x^4 + 34x^5,$$

$$Q = (-13.28p^4 + 209.11p^3 - 318.19p^2 - 237.9p + 549.8 + 90.75px'^2)y^2$$

$$+ (129p^2 - 288p + 348x'^2)y\eta + (-450.74p + 952.37)\eta^2 - 96py\xi$$

$$- 268.8\eta\xi + (1432.2 - 576.1p)\xi^2 - 96y\varphi + (1841.4 - 740.7p)\varphi^2,$$

$$R = (193.7975p^5 - 561.6p^4 + 741.76p^3 - 188.5887p^2 + 170.6584p$$

$$+ 204.6 - 3.4225p^3x'^2 - 341.28p^2x'^2 - 14.4598x'^2 + 31.3125px'^4)x'^2$$

$$+ (408p^2 - 192x'^2)x'\xi' + (1418.9 - 576.1p)\xi'^2 + 336px'\varphi'$$

$$+ (1841.4 - 740.7p)\varphi'^2 + 192x'\tau' + (2250.6 - 905.3p)\tau'^2.$$

$P(x)$ is monotone increasing for $0.3 \leqq x \leqq 0.6$ and $P(0.3) > 0$. Hence $P(x) > 0$ for $0.3 \leqq x \leqq 0.6$.

Since $(5.8p - 5.2)y^2 - 96y\varphi + (1841.4 - 740.7p)\varphi^2 \geqq 0$ and $(19.3p - 17.6)x'^2 + 192x'\tau'$ $+ (2250.6 - 905.3p)\tau'^2 \geqq 0$ for $1.4 \leqq p \leqq 1.7$ we consider $Q^* = Q - (5.8p - 5.2)y^2 + 96y\varphi$ $- (1841.4 - 740.7p)\varphi^2$ and $R^* = R - (19.3p - 17.6)x'^2 - 192x'\tau' - (2250.6 - 905.3p)\tau'^2$. We can prove the positive definiteness of the symmetric matrices of Q^* and R^* for $1.4 \leqq p \leqq 1.7$, $1/10 \leqq x'^2/p^2 < 0.2379$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8 < 8$ for $1.4 \leqq p \leqq 1.7$, $1/10 \leqq x'^2/p^2 \leqq \tan^2(\pi/7)$, $y \geqq 0$.

Case iv) $1 \leqq p \leqq 1.4$.

We start from (G) with $\mathfrak{A} = 0$, $\mathfrak{B} = 37.8$, $\alpha = 1.1$. We remark the following facts:

$$\left(-\frac{115.33}{48}p^3 - 1.83p^2 - \frac{5}{6} + \frac{63}{64}px'^2\right)x'^2y + \frac{37}{10}p^3x'^2y \leqq 0,$$

$$\left(-\frac{613.8}{320}p^2 + 0.18p + \frac{11}{64}x'^2\right)x'^2\eta$$

$$\leqq \frac{1}{96}(142.634844p^2 - 13.385088p - 12.7809x'^2)x'^2,$$

$$-\frac{37}{10}p^3x'^2y - \frac{5}{2}x'y\xi' \leqq \frac{125}{296p^3}y\xi'^2 \leqq \frac{125}{296p^3}\sqrt{\frac{4-p^2}{3}}\xi'^2 \leqq \frac{40.6}{96}\xi'^2,$$

$$-\frac{3}{8}py^3 - \frac{1}{5}y^2\eta \leqq \frac{2}{75p}y\eta^2 \leqq \frac{2}{75p}\sqrt{\frac{4-p^2}{3}}\eta^2 \leqq \frac{2.56}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96 - 490x^2/96$ we have

$$\Re a_8 \leqq 8 - \frac{x^2}{96}P(x) - \frac{1}{96}Q - \frac{1}{96}R,$$

$$P(x) = -564.16 + 2021.42x - 2498.54x^2 + 1357.83x^3 - 347.085x^4 + 34x^5,$$

$$Q = (-13.28p^4 + 209.11p^3 - 318.19p^2 - 358.5p + 791 + 90.75px'^2)y^2$$

$$+ (129p^2 - 288p + 348x'^2)y\eta + (-651.74p + 1353.32)\eta^2 - 96py\xi - 268.8\eta\xi$$

$$+ (1995 - 857.5p)\xi^2 - 96y\varphi + (2565 - 1102.5p)\varphi^2,$$

$$R = (193.7975p^5 - 561.6p^4 + 741.76p^3 - 213.5949p^2 + 132.805p + 285$$

$$- 3.4225p^3x'^2 - 341.28p^2x'^2 - 12.2191x'^2 + 31.3125px'^4)x'^2$$

$$+ (408p^2 - 192x'^2)x'\xi' + (1954.4 - 857.5p)\xi'^2 + 336px'\varphi'$$

$$+ (2565 - 1102.5p)\varphi'^2 + 192x'\tau' + (3135 - 1347.5p)\tau'^2.$$

$P'(x)$ is monotone decreasing for $0.6 \leqq x \leqq 1$ and $P'(1) < 0$, $P'(0.6) > 0$. Further $P(1) > 0$, $P(0.6) > 0$. Hence $P(x) > 0$ for $0.6 \leqq x \leqq 1$.

Since $(4.8p - 3.2)y^2 - 96y\varphi + (2565 - 1102.5p)\varphi^2 \geqq 0$ and $(15.8p - 10.6)x'^2 + 192x'\tau' + (3135 - 1347.5p)\tau'^2 \geqq 0$ for $1 \leqq p \leqq 1.4$ we consider $Q^* = Q - (4.8p - 3.2)y^2 + 96 y\varphi - (2565 - 1102.5p)\varphi^2$ and $R^* = R - (15.8p - 10.6)x'^2 - 192x'\tau' - (3135 - 1347.5p)\tau'^2$. We can prove the positive definiteness of the symmetric matrices of Q^* and R^* for $1 \leqq p \leqq 1.4$, $1/10 \leqq x'^2/p^2 < 0.2379$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8 < 8$ for $1 \leqq p \leqq 1.4$, $1/10 \leqq x'^2/p^2 \leqq \tan^2(\pi/7)$, $y \geqq 0$.

9. Case $1 \leqq p$, $1/10 \leqq x'^2/p^2 \leqq \tan^2(\pi/7)$ and $y \leqq 0$. In this case we put $A = 1/2$, $B = 3/2$, $C = 3/5$ in (A).

