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ON THE ELEMENTARY RELATION BETWEEN PITCH, SLIP, AND PROPULSTION EFFICIENCY.

By

W.Froude.

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ON THE ELEMENTARY RELATION BETWEEN PITCH, SLIP, AND PROPULSIVE EFFICIENCY.

By W. Froude.

RESUME *

The author remarks that the opinions on the theory of the propeller prevailing at the time he read his Paper, consisted in assuming that the waste of motive power used in working a propeller is proportional to the relative slip, and that, therefore, this slip should be diminished in as large a measure as possible. If we wish to determine a propeller by taking the resistance of the ship and its speed, two methods may be employed:

1st. To increase the area of the propeller and especially its diameter.

2nd. To reduce the pitch of a propeller of given diameter.

In this connection there are definite limiting considerations. As regards the diameter of the propeller, we are limited by the amount of space available, and as regards the reduction of pitch, by the speed of rotation convenient or safe to give to the engines.

Under these conditions the author asks: "Would an unlimited area of propeller be theoretically valuable?" "How much do we lose by the limitation of area imposed by mere practical convenience?" adding that he considers that a very exaggerated importance is attached to slip.

In fact, considering that the reduction of slip is obtained either by increasing the area of the propeller or by increasing the number of revolutions, we see that both these methods lead to an increase of friction which cannot be neglected.

"But however confidently on the strength of known data we might assure ourselves of the great loss of power involved in surface friction, we could not thereby arrive at any definite data as to the pattern and dimensions of the screw which would on the whole minimize the waste of power, unless we could bring also into the calculation the co-related propulsive action. Now the pressure or reaction of a fluid on an area moving obliquely through it, has not till lately been reduced to a true theoretical solution; and though it had come to be understood that the old law which made the pressure vary as the square of the sine of the obliquity was entirely in error, and that in reality the resistance was pretty certainly in proportion to the first power of the sine, it is only quite recently that the question has received a sound theoretical solution.

* The passages in quotation marks are extracts from the original. As in all our Résumés, we have retained the notation of the author. (W.M.)

"An eminent mathematician of the day, Lord Rayleigh, has determined the law on streamline principles, rigorously so far as pressure on the advancing surface is concerned, for a plane relatively narrow in the line of motion. According to his solution, if P be the normal pressure acting on the face of the plane $P = \frac{2\pi \sin\theta}{4 + \pi \sin\theta} P'$, where P' is the pressure of a head due to the speed, acting on the plane, and θ is the angle between the plane and the line of motion.

"It appears pretty conclusively, however, by Beaufoy's experiments that, when the plane is moving normally through the water, so that $\theta = 90^\circ$, the resistance actually experienced exceeds P in the ratio of 112 to 96, and it is not improbable that a proportionate excess, beyond P as given by Lord Rayleigh's formula, will be experienced also when the motion is oblique; and in the calculations I have made I have assumed this to be the case.

"As regards surface friction, the experiments I have conducted for the Admiralty show that it varies about as the power 1.85 or 1.9 of the speed; but for convenience we may adhere to the usual expression that it varies as the square of the speed. The coefficient or frictional force per square foot at unit speed, varies greatly with the length of the plane in the line of motion and with the quality of the surface.

"The pressure and the friction may be respectively expressed by the equations $P = p\ A\ v^2 \sin\theta$, and $F = f\ A\ v^2$, where p and f are respectively the pressure and the friction per unit of surface, A the area of the plane, v the speed in the line of motion, and θ the angle between the plane and the line of motion; and if we take the forces in pounds, the area in square feet, and the speed in feet per second, the available data suggest 1.7 (1) as the value of p, and 0.008 as the value of f. bearing in mind, as regards the latter figure, that it provides for the circumstance that the the area of a screw blade has a double surface, the back and the front; and that it is appropriate to a fairly smooth surface, measuring 3 feet in the line of motion. I must, however, add that although it is very important to be pretty correctly informed as to the true measure both of surface friction and of normal pressure, so as to be assured that we are dealing with real and tangible amounts and not with shadowy tendencies, the investigation, even when carried out with the mere abstract coefficients, proves in the highest degree interesting and instructive.

