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# On the elimination of Morin singularities

By Yoshifumi ANDO

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## Introduction.

In this paper we study the problem of finding a smooth map between smooth manifolds with nice Morin singularities in a given homotopy class. A geometric interpretation of Morin singularities of a smooth map $f: N \to P$ is as follows. Let $S^i(f)$ denote the set of all points $x$ of $N$ such that the kernel rank of $df_x$ is $i$. For a certain map $f$, $S^i(f)$ becomes a submanifold of $N$ and we may define $S^{i,j}(f)$ as the set $S^j(f|S^i(f))$ for $f|S^i(f): S^i(f) \to P$ similarly. Let $n = \dim N$, $p = \dim P$ and $i = \max(1, n-p+1)$. Let $I_r$ be the $r$-sequence $(i, 1, \cdots, 1)$. Then we may continue to define $S^{I_r}(f)$ as $S^1(f|S^{I_{r-1}})$ inductively. A point of $S^{i,0}(f)$ or $S^{I_r}(f)$ is called a Morin singularity of symbol $(i, 0)$ or $I_r$ respectively. However this approach does not make it clear for what part of smooth maps $f$, $S^{I_r}(f)$ can be defined. For this we review the following important observation due to Boardman [2].

There exist a submanifold $\Sigma^{i,0}(N, P)$ and a series of submanifolds; $\Sigma^{I_1}(N, P) \supset \Sigma^{I_2}(N, P) \supset \cdots \supset \Sigma^{I_r}(N, P) \supset \cdots$ in the infinite jet space $J^\infty(N, P)$. The codimension of $\Sigma^{i,0}(N, P)$ is $i(p-n+i)$ and that of $\Sigma^{I_r}(N, P)$ is $n-p+r$ for $n \geqq p$ and $r(p-n+1)$ for $n < p$. He has shown that if a jet map $j^\infty f: N \to J^\infty(N, P)$ of $f$ is transverse to all submanifolds $\Sigma^{i,0}(N, P)$ and $\Sigma^{I_r}(N, P)$, then $S^{i,0}(f)$ and $S^{I_r}(f)$ coincide with $(j^\infty f)^{-1}(\Sigma^{i,0}(N, P))$ and $(j^\infty f)^{-1}(\Sigma^{I_r}(N, P))$ respectively. Therefore for generic maps $f$ we may consider $S^{i,0}(f)$ and $S^{I_r}(f)$.

For any integer $r \geqq 1$ we define a subset $\Omega_r(N, P)$ of $J^\infty(N, P)$ as the set of all jets $z$ such that either $z$ is of maximal rank or a point of $\Sigma^{i,0}(N, P)$ or $\Sigma^{I_2}(N, P) \setminus \Sigma^{I_{r+1}}(N, P)$. Then $\Omega_r(N, P)$ becomes an open subbundle of the fibre bundle $J^\infty(N, P)$ over $N$. The first result of this paper is the following.

THEOREM 1. *Let $p \geqq 2$. Then for any continuous section $s$ of $N$ into $\Omega_r(N, P)$, there exists a smooth map $g: N \to P$ such that $j^\infty g$ becomes a section of $N$ into $\Omega_r(N, P)$ homotopic to $s$ in $\Omega_r(N, P)$.*

Next we will study the problem of eliminating the Morin singularities $S^{I_r}(f)$ with $\operatorname{codim} S^{I_r}(f) = n$ from $f$ admitting only Morin singularities. Theorem 1 reduces it to a problem of finding a continuous section of $N$ into $\Omega_{r-1}(N, P)$ homotopic to $j^\infty f$. We will show that if $j^\infty f$ is transverse to $\Sigma^{I_r}(N, P)$ for a connected and closed manifold $N$, then the number of points of $S^{I_r}(f)$ modulo 2 is the unique obstruction of finding the above section. We should note that this number is just the Thom polynomial of the topological closure $\overline{\Sigma^{I_r}(N, P)}$ for $f$ (see the definition of [**10**]).

THEOREM 2. *Let* $r \geqq 2$, $p \geqq 2$ *and* $\operatorname{codim} \Sigma^{I_r}(N, P) = n$. *Let* $N$ *and* $P$ *be orientable manifolds. Then*

(1) *A smooth map* $f$ *with* $j^\infty f(N) \subset \Omega_r(N, P)$ *is homotopic to a smooth map* $g$ *such that* $j^\infty g(N) \subset \Omega_{r-1}(N, P)$ *and* $j^\infty f$ *and* $j^\infty g$ *are homotopic as continuous sections of* $N$ *into* $\Omega_r(N, P)$ *if and only if the Thom polynomial of* $\overline{\Sigma^{I_r}(N, P)}$ *for* $f$ *vanishes.*

(2) *In particular* $f$ *is homotopic to such a smooth map* $g$ *in the following cases;*

i) $n > p$ *and* $r \equiv 1 \pmod 4$
ii) $n > p$, $r \equiv 2$, 3 *or* 4 $\pmod 4$ *and* $n - p \equiv 1 \pmod 2$ *and*
iii) $n \leqq p$ *and* $n + p + r + r(r+1)/2 \equiv 0 \pmod 2$.

It will be shown by the Morse inequalities that the similar statement of Theorem 1 for $p = 1$ is not true. If $N$ is an open manifold, then Theorem 1 is a direct consequence of Gromov [**8**, Theorem 4.1.1] and if $n < p$, it is also a special case of du Plessis [**4**, Theorem B]. So the rest cases will be treated in this paper. The case $r = 2$ of Theorem 1 should be compared with [**6**, Theorem 1.3] which will play an important role in a proof of Theorem 1 (Sections 2 and 3).

The case $n \geqq p$ and $p = 2$ of Theorem 2 has been proved by Levine [**12**, Theorems 1 and 2] for $n > 2$ and by Eliasberg [**5**, Corollary of Theorem 4.9] for $n = 2$. Let $\Sigma^{I_r}(n, p)$ be a fibre of $\Sigma^{I_r}(N, P)$ over $N \times P$. To estimate the primary obstruction class of $j^\infty f$ to be deformed to a section of $\Omega_{r-1}(N, P)$ over $N$ (see [**18**]) we will calculate the number of connected components of $\Sigma^{I_r}(n, p)$ and their orientability in Section 4. In Section 5 we will define the dual class of each connected component of $\Sigma^{I_r}(N, P)$ for $j^\infty f$ in $H^{\operatorname{codim} \Sigma^{I_r}(N, P)}(N; G)$ where $G$ is either $\boldsymbol{Z}$ or $\boldsymbol{Z}/2$ depending on its orientability. It will be shown that these dual classes vanish except for at most one class and that their sum is equal to the Thom polynomial modulo 2. A proof of Theorem 2 in Section 6 is based on these facts. For the calculation of Thom polynomials of Morin singularities see, for example, [**7**, **11**, **15**, **17** and **19**].

## §1. Morin singularities.

Let $N$ and $P$ be paracompact and Hausdorff $C^\infty$ manifolds (simply smooth manifolds) of dimensions $n$ and $p$ respectively. Let $J^k(N, P)$ denote the $k$ jet space $(0 \leqq k \leqq \infty)$. If $s \geqq t$, then we have the canonical projection $\pi^s_t : J^s(N, P) \to J^t(N, P)$. Let $\pi^k_N$ and $\pi^k_P$ be the projections of $J^k(N, P)$ onto $N$ and $P$ mapping a $k$ jet onto its source and target respectively. In this section we review the definition of Boardman submanifold $\Sigma^I(N, P)$ only for $I=(i, 0)$ and $I_r$ of $J^k(N, P)$ in [2].

We begin with recalling the total tangent bundle $\boldsymbol{D}$ over $J^\infty(N, P)$ ([2, Definition 1.9]). This notion is related to the derivation of functions on $J^\infty(N, P)$. A function $\phi$ from an open set $U$ of $J^\infty(N, P)$ into $\boldsymbol{R}$ is called smooth if there exists a smooth function $\psi$ on some open subset of $J^k(N, P)$ with $\phi=\psi\circ\pi_k^\infty$. Any smooth section $D$ of $\boldsymbol{D}$ over $U$ determines a homomorphism between the module of smooth functions on $U$. That is, $D\phi$ is a smooth function on $U$ with property $D(\phi_1+\phi_2)=D\phi_1+D\phi_2$. Any vector field $d$ on an open set $V$ of $N$ defines a smooth section $D$ of $\boldsymbol{D}$ on $(\pi_N^\infty)^{-1}(V)$ characterized by the following equality for any smooth map $f: V\rightarrow P$

$$(*) \qquad D\phi\circ J^\infty f=d(\phi\circ j^\infty f).$$

The total tangent bundle $\boldsymbol{D}$ is identified with $(\pi_N^\infty)^*TN$ by (*). Hence any smooth section $d$ of $(\pi_N^k)^*TN$ yields a smooth section of $\boldsymbol{D}$ denoted by $(\pi_k^\infty)^*d$. If $d$ is a section of $(\pi_N^k)^*TN$ and $\psi$ is a function on $J^k(N, P)$, then $((\pi_k^\infty)^*d)(\psi\circ\pi_k^\infty)$ is of the form $(\phi\circ\pi_{k+1}^\infty)$ for some function $\phi$ on $J^{k+1}(N, P)$. In the sequel we simply write $d(\psi)$ for $\phi$. Most of the arguments in the definition of $\Sigma^I(N, P)$ in [2] are treated over $J^\infty(N, P)$. However we will work over $J^k(N, P)$ where $k$ is not less than the length of symbol $I$, for we will need finiteness of the dimension of $J^k(N, P)$. This approach is guaranteed mainly by [2, Lemmas 1.12, 1.20 and 2.20] and commented in [2, p. 412, line 33].