Applying the trivial inequality $2AXY \leqq |A|\alpha X^2 + |A|\alpha' Y^2$, $\alpha' = 1/\alpha$, $\alpha > 0$ we have

$$\left(-\frac{413}{384}p^5 + \frac{9}{4}p^4 + \frac{5}{12}p^2 + \frac{151}{192}p^3 x'^2 - 9p^2 x'^2 - \frac{5}{12}x'^2 + \frac{149}{128}px'^4 \right) y$$

$$\leqq -\frac{1}{384}(1433.839p^3 - 175.392p^2 - 512.48 - 447px'^2)x'^2 y$$

$$\leqq \frac{\alpha}{96}(179.23p^3 - 21.92p^2 - 64.06 - 55.87px'^2)y^2$$

$$+ \frac{1}{96\alpha}(179.23p^3 - 21.92p^2 - 64.06 - 55.87px'^2)x'^4,$$

$$-2x'y\xi' \leqq \gamma y^2 + \frac{1}{\gamma}x'^2\xi'^2, \qquad \gamma > 0.$$

Making use of these remarks we have

$$\Re a_8 \leqq 8 - \frac{x}{96}(744 + 972x - 2222x^2 + 1726x^3 - 687.3x^4 + 141.55x^5 - 12.04x^6)$$

$$+ \left(-\frac{129}{320}p^4 + \frac{9}{10}p^3 - \frac{27}{160}p^2 x'^2 - \frac{18}{5}px'^2 + \frac{11}{64}x'^4 \right)\eta + \frac{3}{20}y^2\eta$$

$$- \frac{1}{96}[\{(254 - 179.23\alpha)p^3 - (432 - 21.92\alpha)p^2 - (16 - 64.06\alpha + 96\gamma)$$

$$+ (210 + 55.87\alpha)px'^2\}y^2 + (124.8p^2 - 345.6p + 432x'^2)y\eta$$

(J)

$$+ (-66.24p - 69.12)\eta^2 - 96py\xi - 268.8\eta\xi - 192y\varphi]$$

$$- \frac{1}{96}\left[\left\{ -14.125p^5 + 216p^4 + 50p^2 + \left(41.375 - \frac{179.23}{\alpha} \right)p^3 x'^2 \right. \right.$$

$$-\left(432-\frac{21.92}{\alpha}\right)p^2x'^2-\left(25-\frac{64.06}{\alpha}\right)x'^2+\left(31.125+\frac{55.87}{\alpha}\right)px'^4\Big]x'^2$$

$$+(408p^2-192x'^2)x'\xi'-\frac{96}{\gamma}x'^2\xi'^2+336px'\varphi'+192x'\tau'\Big].$$

Case i) $1.9\leqq p$.

If $\eta\leqq 0$, we have

$$\left(-\frac{129}{320}p^4+\frac{9}{10}p^3-\frac{27}{160}p^2x'^2-\frac{18}{5}px'^2+\frac{11}{64}x'^4\right)\eta+\frac{3}{20}y^2\eta$$

$$\leqq-\frac{\eta}{320}(596.187p^2-58.464p-55x'^2)x'^2.$$

If $\eta\geqq 0$, by applying Lemma 1 we have

$$\left(-\frac{129}{320}p^4+\frac{9}{10}p^3-\frac{27}{160}p^2x'^2-\frac{18}{5}px'^2+\frac{11}{64}x'^4\right)\eta+\frac{3}{20}y^2\eta$$

$$\leqq\frac{\eta}{320}\left(-1344p^2+1728p-176+640\frac{1}{p^2}+55x'^2\right)x'^2$$

$$\leqq\frac{\eta}{320}(596.187p^2-58.464p-55x'^2)x'^2.$$

Hence we have

$$\left(-\frac{129}{320}p^4+\frac{9}{10}p^3-\frac{27}{160}p^2x'^2-\frac{18}{5}px'^2+\frac{11}{64}x'^4\right)\eta+\frac{3}{20}y^2\eta$$

$$\leqq\frac{\beta}{96}(89.43p^2-8.76p-8.25x'^2)\eta^2+\frac{1}{96\beta}(89.43p^2-8.76p-8.25x'^2)x'^4.$$

Thus, from (J), we have

$$\Re a_8\leqq 8-\frac{x}{96}(744+972x-2222x^2+1726x^3-687.3x^4+141.55x^5-12.04x^6)$$

$$-\frac{1}{96}[\{(254-179.23\alpha)p^3-(432-21.92\alpha)p^2-(16-64.06\alpha+96\gamma)$$

$$+(210+55.87\alpha)px'^2\}y^2+(124.8p^2-345.6p+432x'^2)y\eta$$

$$+\{-89.43\beta p^2+(-66.24+8.76\beta)p-69.12+8.25\beta x'^2\}\eta^2$$

$$(K)\qquad -96py\xi-268.8\eta\xi-192y\varphi]$$

$$-\frac{1}{96}\Bigg[\Big\{-14.125p^5+216p^4+50p^2+\Big(41.375-\frac{179.23}{\alpha}\Big)p^3x'^2$$

$$-\Big(432-\frac{21.92}{\alpha}+\frac{89.43}{\beta}\Big)p^2x'^2+\frac{8.76}{\beta}px'^2-\Big(25-\frac{64.06}{\alpha}\Big)x'^2$$

$$+\Big(31.125+\frac{55.87}{\alpha}\Big)px'^4+\frac{8.25}{\beta}x'^4\Big\}x'^2+(408p^2-192x'^2)x'\xi'$$

$$-\frac{96}{\gamma}x'^2\xi'^2+336px'\varphi'+192x'\tau'\Bigg].$$

Here we put $\alpha=4/5$, $\beta=12/5$, $\gamma=1/2$. Applying Lemma 1 to the term $-744x/96$ $-976x^2/96$ we have

$$\Re a_8\leqq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=182-1978x+1726x^2-687.3x^3+141.55x^4-12.04x^5,$$

$$Q=(110.61p^3-414.47p^2-732p+2009.24+254.69px'^2)y^2$$

$$+(124.8p^2-345.6p+432x'^2)y\eta+(-214.64p^2-1265.22p+3300.88+19.8x'^2)\eta^2$$

$$-96py\xi-268.8\eta\xi-(4718-1708p)\xi^2-192y\varphi+(6066-2196p)\varphi^2,$$

$$R=(-14.125p^5+216p^4+50p^2-244p+674-182.67p^3x'^2-442.17p^2x'^2$$

$$+3.59px'^2+55.07x'^2+100.96px'^4+3.38x'^4)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+(4718-1708p-192x'^2)\xi'^2+336px'\varphi'$$

$$+(6066-2196p)\varphi'^2+192x'\tau'+(7414-2684p)\tau'^2.$$

$P(x)$ is monotone decreasing for $0.093\leqq x\leqq 0.1$ and $P(0.1)>0$. Hence $P(x)>0$ for $0.093\leqq x\leqq 0.1$.

