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On the Elimination of Non-Resonance Harmonics

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Abstract. Given a weakly coupled Hamiltonian system with short range, one dimensional interactions, and *any* initial conditions a canonical change of variables is constructed which yields a new Hamiltonian consisting of three parts—an integrable term, a resonant term whose effects are localized in those regions of the system which give small denominators in the Kolmogorov–Arnol'd–Moser iteration scheme and a non-resonant interaction term which is very small. (In particular, much, much smaller than our original interactions.) The conditions which allow such a transformation to be constructed are independent of the number of degrees of freedom in the system, as are the estimates on the size of the various terms. Thus, if the resonances are "sparsely" distributed through the system most of the sites in the transformed Hamiltonian behave essentially like an integrable system, at least for as long a time as the trajectory of the system lies within the region where the canonical transformation is defined. In subsequent work it is shown that this time is long, and once again independent of the number of degrees of freedom in the system.

1. Introduction

In the present paper we continue the study of Hamiltonian systems with short range interactions begun in [6]. We prove a theorem which we call the elimination of non-resonance harmonics, because of its similarity to the lemma of the same name in [5]. Roughly speaking our result is as follows. Take a Hamiltonian in action-angle coordinates with short range interactions, e.g.

$$H(I, \phi) = \tfrac{1}{2}\langle I, I\rangle + \varepsilon \sum_{i=1}^{N-1} \cos(\phi_{i+1} - \phi_i), \tag{1.1}$$

and N degrees of freedom. Given some initial condition (I_0, ϕ_0) define a set of "primary resonance vectors of order zero" to be $\nu \in \mathbb{Z}^N$ such that the denominators, in the expression for the generating function of the canonical change of variables that

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solves the Hamilton–Jacobi equation, corresponding to these v's are "very small." Given the Hamiltonian (1.1), the solution of the Hamilton–Jacobi equation given by classical perturbation theory is the change of variables with generating function $\langle I', \phi \rangle + S(I', \phi)$, where

$$S(I', \phi) = \frac{\varepsilon}{2} \sum_v \frac{e^{iv\cdot\phi}}{i\langle I', v\rangle}, \tag{1.2}$$

and the sum over v runs over all vectors of the form $(0,\ldots,0, \pm 1, \mp 1, 0,\ldots,0)$. We are interested in I', near I_0, so we define the resonant vectors, R, to be those v in (1.2) such that $|\langle I_0, v\rangle| < c_0$, where c_0 is a constant determined in the course of the proof. We then define $S(I', \phi)$ by (1.2) but we restrict the sum over v to those $v \notin R$. If we then make the canonical change of variables defined by this generating function (it will be well defined provided ε is sufficiently small) we find that the Hamiltonian (1.1) is transformed into a Hamiltonian $H^1(I', \phi') = h^1(I') + f^{1,\text{resonant}}(I', \phi') + f^{1,\text{nonresonant}}(I', \phi')$. The resonant part of the interaction, $f^{1,\text{resonant}}$, has a Fourier series $\sum f_v^1(I') e^{iv\cdot\phi}$, which has non-zero contributions only if $v \in R$, or for those v' with supp v' "close" to supp v, for some $v \in R$. Since our change of variables was forced to ignore the contributions of the resonant vectors, we expect $f^{1,\text{resonant}}$ will still be $\mathcal{O}(\varepsilon)$. On the other hand, we have chosen $S(I', \phi)$ so that the change of variables will "kill" the $\mathcal{O}(\varepsilon)$ terms in the non-resonant interaction. Thus, we expect $f^{1,\text{nonresonant}} \sim \mathcal{O}(\varepsilon^2)$. We now iterate this procedure (a finite number of times) and we find that there is a canonical change of variables C, such that

$$\tilde{H}(I, \phi) = H \circ C(I, \phi) = \tilde{h}(I') + \tilde{f}^{\text{resonant}}(I', \phi') + \tilde{f}^{\text{nonresonant}}(I', \phi'), \tag{1.3}$$

where the Fourier series of $\tilde{f}^{\text{resonant}}$ contains the resonant harmonics we encountered at the various steps in the iterative procedure while $\tilde{f}^{\text{nonresonant}} \sim \mathcal{O}(e^{-1/\varepsilon^a})$ for some positive constant a. The conditions which allow this change of variables to be constructed (and also the constant a) are independent of the number of degrees of freedom in the system.

This would be of little interest if all vectors, v, were resonant. However, in [8] we demonstrate that for "typical" initial conditions, (I_0, ϕ_0), the resonant vectors are quite "sparse" and, hence, that the motion of the system is governed largely by $\tilde{f}^{\text{nonresonant}}$. We then show that this implies that for most sites j in the system, the motion of $I_j(t)$ is indistinguishable from that of an integrable system for a long (but finite) period of time. Thus, while the system may well become ergodic as the number of degrees of freedom $N \to \infty$, the irregular motion tends to be localized in the vicinity of the resonances, (at least for ε small), while large parts of the system undergo very regular motion, for a long time.