Let $\boldsymbol{K}_0=(\pi_N^k)^*TN$ and $\boldsymbol{P}=(\pi_P^k)^*TP$. First we recall a homomorphism

$$d_1: \boldsymbol{K}_0 \longrightarrow \boldsymbol{P} \qquad \text{over} \quad J^k(N, P).$$

Let $z$ be any $k$ jet of $J^k(N, P)$ with target $y$ of $P$. Let $m_y$ denote the ideal of smooth function germs vanishing at $y$. For any section $d$ of $\boldsymbol{K}_0$ near $z$ and a smooth function $\phi$ in a neighbourhood of $y$ we obtain a smooth function $d\phi$ on a neighbourhood of $z$ in $J^k(N, P)$. This defines a homomorphism $h_1: \boldsymbol{K}_{0,z}\otimes m_y \rightarrow\boldsymbol{R}$ by mapping $(d(z), \phi)$ onto $d\phi(z)$ where $\boldsymbol{K}_{0,z}$ is a fibre of $\boldsymbol{K}_0$ over $z$. Since $d$ annihilate $m_y^2$ at $y$, $h_1$ induces $h_1': \boldsymbol{K}_{0,z}\otimes m_y/m_y^2\rightarrow\boldsymbol{R}$. By identifying $m_y/m_y^2$ with $\mathrm{Hom}(TP_y, \boldsymbol{R})$ $h_1'$ yields a homomorphism $d_{1,z}: \boldsymbol{K}_{0,z}\rightarrow TP_y$ which is what we want to define. Let $\Sigma^i(N, P)$ denote the set of all $k$ jets $z$ such that the kernel rank $d_{1,z}$ is $i$. We define bundles $\boldsymbol{K}_1$ and $\boldsymbol{Q}_1$ over $\Sigma^i(N, P)$ as the kernel bundle $\mathrm{Ker}(d_1)$ and the cokernel bundle $\mathrm{Cok}(d_1)$ respectively. Let $e$ be the canonical projection of $\boldsymbol{P}$ onto $\boldsymbol{Q}_1$ over $\Sigma^i(N, P)$.

Next we define $\Sigma^{(i,j)}(N, P)$ for $i=\max(1, n-p+1)$ and $j=0$ or 1. There has been defined a symmetric homomorphism $h_2: \boldsymbol{K}_1\otimes\boldsymbol{K}_1\rightarrow\boldsymbol{P}$ over $\Sigma^i(N, P)$ in [2, Corollary 4.5] (Later we will see briefly the definition of $h_2$ together with

$h_r$, $r \geqq 3$). Then $e \circ h_2$ yields a homomorphism

$$d_2 : \boldsymbol{K}_1 \longrightarrow \mathrm{Hom}(\boldsymbol{K}_1, \boldsymbol{Q}_1) \quad \text{over} \quad \Sigma^i(N, P).$$

We define $\Sigma^{(i,j)}(N, P)$ to be the set of all $k$ jets $z$ of $\Sigma^i(N, P)$ such that the kernel rank of $d_2$ over $z$ is $j$. If $j=1$, we put $\boldsymbol{K}_2 = \mathrm{Ker}(d_2)$ and $\mathrm{Cok}(d_2)$ becomes the line bundle $\mathrm{Hom}(\boldsymbol{K}_2, \boldsymbol{Q}_1)$. The definition of $\Sigma^{I_r}(N, P)$ goes by induction on $r$ as follows. Again we have a symmetric homomorphism ($t \geqq 2$)

$$h_{t+1} : \overset{t}{\otimes} \boldsymbol{K}_2 \otimes \boldsymbol{K}_1 \longrightarrow \boldsymbol{P} \quad \text{over} \quad \Sigma^{I_t}(N, P).$$

The composition

$$c_{t+1} : \overset{t+1}{\otimes} \boldsymbol{K}_2 \subset \overset{t}{\otimes} \boldsymbol{K}_2 \otimes \boldsymbol{K}_1 \longrightarrow \boldsymbol{P} \xrightarrow{e} \boldsymbol{Q}_1$$

induces a homomorphism

$$d_{t+1} : \boldsymbol{K}_2 \longrightarrow \mathrm{Hom}(\overset{t}{\otimes} \boldsymbol{K}_2, \boldsymbol{Q}_1) \quad \text{over} \quad \Sigma^{I_t}(N, P).$$

Then $\Sigma^{I_{t+1}}(N, P)$ denotes the set of all $k$ jets $z$ of $\Sigma^{I_t}(N, P)$ such that $d_{t+1}$ is a null homomorphism over $z$. Let $\Sigma^{(I_t, 0)}(N, P)$ denote the set $\Sigma^{I_t}(N, P) \setminus \Sigma^{I_{t+1}}(N, P)$ over which $d_{t+1}$ is an isomorphism.

Here we see what a map $h_{t+1}$ is ($h_2$ will be defined similarly). See the details in [2, Theorem 4.1]). Extend the vector bundles $\boldsymbol{K}_i$ to $\overline{\boldsymbol{K}_i}$ over a small neighbourhood of $\Sigma^{I_t}(N, P)$ ($i=1$ and 2) and take any smooth sections $D_1$ of $\overline{\boldsymbol{K}_1}$ and $D_2, \cdots, D_{t+1}$ of $\boldsymbol{K}_2$. For a germ $\phi$ near $y$ of $P$ we obtain a smooth function $D_{t+1}(\cdots D_2(D_1\phi))$ on a neighbourhood of $\Sigma^{I_t}(N, P)$. Furthermore it follows that if either $\phi$ is a germ of $m_y^2$ or one of $\{D_i\}$'s vanishes on $z$, then $D_{t+1}(\cdots D_2(D_1\phi))$ vanishes on $z$. Therefore we obtain a map $\overset{t}{\otimes}\boldsymbol{K}_{2,z} \otimes \boldsymbol{K}_{1,z} \otimes m_y/m_y^2 \rightarrow \boldsymbol{R}$ where $\boldsymbol{K}_{i,z}$ is a fibre of $\boldsymbol{K}_i$ over $z$. By identifying $m_y/m_y^2$ with $\mathrm{Hom}(TP_y, \boldsymbol{R})$ we have

$$h_{t+1,z} : \overset{t}{\otimes} \boldsymbol{K}_{2,z} \otimes \boldsymbol{K}_{1,z} \longrightarrow TP_y.$$

Since the operation of $\{D_i\}$'s on a function is a derivation, $h_{t+1}$ becomes symmetric.

The next important fact is that $d_{t+1}$ is extendable to a surjective homomorphism ([2, (7.6)])

$$d_{t+1} : T(\Sigma^{I_{t-1}}(N, P)) \longrightarrow \mathrm{Hom}(\overset{t}{\otimes} \boldsymbol{K}_2, \boldsymbol{Q}_1) \quad \text{over} \quad \Sigma^{I_t}(N, P).$$

The kernel bundle of $d_{t+1}$ over $\Sigma^{I_t}(N, P)$ is equal to $T(\Sigma^{I_t}(N, P))$. This means that the normal bundle of $\Sigma^{I_t}(N, P)$ in $\Sigma^{I_{t-1}}(N, P)$ is given by $\mathrm{Hom}(\overset{t}{\otimes} \boldsymbol{K}_2, \boldsymbol{Q}_1)$. By [2, (7.7)] we have that $\boldsymbol{K}_1 \cap T(\Sigma^{I_{t-1}}(N, P)) = \boldsymbol{K}_t$ over $\Sigma^{I_t}(N, P)$.

REMARK 1.1. Although we have reviewed the definition of $\Sigma^{I_t}(N, P)$ over $J^k(N, P)$, more careful arguments show that we can actually construct a submanifold $\Sigma^{I_t}(N, P)'$ in $J^t(N, P)$ together with the bundles $\boldsymbol{K}_1'$, $\boldsymbol{Q}_1'$ over $\Sigma^i(N, P)'$, $\boldsymbol{K}_2'$ over $\Sigma^{I_2}(N, P)'$ and the homomorphisms $d_{t+1}'$ over $(\pi_t^{t+1})^{-1}(\Sigma^{I_t}(N, P)')$ so that $d_{t+1}$ comes from $d_{t+1}'$ by $\pi_{t+1}^k$ and consequently $\Sigma^{I_t}(N, P)$ coincides with $(\pi_t^k)^{-1}(\Sigma^{I_t}(N, P)')$. See [2, Lemma 2.20, 3.6 and 3.10].