Since for $1.9\leqq p\leqq 1.907$, $1/10\leqq x'^2/p^2\leqq 0.2379$

$$(6.38p-7.25)y^2-192y\varphi+(6066-2196p)\varphi^2\geqq 0,$$

$$22.58\eta^2-268.8\eta\xi+800\xi^2\geqq 0,$$

$$(-1.35p+15.27)y^2-96py\xi+(3918-1708p)\xi^2\geqq 0$$

and

$$(110.61p^3-414.47p^2-737.03p+2001.22+254.69px'^2)y^2$$

$$+(124.8p^2-345.6p+432x'^2)y\eta+(-214.64p^2-1265.22p$$

$$+3278.3+19.8x'^2)\eta^2\geqq 0,$$

Q is positive definite there. Since for $1.9 \leqq p \leqq 1.907$, $1/10 \leqq x'^2/p^2 \leqq 0.2379$

$$(5.23p - 5.95)x'^2 + 192x'\tau' + (7414 - 2684p)\tau'^2 \geqq 0,$$

$$(-12.86p + 79.21)x'^2 + 336px'\varphi' + (6066 - 2196p)\varphi'^2 \geqq 0$$

and

$$(-14.125p^5 + 216p^4 + 50p^2 - 236.37p + 600.74 - 182.67p^3x'^2$$

$$-442.17p^2x'^2 + 3.59px'^2 + 55.07x'^2 + 100.96px'^4 + 3.38x'^4)x'^2$$

$$+(408p^2 - 192x'^2)x'\xi' + (4718 - 1708p - 192x'^2)\xi'^2 \geqq 0,$$

R is positive definite there.

Thus we have $\Re a_8 < 8$ for $1.9 \leqq p$, $1/10 \leqq x'^2/p^2 \leqq \tan^2 (\pi/7)$, $y \leqq 0$.

Case ii) $1.7 \leqq p \leqq 1.9$.

We put $\alpha = 3/4$, $\beta = 2$, $\gamma = 1/2$ in (K). Applying Lemma 1 to the term $-774x/96$ $-680x^2/96$ we have

$$\Re a_8 \leqq 8 - \frac{x^2}{96} P(x) - \frac{1}{96}Q - \frac{1}{96}R,$$

$$P(x) = 478 - 2052x + 1726x^2 - 687.3x^3 + 141.55x^4 - 12.04x^5,$$

$$Q = (119.57p^3 - 415.56p^2 - 510p + 1562.04 + 251.9px'^2)y^2$$

$$+(124.8p^2 - 345.6p + 432x'^2)y\eta + (-178.86p^2 - 898.72p + 2560.88 + 16.5x'^2)\eta^2$$

$$-96py\xi - 268.8\eta\xi + (3682 - 1190p)\xi^2 - 192y\varphi + (4734 - 1530p)\varphi^2,$$

$$R = (-14.125p^5 + 216p^4 + 50p^2 - 170p + 526 - 198.8p^3x'^2 - 447.57p^2x'^2$$

$$+4.38px'^2 + 60.19x'^2 + 105.43px'^4 + 4.12x'^4)x'^2$$

$$+(408p^2 - 192x'^2)x'\xi' + (3682 - 1190p - 192x'^2)\xi'^2 + 336px'\varphi'$$

$$+(4734 - 1530p)\varphi'^2 + 192x'\tau' + (5786 - 1870p)\tau'^2.$$

$P(x)$ is monotone decreasing for $0.1 \leqq x \leqq 0.3$ and $P(0.3) > 0$. Hence $P(x) > 0$ for $0.1 \leqq x \leqq 0.3$.

It is not so difficult to prove the positive definiteness of Q and R for $1.7 \leqq p \leqq 1.9$, $1/10 \leqq x'^2/p^2 \leqq 0.2379$.

Therefore we have $\Re a_8 < 8$ in the present case.

Case iii) $1.4 \leqq p \leqq 1.7$.

We put $\alpha = 4/5$, $\beta = 3$, $\gamma = 1/2$ in (K).   Applying Lemma 1 to the term $-744x/96$ $-370x^2/96$ we have

$$\Re a_8 \leqq 8 - \frac{x^2}{96} P(x) - \frac{1}{96} Q - \frac{1}{96} R,$$

$$P(x) = 788 - 2129.5x + 1726x^2 - 687.3x^3 + 141.55x^4 - 12.04x^5,$$

$$Q = (110.61p^3 - 414.47p^2 - 277.5p + 1100.24 + 254.69px'^2)y^2$$

$$+ (124.8p^2 - 345.6p + 432x'^2)y\eta + (-268.29p^2 - 502.46p + 1785.88 + 24.75x'^2)\eta^2$$

$$- 96py\xi - 268.8\eta\xi + (2597 - 647.5p)\xi^2 - 192y\varphi + (3339 - 832.5p)\varphi^2,$$

$$R = (-14.125p^5 + 216p^4 + 50p^2 - 92.5p + 371 - 182.67p^3x'^2 - 434.41p^2x'^2$$

$$+ 2.92px^2 + 55.07x'^2 + 100.96px'^4 + 2.75x'^4)x'^2$$

$$+ (408p^2 - 192x'^2)x'\xi' + (2597 - 647.5p - 192x'^2)\xi'^2 + 336px'\varphi'$$

$$+ (3339 - 832.5p)\varphi'^2 + 192x'\tau' + (4081 - 1017.5p)\tau'^2.$$

$P(x)$ is monotone decreasing for $0.3 \leqq x \leqq 0.6$ and $P(0.6) > 0$.   Hence $P(x) > 0$ for $0.3 \leqq x \leqq 0.6$.

It is not so difficult to prove the positive definiteness of Q and R for $1.4 \leqq p$ $\leqq 1.7$, $1/10 \leqq x'^2/p^2 \leqq 0.2379$.