"A TRUE CONCEPTION OF THE RELATION BETWEEN THE MODUS OPERANDI OF THE OBLIQUELY MOVING PLANE, AND THE BLADE OF A SCREW PROPELLER, MAY BE FOUND BY IMAGINING THE PLANE TO BE CARRIED ROUND THE SCREW AXIS, BEING SET OBLIQUELY TO THE PLANE OF ROTATION, AS IF IT WERE A UNIT OF AREA IN AN EXTENDED TRUE SPIRAL SURFACE."

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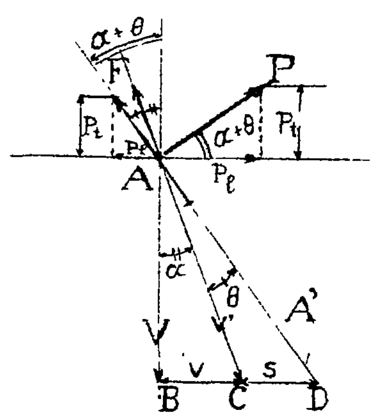
(1) See NOTE I, p. 10.

Fig. 1



Call AA'	the element of a blade of the propeller;
AB = V,	the speed of rotation of this element.
BC = v,	the speed of the forward motion of the ship.
AC = v',	is therefore the resulting speed of the element in the water.
CD = s	is the speed of the slip equal to the difference between the forward motion per revolution and the speed of the ship.

The angle BAC = α, the virtual pitch angle (the angle of the forward motion per revolution.)

The angle CAD = θ, the slip angle.

The angle BAD = $\alpha + \theta$, the actual pitch angle.

The slip ratio is then equal to $\frac{CD}{BD} = \frac{s}{v + s}$

(1) $P = p\,A\,v'^2 \sin\theta$ is the component of the resultant of the normal pressure of the air on the plane.

p is the coefficient of lift equal to 1.7 lbs/sq.ft/ft.;sec. for the water.

A, the area of the element.

$F = f\,A\,v'^2$ is the component of the resultant due to friction; it is directed PARALLEL TO THE PATH AC of the element. (2)

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(1) The calculations which follow are contained in a Mathematical Appendix placed at the end of the original Paper. For convenience of reading we have preferred to insert them among the conclusions which, in the Paper itself, precede the calculations. (W.M.)

(2) "It might at first sight be assumed that this component should be taken account of in the direction of the plane, not of the motion of the plane; but it appears on consideration that all the particles to which the plane frictionally imparts motion along its own plane, must accept at the same time the normal component of the plane's motion, and thus its complete resultant path; the force should therefore be estimated as acting in the direction of the resultant motion of which it is the counterpart. (Since the above matter was in type, I have been led to doubt the correctness of this assumption, and to lean to what was my orig-

(Cont'd on next page.)

f being the coefficient of friction multiplied by 2;
$f = 2 \times 0.004 = 0.008$ lbs/sq.ft/ft.; sec. for the water.

$$K = f/p = 0.008/1.7 = 0.0047.$$

Let us project these two components on the axis of the motion and perpendicular to it. Multiplying the sum of the projections on the axis, that is, the thrust of the propeller, by the speed v of the ship, we obtain the expression of the useful power U_e; multiplying the sum of the projections normal to the axis by the tangential speed of rotation, we obtain the motive power U_g. We thus have:

$$U_e = p\,A\,v^3 \,\operatorname{cosec}^2\alpha \left\{\cos(\alpha + \theta)\sin\theta - k\sin\alpha\right\} \quad (1a)$$

$$U_g = p\,A\,v^3 \,\operatorname{cosec}^2\alpha \quad \cot\alpha\left\{\sin(\alpha + \theta)\sin\theta + k\cos\alpha\right\} \quad \ldots\ldots (2a)$$

whence the element efficiency E is:

$$E\left(= \frac{U_e}{U_g}\right) = \tan\alpha\left[\frac{\cos(\alpha + \theta)\sin\theta - k\sin\alpha}{\sin(\alpha + \theta)\sin\theta - k\cos\alpha}\right] \qquad (4)$$