## § 2. A generalization of a theorem of Eliasberg.

For a jet $z$ of $\Sigma^{(i,0)}(N, P)$ we have a nonsingular homomorphism

$$(e\circ h_2)_z : \boldsymbol{K}_{1,z}\otimes\boldsymbol{K}_{1,z} \longrightarrow \boldsymbol{Q}_{1,z}\,.$$

For each orientation of $\boldsymbol{Q}_{1,z}$ we can consider the index $s$ of $(e\circ h_2)_z$. We define the semi index of $z$ as $\min(s, i-s)$. Let $\Sigma_s^{(i,0)}(N, P)$ denote the set of all jets $z$ of $\Sigma^{(i,0)}(N, P)$ such that the semi index of $z$ is $s$.

We take a sequence of submanifolds $N\supset N_1\supset N_2\supset \cdots \supset N_r$ and an open set $U$ of $N$ as follows. Every $N_j$ is a closed subset in $N$ with codim $N_j=n-p+j$. $N_1\setminus N_2$ is a disjoint union of $N_{1,s}$, $s=0, \cdots, [i/2]$. There exists a smooth map $g$ of a neighbourhood of $N\setminus U$ into $P$. Let $C_{\Omega_r}^\infty(N, P;\ \{N_t\}, g)$ denote the space of all smooth maps $f: N\to P$ for $n\geqq p$ such that

(C-1) $f$ coincides with $g$ on a neighbourhood of $N\setminus U$,

(C-2) $(j^k f)^{-1}(\Sigma_s^{(i,0)}(N, P))=N_{1,s}$, $(j^k f)^{-1}(\Sigma^{I_t}(N, P))=N_t$ for $1\leqq t\leqq r$ and $(j^k f)^{-1}(\Sigma^{I_t}(N, P))=\emptyset$ for $t>r$,

(C-3) $f$ has no other type of singularities.

Let $\mathrm{Hom}_{\Omega_r}(TN, TP;\ \{N_t\}, g)$ denote the space of all homomorphisms $h$ of $TN$ into $TP$ such that

(H-1) $h$ coincides with $dg$ on a neighbourhood of $N\setminus U$,

(H-2) $h$ has a neighbourhood $V$ of $N_1$ where there exists a smooth map $f_h$ in $C_{\Omega_r}^\infty(V, P;\ \{V\cap N_t\}, g|V)$ with $df_h=f|V$,

(H-3) $h$ is of maximal rank outside of $N_1$.

THEOREM 2.1. *Let $N_t$, $g$ and $h$ be as above. Assume that $N$ is connected, $N_{1,s}\cap U$, nonempty for $0\leqq s\leqq[i/2]$ and $n\geqq p\geqq 2$. Then for any homomorphism $h$ there exists a smooth map $f$ of $C_{\Omega_r}^\infty(N, P;\ \{N_t\}, g)$ such that $df$ and $h$ are homotopic in* $\mathrm{Hom}_{\Omega_r}(TN, TP;\ \{N_t\}, g)$.

PROOF. The case $r=1$ of the theorem is [6, Theorem 4.7]. We use it to prove the case $r\geqq 2$. For $h$ we have a smooth map $f_h: V\to P$ in (H-2). Take a sufficiently small tubular disk neighbourhood $W$ of $N_2$ in $V$ so that $(N_{1,s}\cap U)\setminus W$ is nonempty. Then $g$ is extendable to a smooth map $\bar{g}$ on a neighbourhood of $(N\setminus U)\cup W$

$$\bar{g}=\begin{cases} g & \text{on a neighbourhood of } N\setminus U \\ f_h & \text{on } W \end{cases}$$

by the fact that $h=dg$ on a neighbourhood of $N\setminus U$ and $df_h=h|V$. Now we apply the theorem of Eliasberg [**6**, Theorem 4.7] for $(N\setminus N_2, P\,;\, N_1\setminus N_2, \bar{g})$. Then we obtain a smooth map $f'$ of $C^{\infty}_{\Omega_1}(N\setminus N_2, P\,;\, N_1\setminus N_2, \bar{g})$ such that $df'$ is homotopic to $h|(N\setminus N_2)$. Extend $f'$ to a smooth map

$$f=\begin{cases} f' & \text{on } N\setminus N_2 \\ f_h & \text{on } W. \end{cases}$$

Then $f$ has the required properties. Q.E.D.

## §3. Proof of Theorem 1.

First we shall prove the relative form of Theorem 1 for the special case that $N$ is an open set of $\boldsymbol{R}^n$ and $P=\boldsymbol{R}^p$.

PROPOSITION 3.1. *Let $n\geqq p\geqq 2$ and $N$ be an open submanifold of $\boldsymbol{R}^n$. Let $s$ be a smooth section of $\Omega_r(N, \boldsymbol{R}^p)$ over $N$ for which there exists an open set $U$ in $N$ and a smooth map $g$ of a neighbourhood of $N\setminus U$ into $\boldsymbol{R}^p$ such that $j^k g=s$ on $N\setminus U$ and that $j^k g$ is transverse to every submanifold $\Sigma^{I_t}(N, \boldsymbol{R}^p)$ on $N\setminus U$. Then there exists a smooth map $f$ such that $j^k f(N)\subset\Omega_r(N, \boldsymbol{R}^p)$ and $j^k f$ is homotopic to $s$ relative to $N\setminus U$ as sections of $\Omega_r(N, \boldsymbol{R}^p)$ over $N$.*

We will need the following lemma.

LEMMA 3.2. *For a section $s$ given in Proposition 3.1 there exists a homotopy $s_\lambda\,(0\leqq\lambda\leqq 1)$ of sections of $\Omega_r(N, \boldsymbol{R}^p)$ over $N$ such that*

*(1) $s_0=s$ and $s_\lambda|N\setminus U=s|N\setminus U$ for any $\lambda$,*

*(2) $s_1$ is transverse to every $\Sigma^{I_t}(N, \boldsymbol{R}^p)$,*

*(3) there is a smooth map $\bar{g}$ of a neighbourhood $V$ of $(s_1)^{-1}(\Sigma^i(N, \boldsymbol{R}^p))$ and $N\setminus U$ into $\boldsymbol{R}^p$ with $\bar{g}|(N\setminus U)=g|(N\setminus U)$ and $j^k\bar{g}|V=s_1|V$.*

PROOF. We prove the lemma by induction on $r$. In the proof we simply write $\Sigma^{I_r}$ for $\Sigma^{I_r}(N, \boldsymbol{R}^p)$ and $N_r$ for $s^{-1}(\Sigma^{I_r})$. First we show that $s$ is homotopic to a section represented by a smooth map on a neighbourhood of $N_r$ and $N\setminus U$. We may suppose that $N_r\cap U$ is nonempty and that $s$ is transverse to $\Sigma^{I_r}$. We use the same notations in Section 1. Let $d_1''$ denote the homomorphism of $TN$ into $T\boldsymbol{R}^p$ induced from $d_1$ by $s$. It follows from Section 1 that $TN_r\subset (s|N_r)^*(T(\Sigma^{I_r}))$ and that $\boldsymbol{K}_1\cap T(\Sigma^{I_r})=\{0\}$ since $s^{-1}(\Sigma^{I_{r+1}})$ is empty. This means that $d_1''|TN_r$ is an injective homomorphism. By Hirsch's immersion theorem [**9**] we have a homotopy of monomorphisms $k_\lambda: TN_r\to T\boldsymbol{R}^p$ covering a homotopy relative to $(N\setminus U)\cap N_r$, $i_\lambda: N_r\to\boldsymbol{R}^p$ such that $k_0=d_1''|TN_r$ and $i_1$ is an immersion with $k_1=d(i_1)$. Extend $k_\lambda$ to a homotopy $\bar{k}_\lambda: TN|N_r\to T\boldsymbol{R}^p$ of homomorphisms of rank $p-1$ for any $\lambda$ so that $\bar{k}_0=d_1''$ over $N_r$. By using $\bar{k}_\lambda$ we can deform $s$ to $s_1$ in $\Omega_r(N, \boldsymbol{R}^p)$ so that $d_1''|TN_r$ induced from $d_1$ by $s_1$ coincides with $d(i_1)$.

In fact we may apply the covering homotopy property of the fibre bundle $\pi_1^k|\Sigma^{I_r}:\Sigma^{I_r}\to\Sigma^i(N, \boldsymbol{R}^p)'$ (see Remark 1.1) to the following;

$$\begin{tikzcd}
 & \Sigma^{I_r} \arrow[d] \\
N_r \arrow[ru, "s|N_r"] \arrow[r, "\bar{k}_\lambda"] & \Sigma^i(N, \boldsymbol{R}^p)'
\end{tikzcd}$$

where $\bar{k}_\lambda$ is identified with a map sending a point $x$ of $N_r$ into $(\bar{k}_\lambda)|(TN)_x$. Let $s'_\lambda : N_r\to\Sigma^{I_r}$ be the homotopy over $\bar{k}_\lambda$ with $s'_0=s|N_r$. Since $s$ is transverse to $\Sigma^{I_r}$ we can extend $s'_\lambda$ to a homotopy $s_\lambda : N\to\Omega_r(N, \boldsymbol{R}^p)$ relative to $N\setminus U$ so that $(s_\lambda)^{-1}(\Sigma^{I_r})=N_r$ and $s_\lambda$ is transverse to $\Sigma^{I_r}$ for any $\lambda$.