Therefore we have $\Re a_8 < 8$ in the present case.

Case iv) $1 \leqq p \leqq 1.4$.

In this case we put $\alpha = 1$, $\gamma = 1/2$ in (J).   Using the following inequality

$$\left(-\frac{129}{120} p^4 + \frac{9}{10} p^3 - \frac{27}{160} p^2x'^2 - \frac{18}{5} px'^2 + \frac{11}{64} x'^4 \right)\eta + \frac{3}{20} y^2\eta$$

$$\leqq \frac{|\eta|}{48} (89.43p^2 - 8.76p + 50.7 - 8.25x'^2)x'^2$$

$$\leqq \frac{1}{32} (89.43p^2 - 8.76p + 50.7 - 8.25x'^2)\eta^2$$

$$+ \frac{1}{144} (89.43p^2 - 8.76p + 50.7 - 8.25x'^2)x'^4$$

and applying Lemma 1 to the term $-744x/96 - 138x^2/96$ we have

$$\Re a_8 \leqq 8 - \frac{x^2}{96}P(x) - \frac{1}{96}Q - \frac{1}{96}R,$$

$$P(x) = 1020 - 2187.5x + 1726x^2 - 687.3x^3 + 141.55x^4 - 12.04x^5,$$

$$Q = (74.77p^3 - 410.08p^2 - 103.5p + 765.06 + 265.87px'^2)y^2$$

$$+ (124.8p^2 - 345.6p + 432x'^2)y\eta + (-268.29p^2 - 212.46p + 1053.78 + 24.75x'^2)\eta^2$$

$$- 96py\xi - 268.8\eta\xi + (1785 - 241.5p)\xi^2 - 192y\varphi + (2295 - 310.5p)\varphi^2,$$

$$R = (-14.125p^5 + 216p^4 + 50p^2 - 34.5p + 255 - 137.86p^3x'^2 - 439.89p^2x'^2$$

$$+ 2.92px'^2 + 22.16x'^2 + 86.99px'^4 + 2.75x'^4)x'^2$$

$$+ (408p^2 - 192x'^2)x'\xi' + (1785 - 241.5p - 192x'^2)\xi'^2 + 336px'\varphi'$$

$$+ (2295 - 310.5p)\varphi'^2 + 192x'\tau' + (2805 - 379.5p)\tau'^2.$$

$P(x)$ is monotone decreasing for $0.6 \leqq x \leqq 1$ and $P(1) > 0$. Hence $P(x) > 0$ for $0.6 \leqq x \leqq 1$.

It is not so difficult to prove the positive definiteness of Q and R for $1 \leqq p \leqq 1.4$, $1/10 \leqq x^2/p^2 \leqq 0.2379$.

Therefore we have $\Re a_8 < 8$ in the present case.

10. Case $0 \leqq p \leqq 1$. We divide this case into several subcases.

Case i) $0.8 \leqq p \leqq 1$, $x'^2/p^2 \leqq 1/20$ and $y \geqq 0$.

In this case we start from (A) with $A = 1/2$, $B = 3/2$, $C = 3/2$. Then we have

$$\Re a_8 \leqq U(p) + \frac{3}{64}p^4\left(\eta + \frac{13}{6}py\right) + \frac{9}{8}p^3x(\eta + py) + \frac{5}{96}p^2(8 - p^3)y$$

$$+ \left(\frac{151}{192}p^3 - 9p^2 - \frac{5}{12} + \frac{149}{128}px'^2\right)x'^2y + \left(\frac{81}{32}p^2 - 9p + \frac{11}{64}x'^2\right)x'^2\eta$$

$$+ \left(-\frac{127}{48}p^3 + \frac{9}{2}p^2 + \frac{1}{6} - \frac{35}{16}px'^2\right)y^2 + \left(-\frac{41}{8}p^2 + 9p - \frac{27}{8}x'^2\right)y\eta$$

$$+ \left(-3p + \frac{9}{2}\right)\eta^2 + py\xi + \eta\xi + 2y\varphi - \frac{3}{4}y^2\eta$$

$$+ \left(\frac{113}{64 \cdot 12}p^5 - \frac{9}{4}p^4 - \frac{25}{48}p^2 - \frac{331}{64 \cdot 12}p^3x'^2 + \frac{9}{2}p^2x'^2 + \frac{25}{96}x'^2 - \frac{83}{256}px'^4\right)x'^2$$

$$+ \left(-\frac{17}{4}p^2 + 2x'^2\right)x'\xi' - \frac{7}{2}px'\varphi' - 2x'\tau' - 2x'y\xi'.$$

Now we remark the following facts:

$$U(p) \leqq U(1) < 1.101973,$$

$$\frac{3}{64} p^4 \left(\eta + \frac{13}{6} py \right) \leqq \frac{3}{64} p^4 \left\{ \frac{16}{9} p^2(2-p) + \frac{1}{12}(8-p^3) + \frac{1}{4} px'^2 \right\} < 0.111264,$$

$$\frac{5}{96} p^2(8-p^3)y \leqq \frac{5}{96} p^2(8-p^3) \sqrt{\frac{4-p^2}{3}} < 0.364584,$$

$$\left(\frac{151}{192} p^3 - 9p^2 - \frac{5}{12} + \frac{149}{128} px'^2 \right) x'^2 y + \frac{857}{100} p^3 x'^2 y \leqq 0,$$

$$-\frac{857}{100} p^3 x'^2 y - 2x'y\xi' \leqq \frac{100}{857} y\xi'^2 \leqq \frac{100}{857} \sqrt{\frac{4-p^2}{3}} \xi'^2 \leqq \frac{23.16}{96} \xi'^2,$$

$$\left(\frac{81}{32} p^2 - 9p + \frac{11}{64} x'^2 \right) x'^2 \eta \leqq \left(-\frac{81}{32} p^2 + 9p - \frac{11}{64} x'^2 \right) x'^2 \sqrt{\frac{4-p^2}{5}} < 0.250535.$$

Further if $\eta \geqq 0$

$$\frac{9}{8} p^3 x(\eta + py) - \frac{3}{4} y^2 \eta \leqq \frac{9}{8} p^3 x \left\{ \frac{9}{16} p^2(2-p) + \frac{1}{12}(8-p^3) + \frac{1}{4} px'^2 \right\} < 1.303125.$$