Considering that the value of θ is small, we may take

$\sin\theta = \tan\theta = \theta$ and $\cos\theta = 1$; we then have:

$$E = \tan\alpha\left[\frac{\theta - (\theta^2 + k)\tan\alpha}{\theta\tan\alpha + (\theta^2 + k)}\right] \qquad (4a)$$

Differentiating this expression for α and θ, and neglecting the terms lower than θ, we obtain the two conditions of maximum efficiency:

$$\theta = \sqrt{k} \qquad (5)$$

$$\tan 2\alpha = \frac{\theta}{\theta^2 + k} \qquad (6)$$

If we introduce into equation (6) the value of $\theta = \sqrt{k}$ given by expression (5), we obtain the condition of highest maximum efficiency:

$$\tan(\alpha + \theta) = 1 \quad \text{or} \quad \alpha + \theta = 45^\circ \qquad (7a)$$

((2) Cont'd)
inal impression, that the component should be taken in the direction of the plane itself; but the assumption simplifies the solution, and the principal results arrived at are not materially affected by the slight error it involves, as the whole work of skin friction is included under either hypothesis. I had traced the solution far enough under my original impression to know that the more complete solution which I retain as alread y in type is practically admissible.)" (W.F.)

Equation (5) shows that WHATEVER BE THE PITCH, MAXIMUM EFFICIENCY WILL BE OBTAINED BY ADOPTING A CONSTANT SLIP ANGLE (OR A CONSTANT ANGLE OF ATTACK).

The expression (7a) enables us to conclude that IF WE ADOPT THE OPTIMUM SLIP ANGLE, THE HIGHEST MAXIMUM OF EFFICIENCY WILL BE OBTAINED WITH A PITCH ANGLE OF 45°.

On the other hand, substituting in equation (6) the value of the optimum angle $\theta = \sqrt{k}$ for the value of k, and the expression

$$\theta' = \theta - (\theta - \theta')$$

for the value of θ, we obtain the relation which gives the value of the pitch giving maximum efficiency for a slip angle (or an angle of attack) differing little from the optimum angle; we have:

$$\alpha + \theta = 45^\circ + (\theta - \theta')$$

which shows that "any moderate alteration of slip angle would demand that to give maximum efficiency, the pitch angle should receive an increment or decrement in effect equal to that of the slip angle".

The approximate expression of the complete maximum efficiency, say E', is obtained by introducing into the equation (4a) the values of tan α and θ given by equations (5) and (6). We have:

$$E' = 1 - 4\sqrt{k} + 8k - 8k\sqrt{k} \qquad (8a)$$

The author points out that the complete efficiency of a propeller cannot reach this value, since only one section of the blade can have the most effective pitch and that this efficiency tends towards unity if the friction is null $(k = f/p = 0)$ whatever be the pitch, provided that the area be large enough to admit of the slip tending towards 0.

The equation connecting the resistance, R, of the ship and its speed, v, with the area, A, of the propeller, is obtained by substituting in equation (1) Rv for U_e; we thus get the expression:

$$A = \frac{R}{v^2} \times \frac{\sin^2\alpha}{p\left[\cos(\alpha + \theta)\sin\theta - k\sin\alpha\right]} \qquad (9)*$$

In this relation, by putting $45^\circ - \theta$ for α and $\sqrt{k}$ for θ, we have the conditions connecting the resistance, the speed, and the area (A') of the most efficient propeller.