Let $(s_1)^*\boldsymbol{K}_i=K_i$, $(s_1)^*\boldsymbol{Q}_1=Q_1$ and $d''_{t+1}$ be induced from $d_{t+1}$ by $s_1$. Then the normal bundle of $N_t$ in $N$ is $\mathrm{Hom}(K_1\oplus(\overset{t}{\underset{u=2}{\oplus}}\overset{u}{\otimes}K_2), Q_1)$ and $d''_{t+1} : TN_{t-1}\to\mathrm{Hom}(\overset{t}{\otimes}K_2, Q_1)$ over $N_t$ is a surjective homomorphism by Section 1. Therefore we obtain line bundles $L_2, \cdots, L_r$ in $TN|N_r$ such that $L_t$ is mapped isomorphically onto $\mathrm{Hom}(\overset{t}{\otimes}K_2, Q_1)$ by $d''_{t+1}$. Then $K_1, L_2, \cdots, L_r$ are linearly independent in $TN|N_r$ and span $TN|N_r$ together with $TN_r$. Here we fix a diffeomorphism $h$ of a neighbourhood of the zero section of $K_1|N_r\oplus\overset{r}{\underset{t=2}{\oplus}}L_t$ on a sufficiently small neighbourhood $U(N_r)$ of $N_r$ in $N$. Next we consider a system of local coordinates of $P$ near the image of $N_r$. We take a metric of $TP$. Since $d''_1|TN$ is of rank $p-1$ over $N_r$ we have the orthogonal line bundle $Q'_1$ in $(i_1)^*TP$. Let $j : Q'_1\to(i_1)^*TP$ be the inclusion. Then $d''_1|(L_2\oplus\cdots\oplus L_r)\oplus j$ is an injective homomorphism. So we have an immersion $i_P$ of a neighbourhood of the zero section of $L_2\oplus\cdots\oplus L_r\oplus Q'_1$ into $P$ such that $i_P|N_r=i_1$.

Now we construct a smooth map $f_s$ of a small neighbourhood $V$ of $N_r$ in $U(N_r)$ into $P$ such that

(1) $(s_1)^*\boldsymbol{K}_i=(j^kf_s)^*\boldsymbol{K}_i=K_i$ ($i=1$ and 2) and $(s_1)^*\boldsymbol{Q}_1=(j^kf_s)^*\boldsymbol{Q}_1=Q_1$ over $N_r$,

(2) $d''_{t+1}|(TN_{t-1}|N_r)$ and the induced homomorphism of $TN_{t-1}|N_r$ into $\mathrm{Hom}(\overset{t}{\otimes}K_2, Q_1)$ from $d_{t+1}$ by $j^kf_s$ coincide over $N_r$ for $1\leqq t\leqq r$.

In fact, let $x$ be any point of $N_r$ with $s_1(x)\in\Sigma_s^{I_r}(N, \boldsymbol{R}^p)$. We take a system of local coordinates $(t_1, \cdots, t_{p-r}, k, k_1, \cdots, k_{i-1}, l_2, \cdots, l_r)$ near $x$ in $N$ so that

(a) $(t_1, \cdots, t_{p-r})$ is a system of local coordinates of $N_r$ near $x$,

(b) $k_1, \cdots, k_{i-1}$ are local coordinates coming from $K_1/K_2$ by $h$ for which $d''_2(x)$ corresponds to the quadratic form $-\sum_{t=1}^{s}k_t^2+\sum_{t=s+1}^{i-1}k_t^2$,

(c) $k$ comes from $K_2$ and $l_t$ from $L_t$ by $h$.

Since $i_1 : N_r\to P$ is an immersion, we can take $(t_1, \cdots, t_{p-r}, l_2, \cdots, l_r, t_p)$ as a system of local coordinates of $P$ near $i_1(x)$ where $t_p$ comes from $Q'_1$ by $i_P$. Then $f_s$ is given by the following normal form of a smooth map with a Morin singularity of symbol $I_r$ in [**14**] in a small neighbourhood of $x$.

$$(*)\quad (t_1, \cdots, t_{p-r}, k, k_1, \cdots, k_{i-1}, l_2, \cdots, l_r) \longrightarrow$$
$$(t_1, \cdots, t_{p-r}, l_2, \cdots, l_r,$$
$$(1/2)(-\sum_1^s k_t^2+\sum_{s+1}^{i-1} k_t^2)+\sum_{t=2}^{r}(1/(t-1)!)l_t k^{t-1}+(1/(r+1)!)k^{r+1}).$$

We must note the compatibility of $f_s$ and $g$. By [**14**] we can express $g$ near every point of $(N\setminus U)\cap N_r$ as in $(*)$ together with the properties (a), (b) and (c). After expressing $g$ in this way we extend the coordinates to those in (a), (b) and (c) to construct $f_s$.

Now we construct a homotopy $s_\lambda$ of $s_1|N_r$ and $j^k f|N_r$ by using the structure of the vector bundle of $J^k(N, \boldsymbol{R}^p)$. Let

$$s_\lambda=(2-\lambda)s_1|N_r+(\lambda-1)j^k f|N_r \qquad (1\leqq\lambda\leqq 2).$$

It follows from (1) and (2) that $s_\lambda$ gives a homotopy of $N_r$ into $\Sigma^{I_r}$ and that $s_\lambda$ induces the homotopy of bundle maps between normal bundles of $N_r$ and $\Sigma^{I_r}$. Hence $s_\lambda$ is extendable to a homotopy $\bar{s}_\lambda$ of $V$ into a tubular neighbourhood of $\Sigma^{I_r}$ relative to $V\cap(N\setminus U)$ so that $\bar{s}_\lambda$ is transverse to $\Sigma^{I_r}$ and $(\bar{s}_\lambda)^{-1}(\Sigma^{I_r})=N_r$. Then we can extend $\bar{s}_\lambda$ to a homotopy $\bar{s}_\lambda$ of $N$ into $\Omega_r(N, P)$ relative to $N\setminus U$ so that $\bar{s}_2$ is transverse to $\Sigma^{I_r}$ with $(\bar{s}_2)^{-1}(\Sigma^{I_r})=N_r$ which is what we want.

Now we can prove the lemma by induction on $r$. The case of $r=1$ follows from the above result. For the case $r>1$, we use the above homotopy $\bar{s}_\lambda$ $(0\leqq\lambda\leqq 2)$ and the inductive hypothesis of the case $r-1$ for $\bar{s}_2$. Since $\bar{s}_2$ is already represented by a smooth map on a neighbourhood of $N\setminus U$ and $V$, we can construct a homotopy $\bar{s}_\lambda$ $(2\leqq\lambda\leqq 3)$ of $N$ into $\Omega_r(N, P)$ relative to $N\setminus U$ with properties requested. Q.E.D.

PROOF OF PROPOSITION 3.1. Let $U'$ be an open set with $\bar{U}'\subset U$ such that $g$ is defined on $N\setminus U'$. By Lemma 3.2 we may suppose that the given section $s$ has the properties (2) and (3) for $U'$ of Lemma 3.2. Furthermore we may deform $s$ so that $s^{-1}(\Sigma_u^{(i,0)}(N, \boldsymbol{R}^p)\cap U)$ is nonempty for any $u$. Then the section $\pi_1^k\circ s$ of $\mathrm{Hom}(TN, TP)$ over $N$ becomes an element of $\mathrm{Hom}_{\Omega_r}(TN, TP; \{N_t\}, g)$. It follows from Theorem 2.1 that there exists a smooth map $f$ of $C^\infty_{\Omega_r}(N, P; \{N_t\}, g)$ so that $\pi_1^k\circ s$ and $df$ are homotopic in $\mathrm{Hom}_{\Omega_r}(TN, TP; \{N_t\}, g)$ by a homotopy $s_\lambda$ with $s_0=\pi_1^k\circ s$ and $s_1=df$. By the definition every $s_\lambda$ is realized by a smooth map $f_{(s_\lambda)}$ in a sufficiently small neighbourhood $V$ of $N_1$ with Properties (H-1, 2 and 3). Since a fibre of $\pi_1^k$ is contractible, there exists a lift $\bar{s}_\lambda$ of $N$ into $J^k(N, P)$ covering $s_\lambda$ with $s_0=s$, $\bar{s}_1=j^k f$ and $\bar{s}_\lambda|V=j^k f_{(s_\lambda)}$. Since $s_\lambda$ is of maximal rank outside of $N_1$, it follows that $\bar{s}_\lambda$ is a homotopy of sections of $\Omega_r(N, \boldsymbol{R}^p)$ over $N$. This is what we want to prove. Q.E.D.

PROOF OF THEOREM 1. Let $\{V_\alpha\}$ be an open covering of $P$, each of which is diffeomorphic to $\boldsymbol{R}^p$. Since $N$ is compact we may take a finite covering $U_1, \cdots, U_m$ of $N$ such that every $U_t$ is diffeomorphic to an open set of $\boldsymbol{R}^n$ and $\bar{U}_t$ is mapped into some $V_\alpha$, say $V_t$ by $\pi^k_P \circ s$. We show the following assertion $(\mathrm{A}_q)$ by induction on $q$.