To state our results more precisely we introduce some notation. Let V be a set in $\mathbb{R}^N$. A Hamiltonian in action angle form is a function $H^0(I, \phi)$: $V \times \mathbb{R}^N \to \mathbb{R}$ that is periodic with period 2π in each of the ϕ_j variables. Since H^0 is periodic as a function of ϕ we may also regard $H^0(I, \phi)$ as a function on $V \times T^N$, where T^N is the N-torus. We will make no distinction between these two meanings for $H^0(I, \phi)$. Write the Hamiltonian as

$$H^0(I, \phi) = h^0(I) + f^0(I, \phi), \tag{1.4}$$

where $f^0(I,\phi)$ is assumed to be small, in a sense made precise below. Given an N vector ρ^0, and a constant ξ_0 we define the domain $W(\rho^0,\xi_0;V)\equiv \bigcup_{I\in V} \{(I',\phi)\in\mathbb{C}^{2N} \mid |\operatorname{Im}\phi_i|<\xi_0, |I'_i-I_i|<\rho_i^0, \text{ for } i=1,\dots,N\}$. We assume that there are constants ρ_0 and ξ_0 such that $H^0(I,\phi)$ is analytic on $W(\rho^0,\xi_0;V)$, with $\rho^0=\rho_0(1,\dots,1)$. Since we need lots of analyticity in the angular variables to prove our main theorem we assume $\xi^0 \gg 1$. For later convenience define $pr_1 W(\rho^0,\xi_0;V)$ to be $\{I\in\mathbb{R}^N \mid \exists\phi\in\mathbb{R}^N \text{ such that } (I,\phi)\in W(\rho^0,\xi_0;V).\}$. Given initial conditions (I_0,ϕ_0) define the strength of the interaction, ε_0, by

$$\sup\left\{\left|\frac{\partial f^0}{\partial I_j}\right| + \rho_0^{-1}\left|\frac{\partial f^0}{\partial\phi_j}\right|\right\}\leqq \varepsilon_0, \tag{1.5}$$

where the supremum runs over both $j=1,\dots,N$ and $(I,\phi)\in W(\rho^0,\xi_0;\{I_0\})$. We note that it might seem more natural to choose ε to measure the strength of the interaction in (1.1), but a little calculation shows that ε and ε_0 are related by $\varepsilon\rho_0^{-1}e^{2\xi_0}\leqq\varepsilon_0$, and (1.5) proves somewhat more useful in the general case.

The Hamiltonians we consider consist of almost independent almost identical subsystems, lying along a line, with interactions which decrease rapidly in strength as the distance between the points of interaction increases.

(a) nearly independent, nearly identical subsystems: Define $\omega^0(I)=\partial h/\partial I\,(I)$. We require

$$\sup\left|\frac{\partial\omega_i^0}{\partial I_j}\right|\leqq e^{-m|i-j|} \quad \text{if } i\neq j, \tag{1.6}$$

for some constant $m>0$, and

$$\frac{\partial\omega_i^0}{\partial I_i}(I)=1+\chi^0(I;i,i), \tag{1.7}$$

with $\sup|\chi^0(I;i,i)|\leqq c_1, c_1$ some universal constant, say 2^{-3}. The suprema all run over $W(\rho^0,\xi^0;\{I_0\})$.

(b) short range interactions:

$$\begin{aligned}
&\sup\left|\frac{\partial^2 f^0}{\partial\phi_i\partial\phi_j}(I,\phi)\right|\leqq\rho_0^2e^{-m|i-j|},\\
&\sup\left|\frac{\partial^2 f^0}{\partial\phi_i\partial I_j}(I,\phi)\right|\leqq\rho_0e^{-m|i-j|},\\
&\sup\left|\frac{\partial^2 f^0}{\partial I_i\partial I_j}(I,\phi)\right|\leqq e^{-m|i-j|}.
\end{aligned}\tag{1.8}$$

All suprema are taken over $W(\rho^0,\xi_0;\{I_0\})$. This definition of short range interactions differs slightly from that of [6]. Except for the factors of ρ_0, however, which merely serve to keep the dimensions of the two sides the same, (and the corresponding factors of $\varepsilon_0\rho_0$, ε_0, and $\varepsilon_0\rho_0^{-1}$ in [6]) the previous definition follows from (1.8) by applying Cauchy's theorem and setting $m=|\ln(\varepsilon_0\rho_0^{-1})|$. Note also that (1.1) obeys (1.8) provided we choose ε such that $\varepsilon e^{2\xi_0}=\rho_0^2e^{-m}$.