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* In the published Paper there is a printer's error in equations (9) and (9a): the coefficient p is missing.
In equation (9b) the numerical coefficient is 7.9 instead of 8.9, as given in the Paper. (W.M.)

$$A' = \frac{R}{v^2 p}\left(\frac{1}{\sqrt{k}} - 1 - 2\sqrt{k}\right) \qquad (9a)^*$$

where for $k = 0.0047$ and $p = 1.7$

$$A' = 7.9 \times \frac{R}{v^2} \qquad (9b)^*$$

From this relation the author concludes:

1st. That at the low speeds for which the resistance of the ship is PROPORTIONAL TO THE SQUARE OF THE SPEED, THE SLIP RATIO REMAINS CONSTANT.

2nd. THAT GEOMETRICALLY SIMILAR PROPELLERS HAVING AREAS PROPORTIONATE TO THE SQUARES OF THE DIMENSIONS OF TWO SIMILAR SHIPS, WILL GIVE ON THESE SHIPS THE SAME SLIP RATIO. (1)

3rd. As the area giving maximum efficiency (equation 9a) is nearly inversely as the slip, and as efficiency decreases but slowly when the slip is greater than the optimum slip, A GREATLY REDUCED AREA, WITH REFERENCE TO THE AREA A', WOULD BE ADMISSIBLE WITHOUT MUCH LOSS OF EFFICIENCY. (2)

NUMERICAL APPLICATION.

The optimum angle $\theta' = \sqrt{k} = 0.0047 = 3^\circ\ 56'\ 30''$

The optimum slip ratio for $\alpha + \theta = 45^\circ$ and $\theta = 3^\circ\ 56'\ 30''$ is

$$r' = \frac{1 - \tan\alpha}{\tan(\alpha + \theta)} = \frac{1 - \tan(45^\circ - 3^\circ\ 56'\ 30'')}{\tan 45^\circ} = 12\ 3/4\%$$

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(1) See NOTE II, p. 11.

(2) "Experiments, which have been in progress since this Paper has been in type, show conclusively that the decrease of efficiency consequent on increased slip, with screws of ordinary proportion, is scarcely perceptible even when the slip ratio is as large as 30 per cent., with the screw working in undisturbed water. The results so shaped themselves as to point to the conclusion that, for some reason or other, the coefficient of surface friction began to diminish when the slip ratio became as much as 15 per cent., and was about halved when the slip ratio was 30 per cent.; and as it appeared not improbable that with increasing slip a more or less pronounced eddy might become established at the back of the blade, so as more or less completely to neutralize the friction of that surface, a rough experiment was tried by moving a plane obliquely through the water with various angles of slip, and in a position where the effect could be observed; and in point of fact it appeared that when the angle between the plane and its line of motion was about 10 degrees, the water at its back had assumed the form of an eddy, having nearly the speed of the plane, and that it in fact overran the plane when the angle was increased to 15°."

* See foot note p. 6 of this report.

The highest maximum efficiency by equation (8a) and for $k = 0.0047$ is 0.77.

A Plate forming a Supplement to the Paper* contains two figures (5 and 5') relative to the following example:

$f = 0.0085$, $p = 1.7$ $k = f/p = 0.005$;

$v = 24.2$ ft:sec. $R = 20000$ lbs.

In Fig. 5 the author has drawn the curves of efficiency and of the area of the propeller in function of the slip ratio for constant angle of pitch: $\alpha + \theta = 45^{\circ}$.

These curves are drawn according to the relations (4a) and (9) assuming $\theta = 45^{\circ} - \alpha$ and $r = 1 - \tan \alpha$.

We see that THE EFFICIENCY PASSES THROUGH A MAXIMUM FOR A SLIP OF 13% CORRESPONDING TO AN ANGLE OF ATTACK OF ABOUT 4° AND THAT THE DECREASE OF EFFICIENCY IS MORE APPRECIABLE WHEN THE SLIP IS LESS THAN THE OPTIMUM SLIP THAN WHEN IT IS GREATER. As regards the curve of THE AREA, THIS CURVE IS PRACTICALLY IN INVERSE PROPORTION TO THE SLIP, so that the theory is confirmed by the practice which led to an increase of area in order to lessen the slip ratio.