$(\mathrm{A}_q)$ There exists a smooth map $f_q$ of a neighbourhood $W_q$ of $\bigcup_{i=1}^q \bar{U}_i$ into $P$ such that $j^k f_q$ is homotopic to $s|W_q$ in $\Omega_r(W_q, P)$.

Let $W_1$ be a small neighbourhood of $\bar{U}_1$ in $(\pi^k_P \circ s)^{-1}(V_1)$. Then $A_1$ follows from Proposition 3.1 since $V_1$ is diffeomorphic to $\boldsymbol{R}^p$. We prove $A_{q+1}$ under the inductive assumption of $A_q$. Take an open neighbourhood $U$ of $\bigcup_{i=1}^q \bar{U}_i$ such that $\bar{U} \subset W_q$ and a neighbourhood $O_{q+1}$ of $\bar{U}_{q+1}$ in $(\pi^k_P \circ s)^{-1}(V_{q+1})$. We apply Proposition 3.1 to a section $s|O_{q+1}$ and a smooth map $f_q|O_{q+1} \cap W_q$. Then we may extend $f_q$ to a smooth map $f'_{q+1} : O_{q+1} \to V_{q+1}$ so that $j^k f'_{q+1}$ is homotopic to $s|O_{q+1}$ relative to $O_{q+1} \cap \bar{U}$. Then we can define $f_{q+1} : U \cup O_{q+1} \to P$ by

$$f_{q+1} = \begin{cases} f'_{q+1} & \text{on} \quad O_{q+1} \\ f_q & \text{on} \quad U. \end{cases}$$

If we put $W_{q+1} = U \cup O_{q+1}$, then $f_{q+1}$ is a required smooth map. Q.E.D.

REMARK 3.3. By the similar proof of Theorem 1 we can show the relative form of Theorem 1 as Proposition 3.1.

## § 4. Topological properties of $\Sigma^{I_r}(n, p)$.

Let $\Sigma^{I_r}(n, p)$ (or simply $\Sigma^{I_r}$) be the fibre of $\Sigma^{I_r}(\boldsymbol{R}^n, \boldsymbol{R}^p)$ over the origin $(0, 0)$. In order to study the number of connected components of $\Sigma^{I_r}(n, p)$ we use the Boardman's construction of Remark 1.1 for $J^k(\boldsymbol{R}^n, \boldsymbol{R}^p)$ restricted on the fibre $J^k(n, p)$ over $(0, 0)$. By fixing bases of $\boldsymbol{R}^n$ and $\boldsymbol{R}^p$ we obtain a canonical identification

$$h : J^{t+1}(n, p) \longrightarrow J^t(n, p) \times \mathrm{Hom}(S^{t+1}\boldsymbol{R}^n, \boldsymbol{R}^p)$$

mapping a jet $z$ to $\pi^{t+1}_t(z)$ and $t+1$ derivations of $z$. Let $\Sigma^{I_t'}$ be the fibre of $\Sigma^{I_t}(\boldsymbol{R}^n, \boldsymbol{R}^p)'$ over the origins. Let $K_1 = \boldsymbol{K}'_1|\Sigma^{i'}$, $Q_1 = \boldsymbol{Q}'_1|\Sigma^{i'}$ and $K_2 = \boldsymbol{K}'_2|\Sigma^{I_2'}$. We identify $(\pi^t_{\boldsymbol{R}^u})^* T_0 \boldsymbol{R}^u$ with $J^t(n, p) \times \boldsymbol{R}^u$ for $u = n$ or $p$. By the construction of Section 1 and Remark 1.1 we have inclusions

$$i_2 : \overset{2}{\bigcirc} K_1 \longrightarrow \Sigma^{i'} \times S^2 \boldsymbol{R}^n \qquad \text{over} \quad \Sigma^{i'}$$

$$i_{t+1} : \overset{t+1}{\bigcirc} (\pi^t_2)^* K_2 \longrightarrow \Sigma^{I_t'} \times S^{t+1} \boldsymbol{R}^n \qquad \text{over} \quad \Sigma^{I_t'} \quad (t \geqq 2)$$

where $\overset{u}{\bigcirc} K_i$ denote the $u$ symmetric product of $K_i$ and the projection

$$e : \Sigma^{I_t'} \times \boldsymbol{R}^p \longrightarrow (\pi^t_1)^* Q_1 \qquad \text{over} \quad \Sigma^{I_t'}.$$

Then we can define the following homomorphisms by using $i_t$ and $e$ $(t\geqq 2)$

$$k_2 : \Sigma^{i'} \times \mathrm{Hom}(S^2\boldsymbol{R}^n, \boldsymbol{R}^p) \longrightarrow \mathrm{Hom}(\overset{2}{\bigcirc} K_1, Q_1) \qquad \text{over} \quad \Sigma^{i'}$$

$$k_{t+1} : \Sigma^{I_t'} \times \mathrm{Hom}(S^{t+1}\boldsymbol{R}^n, \boldsymbol{R}^p) \longrightarrow \mathrm{Hom}(\overset{t+1}{\bigcirc} (\pi_2^t)^* K_2, (\pi_2^t)^* Q_1) \qquad \text{over} \quad \Sigma^{I_t'}.$$

It follows from the definition of $d'_{t+1}$ that $\Sigma^{I_2'}$ is the set of all 2 jets $z$ in $\Sigma^{i'}\times \mathrm{Hom}(S^2\boldsymbol{R}^n, \boldsymbol{R}^p)$ for which $k_2(z)$ is a quadratic form of rank $i-1$ and $\Sigma^{I_{t+1}'}$ $(t\geqq 2)$ is the set of all $t+1$ jets $z$ in $\Sigma^{I_t'}\times \mathrm{Hom}(S^{t+1}\boldsymbol{R}^n, \boldsymbol{R}^p)$ for which $k_{t+1}(z)$ is a null homomorphism. Now we calculate the number of connected components of $\Sigma^{I_r'}$, that is, of $\Sigma^{I_r}$.

Let $\gamma$ be the canonical $i$ dimensional vector bundle over the grassmann manifold $G_{i,n-i}$ of all $i$ spaces in $\boldsymbol{R}^n$. Then $\gamma$ becomes a subbundle of the trivial $n$ bundle $\theta^n$ over $G_{i,n-i}$. Let $\Sigma$ denote the connected subspace of all homomorphisms of maximal rank in $\mathrm{Hom}(\theta^n/\gamma, \theta^p)$ over $G_{i,n-i}$ $(i=\max(1, n-p+1))$. Let $K_{i,z}$ and $Q_{1,z}$ be fibres determined by $z$. Then $\Sigma^{i'}$ is canonically identified with $\Sigma$ by mapping a 1 jet $z$ to a homomorphism $\bar{d}_{1,z} : \boldsymbol{R}^n/K_{1,z}\to\boldsymbol{R}^p$ induced from $d'_1$ over $K_{1,z}\in G_{i,n-i}$.

Next we see $\Sigma^{I_r'}$ $(r\geqq 2)$. For a jet $z$ of $\Sigma^{I_r'}$ we have the symmetric quadratic form of rank $i-1$, $k_2(\pi_2^r(z))$ over $(\pi_1^r)(z)$. If we fix an orientation of $Q_1$, we can define an index of $k_2(\pi_2^r(z))$, say $s$. So the number $\min(s, i-s)$ is well defined for $z$ which we call the semi index of $z$. Let $\Sigma_s^{I_r}(n, p)$ $(=\Sigma_s^{I_r})$ denote the set of all jets $z$ with the semi index $s$ of $\Sigma^{I_r}$. We show that $\Sigma_s^{I_r}$ is connected. Let $H$ be the subset of $\mathrm{Hom}(\overset{2}{\bigcirc} K_2, Q_1)$ consisting of all quadratic forms of rank $i-1$ with semi index $s$. Let $\tilde{\gamma}$ be the canonical $i$ bundle over the oriented grassmann manifold $\tilde{G}_{i,n-i}$. Let $G_{1,i-1}(\gamma)$ (resp. $G_{i,n-i}(\tilde{\gamma})$) be the associated grassmann bundle of $\gamma$ (resp. $\tilde{\gamma}$). Let $G$ (resp. $\tilde{G}$) denote the fibre product of $\Sigma$ and $G_{1,i-1}(\gamma)$ (resp. $G_{1,i-1}(\tilde{\gamma})$) over $G_{i,n-i}$. For $z$ of $\Sigma_s^{I_r'}$ we define a map

$$g : H \longrightarrow G$$

by $g(k_2(z))=(K_{1,z}, K_{2,z}, \bar{d}_{1,z})$. If $n>p$ and $s\neq(i-1)/2$, then we can define a map $\tilde{g}$ so that the following diagram commutes

$$\begin{tikzcd}
H \arrow[r, "\tilde{g}"] \arrow[dr, "g"'] & \tilde{G} \arrow[d] \\
 & G
\end{tikzcd}$$