We note that the conditions above correspond to considering this system as lying on a one dimensional lattice. We expect our results to extend to higher dimensional lattices, and to systems with periodic boundary conditions, but the technical difficulties encountered are greater there.

Theorem 1.1. *(The elimination of non-resonance harmonics) There exist universal constants* $0 < c \ll 1$, $0 < \sigma$, *and* $K \gg 1$, *and a constant* k_0 *defined below such that if*

$$\varepsilon_0 < c\rho_0(k_0)^{-\sigma} \quad \text{and} \quad m > K, \tag{1.9}$$

we can construct a set, R, of vectors $v \in \mathbb{Z}^N$, *an N vector* $\tilde{\rho}$ *and a change of variables* $C:(I', \phi') \to (I, \phi)$, *analytic and invertible on* $W(\tilde{\rho}, 1; \{\tilde{I}\})$, *where* $\tilde{I}$ *will be defined in the course of the proof. Furthermore, if* $\Gamma = pr_1 W(\tilde{\rho}, 1, \{\tilde{I}\})$, *C is canonical on* $\Gamma \times T^N$. *Defining* $\tilde{H}(I', \phi') = H^0 \circ C(I', \phi')$, *we have*

$$\tilde{H}(I', \phi') = \tilde{h}(I') + \tilde{f}(I', \phi') = \tilde{h}(I') + \tilde{f}^{\text{resonant}}(I', \phi') + \tilde{f}^{\text{nonresonant}}(I', \phi'), \tag{1.10}$$

with $\tilde{f}^{\text{resonant}}(I', \phi') = \sum_{v \in R} \tilde{f}_v(I') e^{iv \cdot \phi}$. *The interactions of* $H(I', \phi')$ *obey the bounds*

$$\begin{aligned}
&\sup \left| \frac{\partial^2 \tilde{f}}{\partial \phi_i' \partial \phi_j'}(I', \phi') \right| \leqq \rho_0^2 e^{-(3m/4)|i-j|}, \\
&\sup \left| \frac{\partial^2 \tilde{f}}{\partial \phi_i' \partial I_j'}(I', \phi') \right| \leqq \rho_0 e^{-(3m/4)|i-j|}, \\
&\sup \left| \frac{\partial^2 \tilde{f}}{\partial I_i' \partial I_j'}(I', \phi') \right| \leqq e^{-(3m/4)|i-j|}
\end{aligned} \tag{1.11}$$

and

$$\begin{aligned}
&\sup \left\{ \left| \frac{\partial \tilde{f}^{\text{nonresonant}}}{\partial I_j'}(I', \phi') \right| + \rho_0^{-1} \left| \frac{\partial \tilde{f}^{\text{nonresonant}}}{\partial \phi_j'}(I', \phi') \right| \right\} \\
&\leqq \rho_0(\varepsilon_0 \rho_0^{-1})^{(3/2)^{k_0}},
\end{aligned} \tag{1.12}$$

for all i and j, with the suprema running over $W(\tilde{\rho}, 1; \{\tilde{I}\})$. *The constant* k_0 *in (1.12) is given by* $k_0 \leqq \min([(\xi_0 - 1)/K_1], [m/K_1])$, *for* K_1 *some universal constant which could be calculated from the proof of the theorem, and* $[x]$ = *integer part of x*. (We remark that one could make the interactions in (1.11) decay as $e^{-(1-\delta)m|i-j|}$, for any $0 < \delta < 1$, by slight modifications in the proof, and by changing the constants in (*1.9*).) We note that this theorem can be trivially proved by picking R to be all of $\mathbb{Z}^N$, in which case, $\tilde{f}^{\text{nonresonant}} = 0$, and we have gained nothing. The point is, that the procedure used to define R yields, for most initial conditions, a "small" set of vectors R, so that the effects $\tilde{f}^{\text{resonant}}$ are localized, and the motion of the system is largely determined by $\tilde{h} + \tilde{f}^{\text{nonresonant}}$. This, as we demonstrate in [8], allows one to give strong bounds on the trajectories of the system.

The proof of this theorem can be used to give bounds on the amount by which the canonical transformation C differs from the identity. Such estimates are implicit in Sects. 3 and 4.