If $\eta \leqq 0$

$$\frac{9}{8} p^3 x(\eta + py) - \frac{3}{4} y^2 \eta = \frac{3}{8} \eta(3p^3 x - 2y^2) + \frac{9}{8} p^4 xy$$

$$\leqq \frac{3}{8} \sqrt{\frac{4-p^2}{5}} \frac{(9p^3 - 2p - 4)x}{3} + \frac{9}{8} p^4 x \sqrt{\frac{4-p^2}{3}} < 1.303125.$$

Making use of these remarks we have

$$\Re a_8 < 8 - 4.868519$$

$$- \frac{1}{96} \{ (254p^3 - 432p^2 - 16 + 210px'^2)y^2 + (492p^2 - 864p + 324x'^2)y\eta$$

$$+ (288p - 432)\eta^2 - 96py\xi - 96\eta\xi - 192y\varphi \}$$

$$- \frac{1}{96} \{ (-14.125p^5 + 216p^4 + 50p^2 + 41.375p^3 x'^2 - 432p^2 x'^2 - 25x'^2 + 31.125px'^4)x'^2$$

$$+ (408p^2 - 192x'^2)x'\xi' + 336px'\varphi' + 192x'\tau' - 23.16\xi'^2 \}.$$

By applying Lemma 1 we have

$$-4.868 \leqq -1.217(p^2 + x'^2 + 3y^2 + 5\eta^2 + 7\xi^2 + 7\xi'^2 + 9\varphi^2 + 9\varphi'^2 + 11\tau'^2)$$

$$-1.217p^2 \leqq -0.30425(p^4 + p^2x'^2 + 3p^2y^2 + 5p^2\eta^2 + 7p^2\xi^2 + 7p^2\xi'^2$$
$$+ 9p^2\varphi^2 + 9p^2\varphi'^2 + 11p^2\tau'^2).$$

Hence we have

$$\Re a_8 < 8 - 0.304p^4 - \frac{1}{96}Q - \frac{1}{96}R,$$

$$Q = (254p^3 - 344.376p^2 + 334.496 + 210px'^2)y^2 + (492p^2 - 864p + 324x'^2)y\eta$$

$$+ (146.04p^2 + 288p + 152.16)\eta^2 - 96py\xi - 96\eta\xi + (204.456p^2 + 817.824)\xi^2$$

$$- 192y\varphi + (262.872p^2 + 1051.488)\varphi^2,$$

$$R = (-14.125p^5 + 216p^4 + 79.208p^2 + 116.832$$

$$+ 41.375p^3x'^2 - 432p^2x'^2 - 25x'^2 + 31.125px'^4)x'^2$$

$$+ (408p^2 - 192x'^2)x'\xi' + (204.456p^2 + 794.664)\xi'^2 + 336px'\varphi'$$

$$+ (262.872p^2 + 1051.488)\varphi'^2 + 192x'\tau' + (321.288p^2 + 1285.152)\tau'^2.$$

Since for $0.8 \leqq p \leqq 1$, $x'^2/p^2 \leqq 1/20$

$$7.556y^2 - 192y\varphi + (262.872p^2 + 1051.488)\varphi^2 \geqq 0,$$

$$5.12y^2 - 96py\xi + 450\xi^2 \geqq 0,$$

$$4.621\eta^2 - 96\eta\xi + (204.456p^2 + 367.824)\xi^2 \geqq 0$$

and

$$(254p^3 - 344.376p^2 + 321.82 + 210px'^2)y^2 + (492p^2 - 864p$$

$$+ 324x'^2)y\eta + (146.04p^2 + 288p + 147.539)\eta^2 \geqq 0,$$

Q is positive definite there. Since for $0.8 \leqq p \leqq 1$, $x'^2/p^2 \leqq 1/20$

$$6.2x'^2 + 192x'\tau' + (321.288p^2 + 1285.152)\tau'^2 \geqq 0,$$

$$21.5x'^2 + 336px'\varphi' + (262.872p^2 + 1051.488)\varphi'^2 \geqq 0$$

and

$$(-14.125p^5+216p^4+79.208p^2+89.132+41.375p^3x'^2-432p^2x'^2$$

$$-25x'^2+31.125px'^4)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+(204.456p^2+794.664)\xi'^2\geqq0,$$

R is positive definite there.   Thus in the present case $\Re a_8<8$.

Case ii) $0.8\leqq p\leqq1$, $x'^2/p^2\leqq1/20$ and $y\leqq0$.

In this case we start from (A) with $A=0$, $B=3/2$, $C=3/2$.   Then we have

$$\Re a_8\leqq U(p)+\frac{3}{64}p^4\left(\eta+\frac{13}{6}py\right)+\frac{9}{8}p^3x(\eta+py)$$

$$+\left(\frac{37}{64}p^3-9p^2+\frac{169}{128}px'^2\right)x'^2y+\left(\frac{81}{32}p^2-9p+\frac{11}{64}x'^2\right)x'^2\eta$$

$$+\left(-\frac{37}{16}p^3+\frac{9}{2}p^2-\frac{59}{16}px'^2\right)y^2+\left(-\frac{19}{4}p^2+9p-\frac{17}{4}x'^2\right)y\eta$$

$$+\left(-3p+\frac{9}{2}\right)\eta^2+\frac{3}{2}py\xi+\eta\xi+3y\varphi$$

$$+\left(\frac{113}{64\cdot12}p^5-\frac{9}{4}p^4-\frac{25}{48}p^2-\frac{331}{64\cdot12}p^3x'^2+\frac{9}{2}p^2x'^2+\frac{25}{96}x'^2-\frac{83}{256}px'^4\right)x'^2$$

$$+\left(-\frac{17}{4}p^2+2x'^2\right)x'\xi'-\frac{7}{2}px'\varphi'-2x'\tau'-\frac{3}{2}x'y\xi'.$$