where the vertical map comes from the covering map. In fact, for $z\in\Sigma_s^{I_2'}$ we can choose orientation of $Q_{1,z}$ so that the index of $k_2(z)$ is $s$. This orientation of $Q_{1,z}$ determines an orientation of $K_{1,z}$ denoted by $o(K_{1,z})$. So we define $\tilde{g}$ by $\tilde{g}(z)=(K_{1,z}, o(K_{1,z}), K_{2,z}, \bar{d}_{1,z})$. Let $\tilde{K}_1$ and $\tilde{K}_2$ be the induced bundles of dimensions $i$ and 1 over $G$ or $\tilde{G}$ of $\gamma$ and the canonical line bundle ever $G_{1,i-1}(\gamma)$ respectively. Let $\tilde{Q}_1$ be the induced bundle of $Q_1$ by the projection of $G$ or $\tilde{G}$ onto $\Sigma^{i'}$. Then we define a map

$$u : \mathrm{Hom}(\tilde{K}_1 \bigcirc \tilde{K}_1,\ \tilde{Q}_1) \longrightarrow \mathrm{Hom}(\tilde{K}_2 \bigcirc \tilde{K}_1,\ \tilde{Q}_1)$$

induced from the inclusion $\tilde{K}_2 \bigcirc \tilde{K}_1 \subset \tilde{K}_1 \bigcirc \tilde{K}_1$ and an injection

$$i_H : H \longrightarrow \mathrm{Hom}((\tilde{K}_1/\tilde{K}_2) \bigcirc (\tilde{K}_1/\tilde{K}_2),\ \tilde{Q}_1) \cong \mathrm{Ker}(u)$$

so that $i_H$ maps $k_2(z)$ onto a nonsingular quadratic form induced from $k_2(z)$ over $g(k_2(z))$ or $\tilde{g}(k_2(z))$.

Then the image of $H$ is the space of all nonsingular quadratic forms of given index $s$ which becomes connected. Hence $\Sigma^{I_2'}_s$ is connected for $n>p$ and $s \neq (i-1)/2$. Other cases for $\Sigma^{I_2'}_s$ follows similarly. The connectedness of $\Sigma^{I_r}_s$ $(r \geqq 3)$ follows by induction on $r$ since $\Sigma^{I_{t+1}'}_{t+1}$ is the inverse image of the zero section of $\mathrm{Hom}(\overset{t+1}{\bigcirc}(\pi^t_2)^* K_2,\ (\pi^t_2)^* Q_1)$ by $k_{t+1}$. Therefore we have

PROPOSITION 4.1. (1) *If* $n \leqq p$, *then* $\Sigma^{I_r}(n, p)$ *is connected. If* $n>p$, *then* $\Sigma^{I_r}_s(n, p)$ *is connected.*

(2) *If* $n>p$, *then* $\Sigma^{I_r}_s(n, p) \setminus \Sigma^{I_{r+1}}_s(n, p)$ *has two connected components* $(r \geqq 2)$.

The next question is to see whether $\Sigma^{I_r}_s(n, p)$ is orientable or not.

PROPOSITION 4.2. (*Case* 1; $n \leqq p$) $\Sigma^{I_r}(n, p)$ *is orientable if and only if either* $n+p+r+r(r+1)/2 \equiv 0$ *or* $n=i=1$.

(*Case* 2; $n>p$) $\Sigma^{I_r}_s(n, p)$ *is orientable in the following cases*

(i) $s \neq (i-1)/2$ *and* $r(r+1)/2 \equiv 1 \pmod 2$

(ii) $s=(i-1)/2$, $r \equiv 1 \pmod 2$ *and* $r(r+1)/2 \equiv 1 \pmod 2$

(iii) $s=(i-1)/2$, $n=i$ *and* $r(r+1)/2 \equiv 1 \pmod 2$.

*Otherwise* $\Sigma^{I_r}_s(n, p)$ *is nonorientable.*

PROOF. In the proof we write $\Sigma^I$ for $\Sigma^{I_r'}$. For the proof we will calculate the first Stiefel-Whitney class of $\Sigma^I$. Let $p_r$ be the projection of $\Sigma^I_s$ onto $G$ $(r \geqq 2)$. Since $H$ is an open set of $\mathrm{Ker}(u)$, we obtain that the tangent bundle of $\Sigma^I$ is isomorphic to the Whitney sum of $(p_r)^* TG$, $(p_r)^*(\mathrm{Ker}(u))$ and $\bigoplus^r_{t=3}(\pi^r_t)^*(\mathrm{Ker}(k_t)) \mid \Sigma^I$. Hence we have

$$W_1(\Sigma^I) = (p_r)^*(W_1(G) + W_1(\mathrm{Ker}(u))) + (\pi^r_t)^*\Big(\sum_{t=3}^{r} W_1(\mathrm{Ker}(k_t))\Big).$$

$$\begin{aligned} W_1(\mathrm{Ker}(u)) &= W_1(\mathrm{Hom}(\tilde{K}_2 \bigcirc \tilde{K}_1,\ \tilde{Q}_1)) \\ &= W_1(\tilde{K}_2 \bigcirc \tilde{K}_1) + iW_1(\tilde{Q}_1) \\ &= iW_1(\tilde{K}_2) + W_1(\tilde{K}_1) + iW_1(\tilde{K}_1) \\ &= iW_1(\tilde{K}_2) + (i+1)W_1(\tilde{K}_1), \end{aligned}$$

$$\begin{aligned} W_1\Big(\sum_{t=3}^{r}\mathrm{Ker}(k_t)\Big) &= \sum_{t=3}^{r}\big(W_1(\overset{t}{\bigcirc}\tilde{K}_2)+W_1(\tilde{Q}_1)\big) \\ &= \sum_{t=3}^{r}\big(tW_1(\tilde{K}_2)+W_1(\tilde{Q}_1)\big) \\ &= (r(r+1)/2-3)W_1(\tilde{K}_2)+(r-2)W_1(\tilde{Q}_1). \end{aligned}$$

It follows from the standard topological properties of grassmann manifolds that $W_1(G)=(n+p+1)W_1(\tilde{K}_1)+iW_1(\tilde{K}_2)$. Therefore we have

$$W_1(\Sigma^I)=p_r^*\{(n+p+r+i)W_1(\tilde{K}_1)+(r(r+1)/2-3)W_1(\tilde{K}_2)\}.$$

If $n>p$, $s\neq(i-1)/2$ $(i>1)$, then it follows that $p_r^*(W_1(\tilde{Q}_1))=0$. Since the fibre of $\tilde{g}$ is connected, $p_r^*: H^1(\tilde{G}\ ;\ \boldsymbol{Z}/2)\to H^1(\Sigma_s^I\ ;\ \boldsymbol{Z}/2)$ is injective. Hence $W_1(\Sigma^I)$ is zero if and only if $r(r+1)/2\equiv 1 \pmod 2$.

If $n>p$, $s=(i-1)/2$ $(i>1)$, then $\Sigma_s^I$ is orientable if and only if (i) either $r\equiv 1 \pmod 2$ or $n=i$ and (ii) $r(r+1)/2\equiv 1 \pmod 2$ by the similar argument. If $n\leqq p$, then $i=1$ and $\tilde{K}_1=\tilde{K}_2$. Hence $\Sigma^I(n, p)$ is orientable if and only if either $n+p+r+r(r+1)/2\equiv 0 \pmod 2$ or $n=i=1$. Q.E.D.

Let $\overline{\Sigma^{I_r}(n, p)}$ denote the topological closure of $\Sigma^{I_r}(n, p)$ in $J^k(n, p)$. It follows that both of $\overline{\Sigma^{I_r}(n, p)}$ and $\overline{\Sigma^{I_r}(n, p)}\setminus\Sigma^{I_r}(n, p)$ are algebraic sets in a Euclidean space $J^k(n, p)$. Let $d_{I_r}(n, p)$ be the dimension of $\overline{\Sigma^{I_r}(n, p)}$ as an algebraic set.

PROPOSITION 4.3. *Let $r\geqq 2$. The dimension of $\overline{\Sigma^{I_r}(n, p)}\setminus\Sigma^{I_r}(n, p)$ is smaller than $d_{I_r}(n, p)-1$.*

For a while we identify $J^k(n, p)$ with $\mathrm{Hom}(\oplus_{i=1}^k S^i\boldsymbol{R}^n, \boldsymbol{R}^p)$ by fixing a basis of $\boldsymbol{R}^n$ and $\boldsymbol{R}^p$. Then we write $z\in J^k(n, p)$ as $(z_1, \cdots, z_k)$ for homomorphisms $z_i$: $S^i\boldsymbol{R}^n\to\boldsymbol{R}^p$. For subspaces $L_2\subset L_1\subset\boldsymbol{R}^n$ with $L_1\subset\mathrm{Ker}(z_1)$ we define

$$h_z : \boldsymbol{R}^n/L_1\oplus L_2\bigcirc L_1\oplus\Big(\bigoplus_{t=3}^{r} S^t L_2\Big) \longrightarrow \boldsymbol{R}^p$$

by restricting $z_i$ $(i\geqq 2)$ and $\bar{z}_1 : \boldsymbol{R}^n/L_1\to\boldsymbol{R}^p$ induced from $z_1$.