We now discuss the application of this theorem to the Hamiltonian (1.1). Since the Hamiltonian is an entire function we can take the size of the analyticity domain as large as we like. Choose $\xi_0 = \frac{1}{4}|\ln \varepsilon|$ and $\rho_0 = 1$. Then condition (1.5) implies $\varepsilon e^{2\xi_0} = \varepsilon^{1/2} \leqq \varepsilon_0$. Set $\varepsilon_0 = \varepsilon^{1/2}$. Then the left hand side of (1.8) is bounded by $\varepsilon e^{2\xi_0} = \varepsilon^{1/2} \leqq e^{-m}$. If we set $m = \frac{1}{2}|\ln \varepsilon| = |\ln \varepsilon_0|$, then (1.8) is satisfied and by making ε small we can make m as large and ε_0 as small as needed. Furthermore, the constant k_0 in Theorem 1.1 can be chosen to be $k_0 = [(1/5K_1)|\ln \varepsilon|]$. Finally note that (1.6) and (1.7) are trivially satisfied for (1.1). Thus we obtain as a corollary to Theorem 1.1:

Corollary 1.2. *Given the Hamiltonian* (1.1) *there is some constant* $0 < c \ll 1$ (*independent of N*) *such that if* $0 \leqq \varepsilon < c$ *we may construct a canonical transformation C, by Theorem 1.1 such that*

$$H \circ C(I', \phi') = \tilde{h}(I') + \tilde{f}^{\text{resonant}}(I', \phi') + \tilde{f}^{\text{nonresonant}}(I', \phi')$$

with

$$\sup\left\{\left|\frac{\partial \tilde{f}^{\text{nonresonant}}}{\partial I'_j}(I', \phi')\right| + \left|\frac{\partial \tilde{f}^{\text{nonresonant}}}{\partial \phi'_j}(I', \phi')\right|\right\} \leqq \varepsilon^{(1/2)(3/2)^{k_0}} \leqq e^{-1/\varepsilon^a} \quad (1.13)$$

for some positive constant a, (*independent of the number of degrees of freedom of the system.*) *The supremum runs over* $W(\tilde{\rho}, 1; \{\tilde{I}_0\})$ (*with* $\tilde{\rho}$ *and* $\tilde{I}$ *defined in the course of the proof.*) *Also, the interactions* $\tilde{f}^{\text{resonant}} + \tilde{f}^{\text{nonresonant}}$, *satisfy* (*1.11*) *with* $m = \frac{1}{2}|\ln \varepsilon|$.

We close this section with some remarks about notation. Throughout the paper $B, B', B_1, B_2, \ldots$ will denote universal constants of magnitude greater than one, while $c, c', c_1, c_2, \ldots$ will denote constants of magnitude less than one. They may represent different constants in different contexts.

The second note concerns the factors of ρ_0 which occur, for instance, in (1.8) and (1.11). These are included to keep the dimensions of the two sides of the inequality the same, since as was pointed out in [4], this often provides a check on one's calculations at intermediate stages of the proof. In the final analysis, however, they are less important and we will often, to save space, write inequalities like (1.8) as follows: Let x_i equal either I_i or ϕ_i depending on the context. Then $f^0(I, \phi) = f^0(x)$ and we write (1.8) as

$$\sup\left|\frac{\partial^2 f^0}{\partial x_i \partial x_j}(x)\right| \leqq \rho_0^n e^{-m|i-j|}, \quad (1.14)$$

where n is chosen to keep the dimensions of the two sides of the inequality the same. If the reader finds this notation ambiguous just set $\rho_0^n = K$ wherever it appears and remember that K is a constant depending on the size of the analyticity domain of the initial Hamiltonian, but independent of the number of degrees of freedom in the system.

We note that Theorem 1.1 is related to Theorem 1.1 of [6]. There are two principal differences in the results. First of all, by more careful estimates it has proved possible to eliminate all dependence on the number of degrees of freedom of the system. This is largely a technical improvement. Of more fundamental importance is the fact that the present work allows resonant regions to exist in the system, but provides a method of isolating (at least for a finite time) their effects. The work of Nekhoroshev [5] provided the motivation for this improvement in the theory.

A final note concerning terminology—when we speak of "dimensional estimates" we mean the standard estimates on derivatives of analytic functions that Cauchy's theorem provides.

Recently, Benettin et al. [1] have studied the model (1.1) by means of classical perturbation theory. They have shown that one can construct a canonical transformation which transforms (1.1) into an integrable system up to errors of arbitrarily high order in ε by this means. Their method (like that of the present work) is restricted to small ε, but thus far the dependence of the size of the allowed perturbation on the parameters of the system, such as the number of degrees of freedom, has not been determined. If these estimates are performed their method may yield an alternate proof of Corollary 1.2.