We remark the following facts;

$$U(p)\leqq U(1)<1.101973,$$

$$\frac{3}{64}p^4\left(\eta+\frac{13}{6}py\right)<0.111264,$$

$$\frac{9}{8}p^3x(\eta+py)<1.303125,$$

$$\left(\frac{37}{64}p^3-9p^2+\frac{169}{128}px'^2\right)x'^2y-\frac{3}{2}x'y\xi'$$

$$\leqq\left(\frac{37}{64}p^3-9p^2+\frac{169}{128}px'^2\right)x'^2y+\frac{9}{4}x'^2|y|+\frac{1}{4}|y|\xi'^2$$

$$<0.533594+\frac{25.4016}{96}\xi'^2,$$

$$\left(\frac{81}{32}p^2 - 9p + \frac{11}{64}x'^2\right)x'^2\eta < 0.250535.$$

Making use of these remarks and applying Lemma 1 to -4.699509 we have

$$\Re a_8 < 8 - 0.2935p^4 - \frac{1}{96}Q - \frac{1}{96}R,$$

$$Q = (222p^3 - 347.472p^2 + 338.112 + 354px'^2)y^2 + (456p^2 - 864p + 408x'^2)y\eta$$

$$+ (140.88p^2 + 288p + 131.52)\eta^2 - 144py\xi + (197.232p^2 + 788.928)\xi^2$$

$$- 96\eta\xi - 288y\varphi + (253.584p^2 + 1014.336)\varphi^2,$$

$$R = (-14.125p^5 + 216p^4 + 78.176p^2 + 112.704$$

$$+ 41.375p^3x'^2 - 432p^2x'^2 - 25x'^2 + 31.125px'^4)x'^2$$

$$+ (408p^2 - 192x'^2)x'\xi' + (197.232p^2 + 763.576)\xi'^2 + 336px'\varphi'$$

$$+ (253.584p^2 + 1014.336)\varphi'^2 + 192x'\tau' + (309.936p^2 + 1239.744)\tau'^2.$$

It is not so difficult to prove the positive definiteness of Q and R for $0.8 \leq p \leq 1$, $x'^2/p^2 \leq 1/20$. Hence we have in the present case $\Re a_8 < 8$.

Case iii) $0.8 \leq p \leq 1$, $1/20 \leq x'^2/p^2 \leq \tan^2(\pi/7)$, $y \geq 0$.

We start from (A) with $A = 1/2$, $B = 3/2$, $C = 1$. Then we have

$$\Re a_8 \leq U(p) + \frac{3}{64}p^4\left(\eta + \frac{13}{6}py\right) + \frac{3}{4}p^3x\left(\eta + \frac{3}{2}py\right) + \frac{5}{96}p^2(8 - p^3)y$$

$$+ \left(\frac{151}{192}p^3 - 9p^2 - \frac{5}{12} + \frac{149}{128}px'^2\right)x'^2y + \left(\frac{33}{32}p^2 - 6p + \frac{11}{64}x'^2\right)x'^2\eta$$

$$+ \left(-\frac{127}{48}p^3 + \frac{9}{2}p^2 + \frac{1}{6} - \frac{35}{16}px'^2\right)y^2 + (-3p^2 + 6p - 4x'^2)y\eta$$

$$+ \left(-\frac{3}{4}p + 2\right)\eta^2 + py\xi + 2\eta\xi + 2y\varphi - \frac{1}{4}y^2\eta$$

$$+ \left(\frac{113}{64 \cdot 12}p^5 - \frac{9}{4}p^4 - \frac{25}{48}p^2 - \frac{331}{64 \cdot 12}p^3x'^2 + \frac{9}{2}p^2x'^2 + \frac{25}{96}x'^2 - \frac{83}{256}px'^4\right)x'^2$$

$$+ \left(-\frac{17}{4}p^2 + 2x'^2\right)x'\xi' - \frac{7}{2}px'\varphi' - 2x'\tau' - 2x'y\xi'.$$

We remark the following facts:

$$U(p) \leqq U(1) < 1.101973,$$

$$\frac{3}{64} p^4 \left(\eta + \frac{13}{6} py \right) < 0.113465,$$

$$\frac{5}{96} p^2 (8 - p^3) y < 0.364584,$$

$$\left(\frac{151}{192} p^3 - 9p^2 - \frac{5}{12} + \frac{149}{128} px'^2 \right) x'^2 y + \frac{167}{20} p^3 x'^2 y \leqq 0,$$

$$-\frac{167}{20} p^3 x'^2 y - 2x'y\xi' \leqq \frac{20}{167 p^3} y\xi'^2 \leqq \frac{23.77}{96} \xi'^2,$$

$$\left(\frac{33}{32} p^2 - 6p + \frac{11}{64} x'^2 \right) x'^2 \eta < 0.95237,$$

$$\frac{3}{4} p^3 x \left(\eta + \frac{3}{2} py \right) - \frac{1}{4} y^2 \eta < 1.232107.$$

Making use of these remarks and applying Lemma 1 to -4.235501 we have

$$\Re a_8 \leqq 8 - 0.2646 p^4 - \frac{1}{96} Q - \frac{1}{96} R,$$

$$Q = (254 p^3 - 355.797 p^2 + 288.92 + 210 px'^2) y^2 + (288 p^2 - 576 p + 384 x'^2) y\eta$$

$$+ (127.005 p^2 + 72 p + 316.2) \eta^2 - 96 py\xi - 192 \eta\xi + (177.807 p^2 + 711.48) \xi^2$$

$$- 192 y\varphi + (228.609 p^2 + 914.76) \varphi^2,$$

$$R = (-14.125 p^6 + 216 p^4 + 75.401 p^2 + 101.64$$

$$+ 41.375 p^3 x'^2 - 432 p^2 x'^2 - 25 x'^2 + 31.125 px'^4) x'^2$$

$$+ (408 p^2 - 192 x'^2) x'\xi' + (177.807 p^2 + 687.71) \xi'^2 + 336 px'\varphi'$$

$$+ (228.609 p^2 + 914.76) \varphi'^2 + 192 x'\tau' + (279.411 p^2 + 1118.04) \tau'^2.$$

It is not so difficult to prove the positive definiteness of Q and R for $0.8 \leqq p \leqq 1$, $1/20 \leqq x'^2 / p^2 \leqq 0.2379$. Hence we have in the present case $\Re a_8 < 8$.

Case iv) $0.8 \leq p \leq 1$, $1/20 \leq x'^2/p^2 \leq 1/10$ and $y \leq 0$.