LEMMA 4.4. *An element $z$ of $\Sigma^{i+s}(n, p)$ belongs to $\overline{\Sigma^{I_r}(n, p)}$ if and only if* (i) $s\geqq 0$ *and* (ii) *there exist an $i$ dimensional subspace $L_1$ in* $\mathrm{Ker}(d_{1,z})$ *and* 1 *dimensional subspace $L_2$ in $L_1$ such that the kernel rank of the homomorphism*

$$h_z : \boldsymbol{R}^n/L_1\oplus L_2\bigcirc L_1\oplus\Big(\overset{r}{\underset{t=3}{\ominus}} S^t L_2\Big) \longrightarrow \boldsymbol{R}^p$$

*is not less than $i+r-2$.*

PROOF. Let $z$ be an element of $\overline{\Sigma^{I_r}(n, p)}$. Then there exists a sequence $\{z^j\}$ in $\Sigma^{I_r}(n, p)$ which converges to $z$. $K_{1,z^j}$ and $K_{2,z^j}$ denote the subspaces of dimensions $i$ and 1 in $\boldsymbol{R}^n$ determined by $z^j$ respectively. Then $\{(K_{1,z^j}, K_{2,z^j})\}$ gives a sequence in $G_{1,i-1}(\gamma)$ which converges to an element, say $(L_1, L_2)$. It is clear that $L_1 \subset K_{1,z}$. Hence we can define a homomorphism $h_z$ induced from $z$. Since

$$h_{zj} : \boldsymbol{R}^n/K_{1,zj} \oplus K_{2,zj} \bigcirc K_{1,zj} \oplus (\bigoplus_{t=3}^{r} S^t K_{2,zj}) \longrightarrow \boldsymbol{R}^p$$

converges to $h_z$ and the kernel rank of $h_{zj}$ is $i+r-2$, it follows that the kernel rank of $h_z$ is not less than $i+r-2$.

Conversely we assume the conditions (i) and (ii) for $z$. Then it follows from the fact $rk(z_1) \leqq n-i$ that we may take a subspace $L$ of dimension $n-i$ which contains the image of $h_z$ and a sequence of homomorphisms $\{a_1^j\}$ of $\boldsymbol{R}^n$ into $L$ of rank $n-i$ converging to $z_1$ such that $\mathrm{Ker}(a_1^j) = L_1$. Let $z^j = (a_1^j, z_2, \cdots, z_k)$ in $J^k(n, p)$. Then the kernel rank of $h_{zj}$ is $i+r-2$ by the definition of $z^j$. This is equivalent to say that $h_{zj}$ induces a zero homomorphism of $L_2 \bigcirc L_1 \oplus (\oplus_{t=3}^{r} S^t L_2)$ into $\boldsymbol{R}^p/L$. This means that $\dim \mathrm{Ker}(d_{t,zj}) = 1$ by Section 1 for $r \geqq t \geqq 2$. Hence $z^j$ becomes a sequence of $\Sigma^{I_r}$. Clearly $\{z^j\}$ converges to $z$.

Q.E.D.

PROOF OF PROPOSITION 4.3. Let $\gamma$ be the $i+s$ dimensional vector bundle over $\Sigma^{i+s}(n, p)$ induced from the canonical $i+s$ dimensional vector bundle over $G_{i+s, n-i-s}$. Let $G_{i,s}(\gamma)$ be its associated grassmann bundle over which we have the canonical $i$ dimensional vector bundle denoted by $\gamma_i$. Let $\gamma_2$ be the canonical line bundle over $G_{1,i-1}(\gamma_i)$ and $\gamma_1$ the induced bundle of $\gamma_i$ over $G_{1,i-1}(\gamma_i)$. Obviously $G_{1,i-1}(\gamma_i)$ consists of all triples $(z, L_1, L_2)$ where $z \in \Sigma^{i+s}(n, p)$, $L_1$ is an $i$ dimensional subspace of $K_{1,z}$ and $L_2$ is a 1 dimensional subspace of $L_1$. The projection of $G_{1,i-1}(\gamma_i)$ onto $\Sigma^{i+s}(n, p)$ is denoted by $p$. We consider the following vector bundle over $G_{1,i-1}(\gamma_i)$,

$$\mathrm{Hom}(\theta^n/\gamma_1 \oplus \gamma_2 \bigcirc \gamma_1 \oplus (\bigoplus_{t=3}^{r} \overset{t}{\bigcirc} \gamma_2),\ \theta^p).$$

Then we can define a smooth section $s$ by mapping $(z, L_1, L_2)$ to a homomorphism of $h_z$. It is clear that $s$ is transverse to every manifold $S^h$ of all homomorphisms of kernel rank $h$. It follows from Lemma 4.4 that $\overline{\Sigma^{I_r}(n, p)} \cap \Sigma^{i+s}(n, p)$ is equal to $p(s^{-1}(\overline{S^{i+r-2}}))$. Now we estimate the codimension of $\overline{\Sigma^{I_r}(n, p)} \cap \Sigma^{i+s}(n, p)$ in the case of either $n \geqq p$, $r > 2$ and $s > 0$ or $n < p$ as follows:

$$\begin{aligned}
&\mathrm{codim}(\overline{\Sigma^{I_r}(n, p)} \cap \Sigma^{i+s}(n, p)) - d_{I_r} \\
&\quad \geqq \mathrm{codim}(\Sigma^{i+s}(n, p)) + \mathrm{codim}(S^{i+r-2}) - \dim G_{1,i-1} - \dim G_{i,s} - d_{I_r} \\
&\quad \geqq (i+s)(p-n+i+s) + (i+r-2)(p-n+i) - is - (i-1) - d_{I_r} \\
&\quad = s(p-n+i) + s^2 + is + (i+r-2)(p-n+i) - is - (i-1) \\
&\quad = (s+i+r-2)(p-n+i) + s^2 - (i-1)
\end{aligned}$$

$$\geqq\begin{cases} s+s^2+r-1 & \text{if } n\geqq p \\ (s+r-1)(p-n+1)+s^2 & \text{if } n<p. \end{cases}$$

If $n\geqq p$, $r=2$ and $s=0$, then we have $\operatorname{codim}\Sigma^{(i,h)}\geqq\operatorname{codim}\Sigma^{(i,1)}+2$ for $h\geqq 2$. This proves the proposition. Q.E.D.

## § 5. Dual classes.

In the rest of the paper $I$ means $I_r$. In this section we assume that $N$ is a connected and closed manifold. Let $c_I$ be $\operatorname{codim}\Sigma^I(N, P)$. We will define the dual class $c^I_{s,u}$ of $\overline{\Sigma^I_s(N, P)}$ for a section $u: N\to\Omega_r(N, P)$ in $H^{c_I}(N; \boldsymbol{Z})$ in case $TN$, $(\pi^k_P\circ s)^*TP$ and $\Sigma^I_s$ are orientable. In the other cases $c^I_{s,u}$ can be definable in $H^{c_I}(N; \boldsymbol{Z}/2)$ similarly and we can also follow the method defining the dual class of $\overline{\Sigma^{I_2}(n, p)}$ in [**15**]. So we omit it. It will be seen that this dual class is the primary obstruction class of $u$ to be homotopic to a section of $\Omega_r(N, P)\setminus\Sigma^I_s(N, P)$ over $N$. By fixing a structure of a vector bundle on $J^k(N, P)$ (see, for example, [**3**, Chapter 2]) we take a metric of $J^k(N, P)$. Note that $\Sigma^I_s(N, P)$ is invariant under the coordinate changes of this bundle structure. Let $J^k_u$ be the induced vector bundle $(\pi^k_0\circ u)^*J^k(N, P)$ and $S^I_{s,u}$, $(\pi^k_0\circ u)^*(\Sigma^I_s(N, P))$. Let $D_u$ (resp. $S_u$) denote the associated disk (resp. sphere) bundle of $J^k_u$ and $D$ (resp. $S$), the unit disk (resp. sphere) of $J^k(n, p)$ for a while. Then $\overline{\Sigma^I}\cap D$ is a cone of $\overline{\Sigma^I}\cap S$. By [**13**, Theorem 1] we can triangulate an algebraic set $\overline{\Sigma^I_s}\cap S$ so that $(\overline{\Sigma^I_s}\setminus\Sigma^I_s)\cap S$ becomes its subcomplex.