2. The Induction Procedure

Consider the initial Hamiltonian $H^0(I,\phi)$ of Sect. 1. We first locate the *primary resonances of order zero*, R_p^0. Let $\mathbb{X}_0 = \{\nu \in \mathbb{Z}^N \mid d(\operatorname{supp}\nu) \leqq L_0,\ 0<|\nu| \leqq M_0\}$, where $d(\operatorname{supp}\nu)$ is the diameter of $\operatorname{supp}\nu$, if ν is considered as an integer valued function on the lattice $\{1,\ldots,N\}$, $|\nu| = \sum_{j=1}^{N} |\nu_j|$, and L_0 and M_0 are constants defined below. Given initial conditions (I_0,ϕ_0), set $\omega^0(I) = \partial h^0/\partial I\ (I)$ and define

$$R_p^0 = \{\nu \in \mathbb{X}_0 \mid |\langle \omega^0(I_0), \nu\rangle| < \rho_0 \lambda(\varepsilon_0)\{B_1 e^{(3/2)|\nu| + L_0}\}^{-1}\},$$

where $\lambda(\varepsilon_0) = (\varepsilon_0 \rho_0^{-1})^\alpha$ for α some small positive constant that will be implicitly defined in the course of the proof and B_1 some large constant. In the KAM theory one attempts to construct a canonical change of coordinates which "kills" the nonintegrable part of the Hamiltonian to order ε_0^2. In the present case we must be content to "kill" only those harmonics, (Fourier coefficients), $f_\nu^0(I)$, with $\nu \notin R_p^0$. Constructing a canonical transformation C^0, via the classical perturbation theory, we obtain a new Hamiltonian

$$H^1(I',\phi') = H^0 \circ C^0(I',\phi') = h^1(I') + f^{1,\mathrm{r}}(I',\phi') + f^{1,\mathrm{nr}}(I',\phi').$$

As expected, the size of $f^{1,nr}(I',\phi') \ll \varepsilon_0$. What is perhaps slightly surprising is that one must include in the "resonant" part of the interaction, $f^{1,r}$, not only harmonics $f_\nu^1(I)$ with $\nu \in R_p^0$, but also those harmonics which are "close" to primary resonances, in a sense made precise below. We call these harmonics the *secondary resonances of zeroth order*. One then iterates this procedure locating at each stage the primary resonances, and choosing the canonical transformation to kill the nonresonant harmonics. A finite number of such iterations suffices to prove Theorem 1.1.

Suppose we have constructed $H^k(I,\phi) = h^k(I) + f^{k,\mathrm{r}}(I,\phi) + f^{k,\mathrm{nr}}(I,\phi)$, with $f^{k,\mathrm{r}}(I,\phi) = \sum_{\nu \in \tilde{R}^k} f_\nu^k(I) e^{i\nu\cdot\phi}$, for some set of harmonics $\tilde{R}^k$, specified below. Define $L_k = 2(3/2)^k |\ln(\varepsilon_0 \rho_0^{-1})|/m$, $M_k = (3/2)^k |\ln(\varepsilon_0\rho_0^{-1})|/\delta$, for δ a universal constant to be specified in the course of the proof. Define $\rho_{k+1} = \rho_k \lambda(\varepsilon_0)[B_2 e^{2M_k + 2L_k}]^{-1}$, for $k = 0,\ldots,$ and set $\mathbb{X}_k = \{\nu \in \mathbb{Z}^N \mid d(\operatorname{supp}\nu) \leqq L_k,\ 0 < |\nu| \leqq M_k\}$. Let $\omega^k(I) = (\partial h^k/\partial I)(I)$. In the course of the proof we will define a sequence of vectors $I_0, I_1, \ldots, I_{k_0}$, whose first element is the initial value of the action variables, I. The *primary resonances of order k* are

$$R_p^k = \{v \in \mathbb{X}_k \backslash \tilde{R}^k \,|\, |\langle \omega^k(I_k), v\rangle| < \rho_0 \lambda(\varepsilon_0)[B_1 e^{(3/2)|v| + L_k}]^{-1}\}. \tag{2.1}$$

The *secondary resonances of order k* are

$$R_s^k = \{v \in \mathbb{X}_{k+1} \backslash (\tilde{R}^k \cup R_p^k) | \overline{\operatorname{supp} v} \cap \overline{\operatorname{supp} v'} \neq \emptyset \text{ for some } v' \in \tilde{R}^k \cup R_p^k\}. \tag{2.2}$$

Here $\overline{\operatorname{supp} v} = \{i, i+1, \ldots, j-1, j\}$, for i and j respectively the leftmost and rightmost sites in $\operatorname{supp} v$. We assume that the set of vectors $\tilde{R}^k$ which defines $f^{k,r}$ above is given by $\tilde{R}^k = \bigcup_{m=0}^{k-1} [R_p^m \cup R_s^m]$ if $k > 0$, and $\tilde{R}^0 = \emptyset$.

The k^{th} *order sites*, S^k, are all sites j such that:

(i) $j \in \overline{\operatorname{supp} v}$ for some $v \in R_p^k \cup R_s^{k-1}$ (for $k = 0$ take $R_s^{-1} \equiv \emptyset$,)

(ii) $j \notin S^m$, $m = 0, \ldots, k-1$.

Roughly speaking, the motion of $(I_j(t), \phi_j(t))$ for $j \in S^m$, is controlled by the m^{th} order resonances.