We start from (A) with $A=0$, $B=5/4$, $C=1$.   Then we have

$$\Re a_8 \leq U(p) + \frac{3}{64} p^4 \left(\eta + \frac{13}{6} py \right) + \frac{3}{4} p^3 x \left(\eta + \frac{5}{4} py \right)$$

$$+ \left(-\frac{11}{64} p^3 - \frac{15}{2} p^2 + \frac{169}{128} px'^2 \right) x'^2 y + \left(\frac{33}{32} p^2 - 6p + \frac{11}{64} x'^2 \right) x'^2 \eta$$

$$+ \left(-\frac{21}{16} p^3 + \frac{25}{8} p^2 - 4px'^2 \right) y^2 + \left(-\frac{13}{8} p^2 + 5p - \frac{39}{8} x'^2 \right) y\eta$$

$$+ \left(-\frac{3}{4} p + 2 \right) \eta^2 + 2py\xi + 2\eta\xi + 3y\varphi + \frac{9}{8} py^3$$

$$+ \left(\frac{113}{64\cdot12} p^5 - \frac{9}{4} p^4 - \frac{25}{48} p^3 - \frac{331}{64\cdot12} p^3 x'^2 + \frac{9}{2} p^2 x'^2 + \frac{25}{96} x'^2 - \frac{83}{256} px'^4 \right) x'^2$$

$$+ \left(-\frac{17}{4} p^2 + 2x'^2 \right) x'\xi' - \frac{7}{2} px'\varphi' - 2x'\tau' - \frac{3}{2} x'y\xi'.$$

We remark the following facts:

$$U(p) < 1.101973,$$

$$\frac{3}{64} p^4 \left(\eta + \frac{13}{6} py \right) < 0.111849,$$

$$\frac{3}{4} p^3 x \left(\eta + \frac{5}{4} py \right) < 1.030469,$$

$$\left(-\frac{11}{64} p^3 - \frac{15}{2} p^2 + \frac{169}{128} px'^2 \right) x'^2 y - \frac{3}{2} x'y\xi'$$

$$\leq \left(-\frac{11}{64} p^3 - \frac{15}{2} p^2 + \frac{169}{128} px'^2 \right) x'^2 y + \frac{9}{4} x'^2 |y| + \frac{1}{4} |y|\xi'^2$$

$$< 0.985586 + \frac{25.4016}{96} \xi'^2,$$

$$\left(\frac{33}{32} p^2 - 6p + \frac{11}{64} x'^2 \right) x'^2 \eta < 0.384214.$$

Making use of these remarks and applying Lemma 1 to -4.385909 we have

$$\Re a_8 < 8 - 0.274 p^4 - \frac{1}{96} Q - \frac{1}{96} R,$$

$$Q = (126 p^3 - 221.07 p^2 + 315.72 + 384 p x'^2) y^2 + (156 p^2 - 480 p + 468 x'^2) y \eta$$

$$+ (131.55 p^2 + 72 p + 334.2) \eta^2 - 192 p y \xi - 192 \eta \xi + (184.17 p^2 + 736.68) \xi^2$$

$$- 288 y \varphi + (236.79 p^2 + 947.16) \varphi^2,$$

$$R = (-14.125 p^6 + 216 p^4 + 76.31 p^2 + 105.24 + 41.375 p^3 x'^2$$

$$- 432 p^2 x'^2 - 25 x'^2 + 31.125 p x'^4) x'^2$$

$$+ (408 p^2 - 192 x'^2) x' \xi' + (184.17 p^2 + 711.27) \xi'^2 + 336 p x' \varphi'$$

$$+ (236.79 p^2 + 947.16) \varphi'^2 + 192 x' \tau' + (289.41 p^2 + 1157.64) \tau'^2.$$

It is not so difficult to prove the positive definiteness of Q and R for $0.8 \le p \le 1$, $1/20 \le x'^2/p^2 \le 1/10$. Hence we have in the present case $\Re a_8 < 8$.

Case v) $0.8 \le p \le 1$, $1/10 \le x'^2/p^2 \le \tan^2 (\pi/7)$ and $y \le 0$.

We start from (A) with $A = 0$, $B = 5/4$, $C = 1$. We remark the following facts;

$$U(p) < 1.101973,$$

$$\frac{3}{64} p^4 \left(\eta + \frac{13}{6} p y \right) < 0.113465,$$

$$\frac{3}{4} p^3 x \left(\eta + \frac{5}{4} p y \right) < 1.056325,$$

$$\left(-\frac{11}{64} p^3 - \frac{15}{2} p^2 + \frac{169}{128} p x'^2 \right) x'^2 y - \frac{3}{2} x' y \xi'$$

$$\le \left(-\frac{11}{64} p^3 - \frac{15}{2} p^2 + \frac{169}{128} p x'^2 \right) x'^2 y + \frac{3}{8} |y| x'^2 + \frac{3}{2} |y| \xi'^2$$

$$< 1.882942 + \frac{152.4096}{96} \xi'^2,$$

$$\left(\frac{33}{32} p^2 - 6 p + \frac{11}{64} x'^2 \right) x'^2 \eta < 0.908094.$$

Making use of these remarks and applying Lemma 1 to -2.937201 we have

$$\Re a_8 < 8 - 0.1835 p^4 - \frac{1}{96} Q - \frac{1}{96} R,$$

$$Q = (126 p^3 - 247.134 p^2 + 211.464 + 384 px'^2) y^2 + (156 p^2 - 480 p + 468 x'^2) y\eta$$

$$+ (88.11 p^2 + 72 p + 160.44)\eta^2 - 192 py\xi - 192\eta\xi + (123.354 p^2 + 493.416)\xi^2$$

$$- 288 y\varphi + (158.598 p^2 + 634.392)\varphi^2,$$

$$R = (-14.125 p^5 + 216 p^4 + 67.622 p^2 + 70.488$$

$$+ 41.375 p^3 x'^2 - 432 p^2 x'^2 - 25 x'^2 + 31.125 px'^4) x'^2$$

$$+ (408 p^2 - 192 x'^2) x'\xi' + (123.354 p^2 + 341.006)\xi'^2 + 336 px'\varphi'$$

$$+ (158.598 p^2 + 634.392)\varphi'^2 + 192 x'\tau' + (193.842 p^2 + 775.368)\tau'^2.$$

It is not so difficult to prove the positive definiteness of Q and R for $0.8 \le p \le 1$, $1/10 \le x'^2/p^2 \le 0.2379$. Hence we have $\Re a_8 < 8$ in the present case.

Case vi) $0 \le p \le 0.8$, $x'^2/p^2 \le 1/20$.