The author decomposes the propulsive power exerted on a propeller shaft into four terms, viz.:

1st. The useful power equal to the product of thrust and speed.
2nd. The power lost on account of slip.
3rd. The power corresponding to the work of the component of friction following the axis of the propeller.
4th. The power corresponding to the work of the component of friction, following the perpendicular to the axis.

The sum of these four terms constitute the propulsive power. (1)

On Fig. 5, the values of these four terms, the useful power being constant and equal to $20000 \times 24.2 = 484000$ lbs/ft/sec., have been laid off in curves in function of the slip.

These curves enable us to note that the power due to slip decreases as slip decreases, but that the powers due to components of friction increase as slip decreases, so that the gross propulsive power passes through a minimum.

Fig. 5' gives the same values of the efficiency, the area, and the various elements composing the propulsive power, in function of the angle of pitch ($\alpha + \theta$) for a constant angle of slip and equal to the optimum angle $\theta = \sqrt{k}$.

We see on the figure:

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* See Plate B.4.

(1) See NOTE III, p. 12.

1st. That efficiency passes through a maximum for an angle of pitch of 45^{o}.

2 nd. That the area (that is, the diameter) increases when the pitch increases.

3rd. That the power due to slip passes through a minimum between $\alpha + \theta = 45$ and 50^{o}.

4th. That the power due to the longitudinal component of friction increases with pitch.

5th. That the power due to the transversal component of friction decreases when the pitch increases.

"It may be useful to observe in conclusion, that whatever may be the effect of the difficulties just referred to as attaching to the extension of the solution from the action of the obliquely propelling plane to that of an actual screw, there are two assertions which may be confidently made in reference to the investigation and its results:

"1st. That the conclusions which have been drawn as regards the plane are in substance incontestable, so far as concerns their character and general bearings; though it is probable that quantitatively they may need some correction on the score of the incomplete exactness of the coefficients of pressure and of friction, which have been provisionally suggested; and

"2nd. That no theoretical treatment of the action of an actual screw can be sound which does not incorporate and mainly rest on the principles embodied in the treatment of the problem of the plane, and indeed that the character of the results must, in their most essential features, be the same in both cases."

NOTES

I. - It is interesting to translate the expression given by Froude for the elements of the resultant of the action of the water on a plane into the notation employed in aviation and especially into the notation of the Eiffel Laboratory, by assuming that the forces are proportionate to the specific weight of the fluid, that is, that they are in the ratio of 800 to 1.

Froude's formulas are:

$$P = p\ A\ V^2 \operatorname{Sin} \theta$$

$$F = f\ A\ V^2$$

where P is the component normal to the plane and F the component directed either tangentially to the plane or along the trajectory; p and f are the coefficients the value of which, for the water is: p = 1.7, (1), f = 0.008, the units being the pound, foot, and second. A is the area of the plane, θ, the angle of attack, and V the speed.

The factor for transforming the coefficients (lbs/sq.ft/ft/sec. into the coefficients (Kg/sq.m/m.sec.) is 52.5 for the water and 52.5/800 = 0.0656 for the air. Thus for the WATER we have:

$$P = 89\ A\ V^2 \operatorname{Sin} \theta$$

$$F = 0.21\ A\ V^2$$

and for the AIR

$$P = 0.111\ A\ V^2 \operatorname{Sin} \theta$$

$$F = 0.000524\ A\ V^2$$

If we wish to determine the values of K_x and K_y by these expressions, we find:

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(1) Neglecting the term $\pi \operatorname{Sin} \theta$ in the formula of Rayleigh, $P = \dfrac{2\pi \operatorname{Sin} \theta \times \delta\ V^2 A}{(4 + \pi \operatorname{Sin} \theta)\ 2g}$ where δ is the specific weight and g the acceleration of gravity, we have $P = \dfrac{\pi}{2}\ \dfrac{\delta \operatorname{Sin} \theta}{2g} = 1.53$ $\operatorname{Sin} \theta$. Now, as the author remarks on p. 3 in Beaufoy's experiments the resistance at 90° is 112/96, or 17% greater than that given by Rayleigh's formula. Multiplying 1.53 by 1.17, we obtain 1.79; the author has adopted the slightly lower figure of 1.7.