One interesting technical difference between the present work and previous work on the KAM theory is that we must allow the size of the neighborhood on which we define our change of variables to vary as we move about through the system. Given a domain $W(\rho, \xi; \{I\})$, $(\rho)_j$ determines the size of the complex neighborhood about I_j, and we must choose that size to be much larger when $j \in S^m$, for m small, than when j lies in the non-resonant regions (i.e. $j \notin S^m$, $m = 0, \ldots, k$). Define

$$(b^m)_i = \begin{cases} c_1 \rho_{m+1}/k_0 & \text{if } i \in S^m, m = 0, \ldots, k \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Then for $k \geqq 1$, define

$$(\rho^k)_i = \begin{cases} \rho_k \text{ if } i \notin \bigcup_{m=0}^{k-2} S^m & \text{(or for all } i \text{ if } k = 1) \\ (\rho^{k-1})_i - 8(b^m)_i & \text{if } \; i \in S^m, m = 0, \ldots, k-2. \end{cases} \tag{2.4}$$

In the angular variables ϕ we give up a fixed amount of analyticity with each iteration and set $\xi^{k+1} = \xi^k - 3(\delta + 2)$.

We assume that $H^k(I, \phi)$ obeys the following estimates, on $W(\rho^k, \xi_k; \{I_k\})$.

$$\sup\left\{\left|\frac{\partial f^{k,\text{nr}}}{\partial I_j}\right| + \rho_0^{-1}\left|\frac{\partial f^{k,\text{nr}}}{\partial \phi_j}\right|\right\} \leqq \varepsilon_k, \tag{2.5}$$

with $\varepsilon_k = \rho_0(\varepsilon_0 \rho_0^{-1})^{(3/2)^k}$, and

$$\sup\left\{\left|\frac{\partial f^{k,\text{r}}}{\partial I_j}\right| + \rho_0^{-1}\left|\frac{\partial f^{k,\text{r}}}{\partial \phi_j}\right|\right\} \leqq A(k,j), \tag{2.6}$$

where

$$A(k,j) = \begin{cases} \rho_0(\varepsilon_k \rho_k^{-1})^{(1-\eta_k)} & \text{if } \; j \notin \bigcup_{n=0}^{k-1} S^n \\ \rho_0(\varepsilon_n \rho_n^{-1})^{(1-\eta_k)} & \text{if } \; j \in S^n, n = 0, \ldots, k-1, \end{cases} \tag{2.7}$$

and $\eta_k = 8\sum_{j=0}^{k-1}\beta_j$, $(\eta_0=0)$ and $\beta_j=c_1(3/2)^{-j}+c_2/k_0$, with c_1 and c_2 chosen so that $\eta_k<1/8$, for $k=0,\dots,k_0$. We note that $\varepsilon_k \leqq A(k,j)$, so the non-resonant part of the interaction is smaller than the resonant part. We also assume that our Hamiltonian retains its short range character, so that

$$\frac{\partial^2 h^k}{\partial I_I \partial I_j}(I) = \delta_{ij} + \chi^k(I;i,j), \tag{2.8}$$

where

$$\sup|\chi^k(I;i,j)| \leqq \theta(k;i,j) = \begin{cases} e^{-(1-\eta_k)m|i-j|} & \text{if} \quad i \neq j \\ c_1 + Bk_0 \sum_{j=0}^{k-1} \varepsilon_j \rho_{j+1}^{-1}, & \text{if} \quad i=j \end{cases} \tag{2.9}$$

and the constant c_1 that appears on the right-hand side of (2.9) is the same as that appearing in the bound on $\chi^0(I;i,i)$ in Sect. 1. We also need

$$\sup\left|\frac{\partial^2 f^k}{\partial x_i \partial x_j}(x)\right| \leqq \rho_0^n e^{-(1-\eta_k)m|i-j|}, \tag{2.10}$$

with x defined as in (1.14) and ρ_0^n chosen to insure that the dimensions of the two sides of (2.10) are the same. Also, in both (2.9) and (2.10) the suprema run over $W(\rho^k, \xi_k; \{I_k\})$. Given these assumptions we have

Proposition 2.1. *Let $H^k(I,\phi)$ be as above (with $k=0,1,\dots,k_0-1$). Then if*

$$\varepsilon_0 < c\rho_0(k_0)^{-\sigma} \quad \textit{and} \quad m > K, \tag{2.11}$$

with c, σ, K the same constants as in Theorem 1.1, *there exists a change of variables $C^k:(I',\phi') \to (I,\phi)$, analytic and invertible on $W(\rho^{k+1}, \xi_{k+1}; \{I_{k+1}\})$. Furthermore $C^k: W(\rho^{k+1}, \xi_{k+1}; \{I_{k+1}\}) \to W(\rho^k, \xi_k; \{I_k\})$. If $\Gamma^{k+1} = \mathrm{pr}_1 W(\rho^{k+1}, \xi_{k+1}; \{I_{k+1}\})$, C^k is canonical on $\Gamma^{k+1} \times T^N$. Define*