LEMMA 5.1. *Let $P$ be a finite simplicial complex of dimension $p$ and $Q$ its subcomplex such that $P\setminus Q$ is a connected topological manifold and* $\dim Q\leqq p-2$. *Then we have the following*

(1) *If $P\setminus Q$ is orientable, then $H_{p+1}(CP, P; \boldsymbol{Z})\cong\boldsymbol{Z}$ and $H^{p+1}(CP, P; \boldsymbol{Z})\cong\boldsymbol{Z}$.*

(2) *If $P\setminus Q$ is nonorientable, then $H_{p+1}(CP, P; \boldsymbol{Z})=\{0\}$ and $H^{p+1}(CP, P; \boldsymbol{Z})$ $=\boldsymbol{Z}/2$.*

The proof of the lemma will be elementary. It follows from Proposition 4.3 and Lemma 5.1 that

$$H_{d_I}(\overline{\Sigma^I_s}\cap D, \overline{\Sigma^I_s}\cap S\ ;\ \boldsymbol{Z})\cong\boldsymbol{Z}.$$

Therefore we have

$$H_{d_I+n}(\overline{S^I_{s,u}}\cap D_u, \overline{S^I_{s,u}}\cap S_u)\ ;\ \boldsymbol{Z})\cong\boldsymbol{Z}.$$

Let a generator of this homology group or its image in $H_{d_I}(D_u, S_u\ ;\ \boldsymbol{Z})$ be denoted by $[\overline{S^I_{s,u}}]$. By the Poincaré duality isomorphism $H_*(D_u; \boldsymbol{Z})\cong H^*(N; \boldsymbol{Z})$, $[\overline{S^I_{s,u}}]$ is mapped onto an element of $H^{c_I}(N; \boldsymbol{Z})$ denoted by $c^I_{s,u}$. We call $c^I_{s,u}$ the dual class of $\overline{\Sigma^I_s(N, P)}$ for $u$. The sum of all $c^I_{s,u}$, $s=0, \cdots, [(i-1)/2]$, is called the dual class of $\overline{\Sigma^I(N, P)}$ for $u$ and denoted by $c_u^I$. If $u$ is a jet section $j^k f$ of a smooth map $f$, this class modulo 2 is called the Thom polynomial of $\overline{\Sigma^I(N, P)}$ for $f$ in [**10**].

Let $\Omega_r$ denote the fibre of $\Omega_r(\boldsymbol{R}^n, \boldsymbol{R}^p)$ over the origin $0\times 0$. Let $c(u)$ denote the primary obstruction class of $u$ to be homotopic to a section of $\Omega_{r-1}(N, P)$ over $N$ ([**18**]). Then $c(u)$ is an element of $H^{c_I}(N\ ;\ \pi_{c_I}(\Omega_r, \Omega_{r-1}))$. Since $\pi_i(\Omega_r, \Omega_{r-1})$ vanishes for $i<c_I$ and $\Omega_{r-1}$ is simply connected except for the case $n=p$ and $I=(1, 1)$, we have

$$\pi_{c_I}(\Omega_r, \Omega_{r-1})\cong H_{c_I}(\Omega_r, \Omega_{r-1}\ ;\ \boldsymbol{Z})$$

by the Hurewicz isomorphism theorem except for the above case. (If this pair is $c_I$ simple, this exception is unnecessary. In fact, for $n=p=r=2$, $\Omega_1(2, 2)$ is simply connected.) On the other hand we have by Alexander duality theorem

$$\begin{aligned} H_{c_I}(\Omega_r, \Omega_{r-1}\ ;\ \boldsymbol{Z}) &\cong H_{c_I}(\Omega_r\cap S, \Omega_{r-1}\cap S) \\ &\cong H^{d_I-1}((\overline{\Sigma^I}\cup\Omega^c)\cap S, \Omega^c\cap S) \\ &\cong H^{d_I-1}(\overline{\Sigma^I}\cap S, (\overline{\Sigma^I}\setminus\Sigma^I)\cap S) \\ &\cong \bigoplus_{s=0}^{[(i-1)/2]} H^{d_I-1}(\overline{\Sigma_s^I}\cap S\ ;\ \boldsymbol{Z}) \end{aligned}$$

where $\Omega^c$ denotes the complement of $\Omega_r$ in $J^k(n, p)$. Hence we have the following proposition.

PROPOSITION 5.2. *Let $r\geqq 2$ and for the case $n=p$ let $r>2$ or $n=p=r=2$. We assume that if one of $\Sigma_s^I(n, p)$, $s=0, \cdots, [(i-1)/2]$ and $N$ are orientable, then $(\pi_0^k\circ u)^*TP$ is orientable. Then the primary obstruction class $c(u)$ is equal to the direct sum of the dual classes $c_{s,u}^I$, $s=0, \cdots, [(i-1)/2]$ in $H^{c_I}(N\ ;\ \bigoplus_{s=0}^{[(i-1)/2]}G_s)$ where $G_s=\boldsymbol{Z}$ or $\boldsymbol{Z}/2$ depending on whether $\Sigma_s^I(n, p)$ is orientable or not.*

Let $u$ be a smooth section of $\Omega_r(N, P)$ over $N$ transverse to every $\Sigma^{I_t}(N, P)$. Let $N_{t,s}$ be $u^{-1}(\Sigma_s^{I_t}(N, P))$. Then $c_{s,u}^{I_t}$ is equal to the dual class of $N_{t,s}$ in $N$. We will use the following fact in a proof of Proposition 5.3. Consider a closed submanifold $M$ such that $N\supset M\supset N_{r,s}$. Then the dual class of $N_{r,s}$ is a cup product of that of $M$ in $N$ and that of $N_{r,s}$ in $M$ coming from the cohomology group of $N$. Here we give a table of orientability of $\Sigma_s^I$ for the case $r\geqq 2$ and $n>p$ by Proposition 4.2.

| $r$ | $4k+2$ | $4k+3$ | $4k+4$ | $4k+5$ |
|---|---|---|---|---|
| $\Sigma_s^I$ $(s\neq(i-1)/2)$ | orientable | non-orientable | non-orientable | orientable |
| $\Sigma_s^I$ $(s=(i-1)/2)$ | non-orientable | non-orientable | non-orientable | orientable |

Proposition 5.3. *Under the assumption as Proposition* 5.2 *in addition to* $c_I=n$, $c_{s,u}^I$ *vanishes in the following cases:*

(i) $n>p$ *and* $s\neq(i-1)/2$,

(ii) $n>p$, $s=(i-1)/2$ *and* $r\equiv 1 \pmod 4$ *and*

(iii) $n\leqq p$ *and* $n+p+r+r(r+1)/2\equiv 0 \pmod 2$.

Proof. First we show that $c_{s,u}^{I_t}$ vanishes for $n>p$, $s\neq(i-1)/2$ and $t\equiv 2$ (mod 4). In this case it follows from Propositions 4.1 and 4.2 that $\Sigma_s^{I_t}$ and $\Sigma_s^{I_{t-1}}$ are orientable and that $\Sigma_s^{I_{t-1}}\setminus\Sigma_s^{I_t}$ has two connected components, say $\Sigma^+$ and $\Sigma^-$. Therefore both of manifolds $N_{t-1,s}$ and $N_{t,s}$ are orientable. Let $N^+$ denote the closure of $(u)^{-1}(\Sigma^+)$. Then it is clear that $N_{t,s}$ bounds $N^+$. Hence the dual class of $N_{t,s}$ vanishes in $H^{c(I_t)}(N\,;\boldsymbol{Z})$. Consider the triple $N\supset N_{t,s}\supset N_{r,s}$ for $t\equiv 2$ (mod 4). Then it follows from the above remark and the above table that the dual class of $N_{r,s}$ for $n>p$ and $s\neq(i-1)/2$ vanishes in $H^{c_I}(N, G)$.

It follows from [**16**] and [**1**, Corollary 5.3] that $\Sigma^i(n, p)$ is orientable if $n+p$ is even and that the dual class of $c_u^i$ is an element of 2 torsion if $i$ is odd. Hence if $n>p$, $s=(i-1)/2$ and $r\equiv 1 \pmod 4$, then $p+n$ is even and the dual class of $N_{r,s}$ is an element of 2 torsion of $H^{c_I}(N\,;\boldsymbol{Z})\cong\boldsymbol{Z}$, that is, vanishes by considering $N\supset N_1\supset N_{r,s}$.

For $n\leqq p$ it follows from Proposition 4.2 that $\Sigma^{I_2}$ is orientable for $n+p$ odd and $\Sigma^1$ is orientable for $n+p$ even. By [**1**, Corollary 5.3] $c_u^1$ and $c_u^{I_2}$ are elements of 2 torsion in both cases. Therefore by the similar arguments $c_u^I$ vanishes.

Q.E.D.

## § 6. Proof of Theorem 2.

We have assumed in Theorem 2 that $N$ and $P$ are always orientable for simplicity. However this assumption can be weakened in the first part (1) of Theorem 2 as 'If $\Sigma^{I_r}(n, p)$ and $N$ are orientable, then $f^*TP$ is also orientable.' We prove Theorem 2 in this form.

Proof of Theorem 2. By Theorem 1 a proof is reduced to the problem of finding a section of $\Omega_{r-1}(N, P)$ homotopic to $j^kf$ in $\Omega_r(N, P)$. This is possible if and only if $c_{s,j^kf}^I$ vanishes for $0\leqq s\leqq[(i-1)/2]$ by Proposition 5.2. However it vanishes except for the case $s=(i-1)/2$ by Proposition 5.3. Therefore this is equivalent to say that the Thom polynomial of $\overline{\Sigma^I(N, P)}$ for $f$ vanishes. This is the first part of Theorem 2. Furthermore $c_{s\,j^kf}^I$ for $s=(i-1)/2$ vanishes in both cases of (i) and (iii). In case (ii) we have always $s\neq(i-1)/2$ since $n-p$ is odd. This is the second part. Q.E.D.

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Yoshifumi ANDO
Department of Mathematics
Yamaguchi University
Yamaguchi 753
Japan