$$K_x = 0.000524 \text{ Cos } \theta + 0.111 \text{ Sin}^2 \theta, \quad K_y = 0.111 \text{ Sin } \theta \text{ (Cos } \theta - 0.0047) \qquad F \text{ parallel to the plane}$$

$$K_x = 0.000524 + 0.111 \text{ Sin}^2 \theta, \quad K_y = 0.111 \text{ Sin } \theta \text{ Cos } \theta \qquad F \text{ parallel to the trajectory.}$$

We thus see at once that for small values of θ, the only ones, moreover, which are of interest, the formulas differ very little in the two cases.

These formulas also indicate results little different from those lately obtained in the aerodynamic laboratories.

Thus the coefficient of friction equal to 0.000524/2 = 0.000262 Kg/sq.m/:sec. is of the order of the values now admitted.

The polar diagram of the plane traced by the above formulas differs little from the polar of the square plane obtained at the Eiffel Laboratory. See "Resistance of the Air and Aviation" p. 231).

The above formulas may be written as follows, assuming Cos θ = 1 and introducing the coefficients p and f.

$$K_x = 0.000524 + 0.111 \text{ Sin}^2 \theta = f + p \text{ Sin}^2 \theta$$

$$K_y = 0.111 \text{ Sin } \theta = p \text{ Sin } \theta$$

The minimum of K_x/K_y is obtained with

$$1 = \sqrt{f/p} = \sqrt{K} = 3^\circ\ 56'\ 30''$$

and

$$\frac{K_x}{K_y} \text{ minimum} = 2 \sqrt{\frac{f}{p}} = 2 \quad i = 0.136$$

We thus see that the term k entering into the formulas is equal to $1/4 \left(\frac{K_x}{K_y} \text{ minimum} \right)^2$.

II. - Here we see appear the notion of the constancy of the characteristic coefficient of the propeller R/v^2D^2 for a given slip.

We know that R. E. Froude (1) was the first to represent test results of a family of propellers geometrically similar but differing by

(1) The determination of the most suitable dimensions for screw propellers. "Transactions of the Institution of Naval Architects," 1886.

diameter, by a single R/v^2D^2 curve showing also the efficiency in function of the slip ratio.

For representing the tests of aerial propellers, D. Rialouchinski was the first to utilize the curves of the characteristic coefficients P_m/n^3D^5 and R/n^2D^4 in function of V/nd. (See "La Technique Aeronautique," 1910.

III. - The decomposition of the motive power into four terms corresponding to the useful power, the power due to slip, and the powers due to the components of friction, seems to us very suggestive and little known. We will therefore give the demonstration of it.

We will call (see Fig. 1, p.4) P_l, P_t and F_l, F_t the longitudinal and transversal components of pressure normal to the plane P, and of the force of friction F. We have:

$$v = V \tan \alpha \quad \text{and} \quad r = 1 - \frac{\tan \alpha}{\tan(\alpha + \theta)}$$

The useful power $U_e = (P_l - F_l)\, v$

The motive power $U_g = \left(P_l \tan(\alpha + \theta) + F_t\right) \quad V =$

$$\frac{P_l \tan(\alpha + \theta)}{\tan \alpha} \,.\, v + F_t \,.\, V$$

or, introducing the slip:

$$U_g = \frac{P_l \,.\, v}{1 - r} + F_t \,.\, V =$$

$$= \frac{P_l\, v}{1 - r} - \frac{F_l \,.\, v}{1 - r} + F_t \,.\, V + \frac{F_l \,.\, v}{1 - r}$$

$$= U_e + U_g \,.\, r + F_l \,.\, v + F_t\, V\, (1 - r) \qquad \text{(A)}$$

We see that the term due to the work of the transversal component of the force of friction comprises the factor (1 - r) which is not mentioned by the author.