$$\begin{aligned} H^{k+1}(I',\phi') &= H^k \circ C^k(I',\phi') = h^{k+1}(I') + f^{k+1}(I',\phi') \\ &= h^{k+1}(I') + f^{k+1,r}(I',\phi') + f^{k+1,nr}(I',\phi'). \end{aligned} \tag{2.12}$$

where $f^{k+1,r}(I',\phi') = \sum_{v \in \tilde{R}^k \cup R_p^k \cup R_s^k} f_v^{k+1}(I') e^{iv\cdot\phi'}$. (The procedure for splitting $H^{k+1}(I',\phi')$ into its integrable (h^{k+1}) and nonintegrable parts is given in Sect. 3.) $H^{k+1}(I',\phi')$ obeys the bounds (2.5)–(2.10), with k replaced by $k+1$, and the suprema are now taken over $W(\rho^{k+1}, \xi_{k+1}; \{I_{k+1}\})$.

We note that I_k is defined inductively by $(I_{k+1}, \phi_{k+1}) = (C^k)^{-1}(I_k, \phi_k)$. (Of course we must check in the course of the proof that (I_k, ϕ_k) lies in the domain of $(C^k)^{-1}$.)

Given Proposition 2.1 we immediately obtain Theorem 1.1. Note that our original Hamiltonian $H^0(I,\phi)$ satisfies (2.5)–(2.10) if we take $\tilde{R}^0 = \emptyset$, and hence $f^{0,r}(I,\phi) = 0$. Now apply Proposition 2.1, until $k = k_0 - 1$. Note that if the constant K_1 in Theorem 1.1 is large enough (in particular $K_1 > 3(\delta+2)$) $\xi_{k_0} \geqq 1$. Define $C = C^0 \circ C^1 \circ \dots \circ C^{k_0-1}$, $\tilde{I} = I_{k_0}$, and $\tilde{\rho} = \rho_{k_0}$. Then C is defined on $W(\tilde{\rho}, 1; \{\tilde{I}\})$ and maps this set into $W(\rho, \xi_0; \{I_0\})$ so

$$\tilde{H}(I',\phi') = H^0 \circ C(I',\phi') = H^{k_0}(I',\phi') = \tilde{h}(I') + \tilde{f}^{\text{resonant}}(I',\phi') + \tilde{f}^{\text{nonresonant}}(I',\phi'), \tag{2.13}$$

with $\tilde{f}^{\text{resonant}}(I',\phi') = f^{k_0,r}(I',\phi')$ and $\tilde{f}^{\text{nonresonant}}(I',\phi') = \tilde{f}^{k_0,nr}(I',\phi')$. The stated bounds on $\tilde{H}(I',\phi')$ then follow from (2.5)–(2.10).

Note that we obtain somewhat more information from Proposition 2.1 than was stated in Theorem 1.1 For instance from (2.6) and (2.7) we see that if δ and m are large enough, $\varepsilon_n \rho_n^{-1} \leqq (\varepsilon_0 \rho_0^{-1})^{(3/4)(3/2)^n}$ (δ and m do *not* need to depend on n for this to be the case,) and we have

Corollary 2.2. *If $\tilde{H}(I,\phi)$ is the Hamiltonian constructed in Theorem* 1.1, *then*

$$\sup \left\{ \left| \frac{\partial \tilde{f}^{\text{resonant}}}{\partial I_j} \right| + \rho_0^{-1} \left| \frac{\partial \tilde{f}^{\text{resonant}}}{\partial \phi_j} \right| \right\} \leqq \begin{cases} \rho_0(\varepsilon_0 \rho_0^{-1})^{(1/2)(3/2)^{k_0}} & \text{if } j \notin \bigcup_{n=0}^{k_0-1} S^n \\ \rho_0(\varepsilon_0 \rho_0^{-1})^{(1/2)(3/2)^n} & \text{if } j \in S^n, n = 0, \ldots, k_0 - 1, \end{cases} \tag{2.14}$$

and the supremum runs over $W(\tilde{\rho}, 1; \{\tilde{I}\})$.

We remark that at many places in the proof we will state inequalities which are true, provided m and δ are sufficiently large (where "sufficiently large" does not depend on the number of degrees of freedom of the system) without explicitly stating this assumption. Note also, that because of the definition of m, k_0, and β_k, $m \cdot \beta_k \geqq c_2 K_1$, for all $k = 0, \ldots, k_0$, where K_1 is some universal constant which we may choose as large as we need, which means that $m \cdot \beta_k$ can be chosen to be some large constant.

We note that one could replace the factors of (1/2) in the exponents in (2.14) with any number less than one, by making only minor changes in the proof.