In this case we start from (A) with $A=0$, $B=3/2$, $C=3/2$. We remark the following facts;

$$U(p) \le U(0.8) < 0.695811,$$

$$\frac{3}{64} p^4 \left(\eta + \frac{13}{6} py\right) < 0.038319,$$

$$\frac{9}{8} p^3 x(\eta + py) < 0.73728,$$

$$-\frac{3}{2} x'y\xi' \le \frac{3}{2} \sqrt{\frac{1}{20}} p \sqrt{\frac{4-p^2}{3}} \sqrt{\frac{4-p^2}{7}} < 0.292771,$$

$$\left(\frac{81}{32} p^2 - 9 p + \frac{11}{64} x'^2\right) x'^2 \eta < 0.14642.$$

Further if $y \ge 0$,

$$\left(\frac{37}{64} p^3 - 9 p^2 + \frac{169}{128} px'^2\right) x'^2 y + \frac{9}{8} py^3 \le \frac{9}{8} py^3 < 1.066868.$$

If $y \le 0$,

$$\left(\frac{37}{64}p^3 - 9p^2 + \frac{169}{128}px'^2\right)x'^2y + \frac{9}{8}py^3 \leqq \left(\frac{37}{64}p^3 - 9p^2 + \frac{169}{128}px'^2\right)x'^2y$$

$$< 0.18506 < 1.066868.$$

Making use of these remarks and applying Lemma 1 to -5.022531 we have

$$\Re a_8 < 8 - 0.3138p^4 - \frac{1}{96}Q - \frac{1}{96}R,$$

$$Q = (222p^3 - 341.604p^2 + 361.584 + 354px'^2)y^2 + (456p^2 - 864p + 408x'^2)y\eta$$

$$+ (150.66p^2 + 288p + 170.64)\eta^2 - 144py\xi + (210.924p^2 + 843.696)\xi^2$$

$$- 96\eta\xi - 288y\varphi + (271.188p^2 + 1084.752)\varphi^2,$$

$$R = (-14.125p^5 + 216p^4 + 80.132p^2 + 120.528$$

$$+ 41.375p^3x'^2 - 432p^2x'^2 - 25x'^2 + 31.125px'^4)x'^2$$

$$+ (408p^2 - 192x'^2)x'\xi' + (210.924p^2 + 843.696)\xi'^2 + 336px'\varphi'$$

$$+ (271.188p^2 + 1084.752)\varphi'^2 + 192x'\tau' + (331.452p^2 + 1325.808)\tau'^2.$$

It is not so difficult to prove the positive definiteness of Q and R for $0 \leqq p \leqq 0.8$, $x'^2/p^2 \leqq 1/20$. Hence we have in the present case $\Re a_8 < 8$.

Case vii) $0 \leqq p \leqq 0.8$, $1/20 \leqq x'^2/p^2 \leqq \tan^2(\pi/7)$.

We start from (A) with $A = 0$, $B = 5/4$, $C = 1$. We remark the following facts:

$$U(p) \leqq U(0.8) < 0.695811,$$

$$\frac{3}{64}p^4\left(\eta + \frac{13}{6}py\right) < 0.03878,$$

$$\frac{3}{4}p^3x\left(\eta + \frac{5}{4}py\right) < 0.572523,$$

$$-\frac{3}{2}x'y\xi' < 0.536495,$$

$$\left(\frac{33}{32}p^2 - 6p + \frac{11}{64}x'^2\right)x'^2\eta < 0.516068.$$

Further if $y \geqq 0$,

$$\left(-\frac{11}{64}p^3 - \frac{15}{2}p^2 + \frac{169}{128}px'^2\right)x'^2y + \frac{9}{8}p^3y \leqq \frac{9}{8}p^3y < 1.066868.$$

If $y \leqq 0$,

$$\left(-\frac{11}{64}p^3-\frac{15}{2}p^2+\frac{169}{128}px'^2\right)x'^2y+\frac{9}{8}p^3y \leqq \left(-\frac{11}{64}p^3-\frac{15}{2}p^2+\frac{169}{128}px'^2\right)x'^2y$$

$$<0.78343<1.066868.$$

Making use of these remarks and applying Lemma 1 to -4.573455 we have

$$\Re a_8 < 8-0.2858p^4-\frac{1}{96}Q-\frac{1}{96}R,$$

$$Q=(126p^3-217.686p^2+329.256+384px'^2)y^2+(156p^2-480p+468x'^2)y\eta$$

$$+(137.19p^2+72p+356.76)\eta^2-192py\xi-192\eta\xi$$

$$+(192.066p^2+768.264)\xi^2-288y\varphi+(246.942p^2+987.768)\varphi^2,$$

$$R=(-14.125p^5+216p^4+77.438p^2+109.572$$

$$+41.375p^3x'^2-432p^2x'^2-25x'^2+31.125px'^4)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+(192.066p^2+768.264)\xi'^2+336px'\varphi'$$

$$+(246.942p^2+987.768)\varphi'^2+192x'\tau'+(301.818p^2+1207.272)\tau'^2.$$

It is not so difficult to prove the positive definiteness of Q and R for $0\leqq p\leqq 0.8$, $1/20\leqq x'^2/p^2\leqq 0.2379$. Hence we have in the present case $\Re a_8<8$.

Summing up the results we have completed the proof of our theorem.

REFERENCES

[1] GARABEDIAN, P.R., G. G. ROSS, AND M. SCHIFFER, On the Bieberbach conjecture for even n. Journ. Math. Mech. 14 (1965), 975–989.

[2] GOLUSIN, G. M., On p-valent functions. Mat. Sbornik (N.S.) 8 (1940), 277–284.

[3] GRUNSKY, H., Koeffizientenbedingungen über schlicht abbildende Funktionen. Math. Zeits. 45 (1939), 29–61.

[4] JENKINS, J. A., AND M. OZAWA, On local maximality for the coefficient a_8. Ill. Journ. Math. 11 (1967), 596–602.

[5] OBROCK, A., An inequality for certain schlicht functions. Proc. Amer. Math. Soc. 17 (1966), 1250–1253.

[6] OZAWA, M., An elementary proof of local maximality for the coefficient a_8. Kōdai Math. Sem. Rep. 21 (1969), 459–462.

[7] OZAWA, M., AND Y. KUBOTA, On the eighth coefficient of univalent functions. Journ. Analyse Math. 23 (1970), 323–352.

[8] SCHIFFER, M., Univalent functions whose n first coefficients are real. Journ. Analyse Math. 18 (1967), 329–349.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY, AND
TOKYO GAKUGEI UNIVERSITY.