We would also point out that the force of friction, F, may be replaced in formula (A) by any component of the resultant of the forces of the air on the plane, provided that the other component be normal to the plane; otherwise stated: $F_l \,.\, v + F_t\, V\, (1 - r)$ is a constant, whatever be the value of F.

W. M.

To Illustrate Mr. W. Froude's Paper on the Elementary Relation be

Pitch, Slip, and Propulsive Efficiency

Curves showing as apportioned between its several elements, the varying energy
oblique propelling plane, driving a ship at constant speed with constant
variation of angle of actual pitch (Fig 5) and of angle of slip (Fig 5')

Assumed speed of ship 24·2 ft. per second. Assumed thrust, 20.000 lbs. Assumed
Corresponding curves of "efficiency" and of "area of plane necessary" are addec

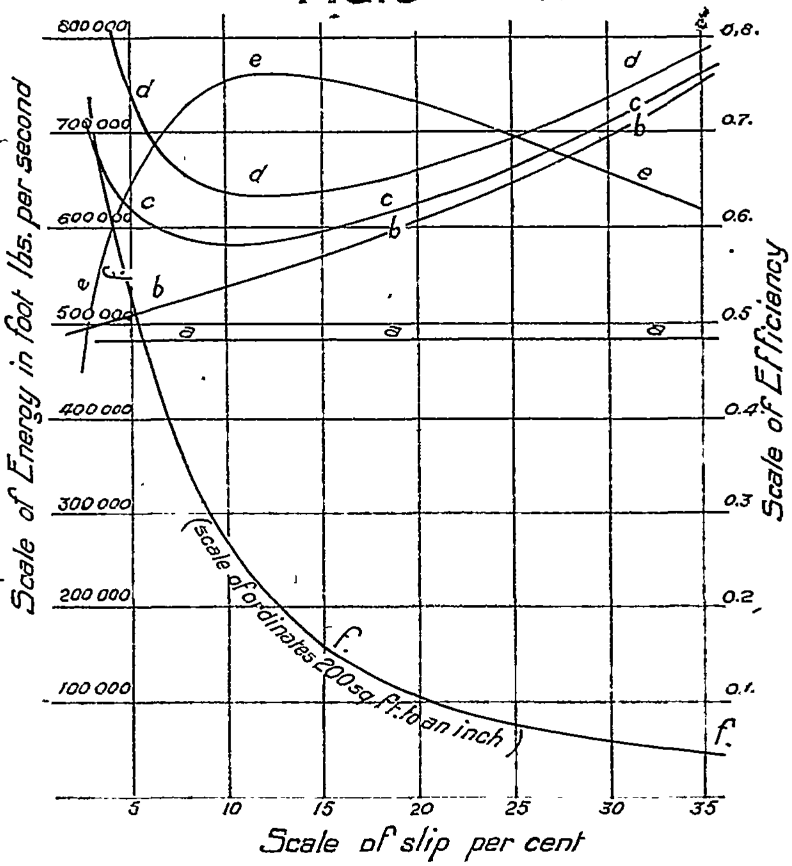
Ordinate from base to aaa = Useful energy. — Ordinate from a
" " bbb to ccc = Energy expended in overcoming lo
" " ccc to ddd = " " " " f

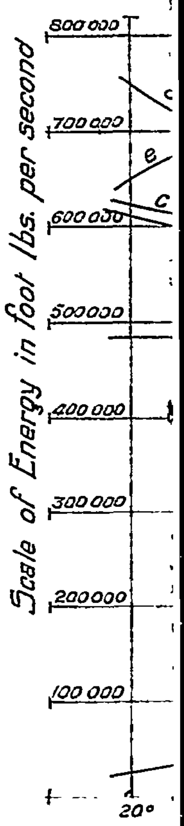
Line { eee is curve of Efficiency
fff " curve of area of plane necessary.

FIG. 5

Fig. 5. Angle of slip (or θ) varying
" of actual pitch
(or α+θ) constant = 45°

Note. The latter condition gives maximum efficiency when $\theta = \sqrt{k}$





·tween

xpended by an
thrust, but with

value of $\frac{f}{p}$ or $k = \cdot 005$ ($p = 1{\cdot}7$; $f = \cdot 0085$)

a to bbb = energy expended in slip
gitudinal component of surface friction
ansverse " " " "

ELEMENTARY RELATION BETWEEN PITCH. SLIP. AND PROPULSIVE EFFICIENCY

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS_ PARIS OFFICE

DESIGNED :
DRAWN : [signature] 3/12/920
CHECKED W. Margoulis
APPROVED W.K. 4/23/20

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FIG. 5'

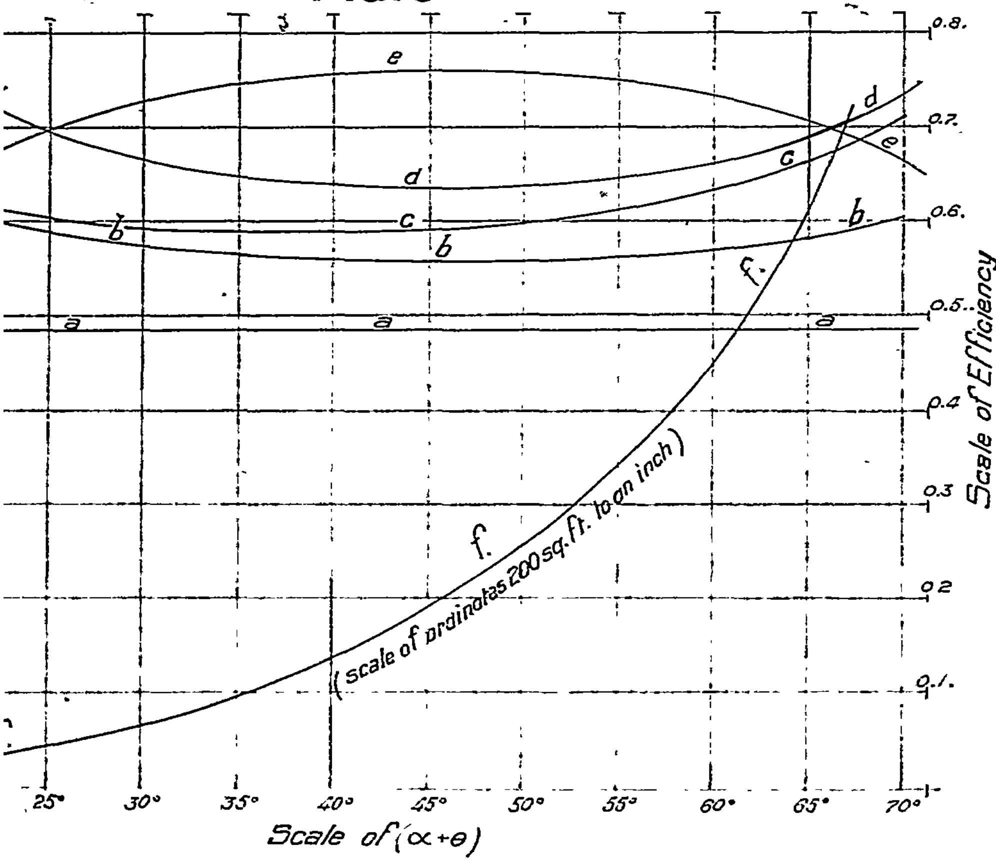

Fig. 5' Angle of actual pitch (or $\alpha+\theta$) varying.
Angle of slip (or θ) constant $=\sqrt{k}$

Note. The latter condition gives maximum efficiency for all values of ($\alpha+\theta$)