3. The Canonical Transformation

In this section we construct the canonical transformation, C^k, whose existence is asserted in Proposition 2.1. Let $H^k(I,\phi) = h^k(I) + f^k(I,\phi)$ be a Hamiltonian satisfying the inductive hypotheses of the previous section. Define the generating function for the desired change of variables by

$$S(I',\phi) = \sum_{\substack{v \in X_k \\ v \notin \tilde{R}^k \cup R_p^k}} \frac{f_v^k(I') e^{iv \cdot \phi}}{i \langle \omega^k(I'), v \rangle}, \tag{3.1}$$

with $f_v^k(I')$ the Fourier coefficients of $f^k(I',\phi')$.

This is the generating function that one is led to by classical perturbation theory if one only attempts to "kill" the non-resonant harmonics in the interactions. Since there are only a finite number of terms in the definition of S, it can fail to be well defined only if the denominator of one of the terms vanishes. That this does not occur is guaranteed by

Lemma 3.1. *Define the N-vector* r_k^n *by* $(r_k^n)_i = 2^{8-n}\rho_{k+1}$ *if* $i \notin \bigcup_{m=0}^{k-2} S^m$, $(r_k^n)_i = (\rho^k)_i - n(b^m)_i$ *if* $i \in S^m$, $m = 0, \dots, k-2$. *Then on* $W(r_k^1, \xi_k; \{I_k\})$,

$$\sup[|\langle \omega^k(I'), v\rangle|^{-1}] \leqq 2B_1(\rho_0\lambda(\varepsilon_0))^{-1}\exp[(3/2)|v| + L_k], \tag{3.2}$$

for all $v \in \mathbb{X}_k$, *but* $v \notin \tilde{R}^k \cup R_p^k$. *The constant* B_1 *in this inequality is the same as that in* (2.1).

Remark. We could actually define the generating function S on a larger domain than $W(r_k^1, \xi_k; \{I_k\})$. If R_p^k is defined by (2.1), let V_k be the largest connected set in $\mathbb{R}^N$ containing I_k, such that if $v \in \mathbb{X}_k \backslash (\tilde{R}^k \cup R_p^k)$, and $I \in V_k$, $|\langle \omega^k(I), v\rangle|^{-1} \leqq B_1(\rho_0\lambda(\varepsilon_0))\exp[(3/2)|v| + L_k]$. (Roughly speaking V_k is the set of I's with the same resonant vectors as I_k). Then (3.2) holds on the larger domain $W(r_k^1, \xi_k; V_k)$, with an attendant increase in the size of the domains on which the canonical transformations in Proposition 2.1 and Theorem 1.1 are defined. At present, however, I have found no use for this larger domain.

Proof. Since $v \in \mathbb{X}_k$, $v \notin \tilde{R}^k \cup R_p^k$,

$$|\langle \omega^k(I_k), v\rangle^{-1}| \leqq B_1(\rho_0\lambda(\varepsilon_0))^{-1}\exp[(3/2)|v| + L_k]. \tag{3.3}$$

Furthermore there is a path γ, consisting of N components, γ_j, along which only one coordinate of I varies, joining I_k to I' for every I' such that (I', ϕ) is in $W(r_k^1, \xi_k; \{I_k\})$ for some ϕ. Also, the length of γ_j is bounded by $(r_k^1)_j$. By the fundamental theorem of calculus,

$$\langle \omega^k(I'), v\rangle^{-1} = \langle \omega^k(I_k), v\rangle^{-1}\left\{1 + \langle \omega^k(I_k), v\rangle^{-1} \times \int_\gamma dI'' \left\langle \frac{\partial \omega^k}{\partial I}(I''), v\right\rangle\right\}^{-1}. \tag{3.4}$$

Since $v \notin \tilde{R}^k \cup R_p^k$, $(r_k^1)_j = 2^7\rho_{k+1}$ for all j such that $\operatorname{dist}(j, \operatorname{supp} v) \leqq L_{k-1}$. (If $k = 0$ or 1, $(r_k^1)_j = 2^7\rho_{k+1}$ for *all* sites j.) This follows since if $j \in \bigcup_{m=0}^{k-2} S^m$, the definition of the secondary resonances would force v to be an element of R_s^{k-1} if $\operatorname{dist}(j, \operatorname{supp} v) \leqq L_{k-1}$. Write

$$\int_\gamma dI'' \left\langle \frac{\partial \omega^k}{\partial I}(I''), v\right\rangle = \sum_{i,j} \int_{\gamma_j} \frac{\partial^2 h^k}{\partial I_i \partial I_j}(I'') v_i dI''_j. \tag{3.5}$$

For each $i \in \operatorname{supp} v$, bound the sum over j by breaking it into two parts. For those j's with $|j - i| \leqq L_{k-1}$, bound the length of γ_j by $2^7\rho_{k+1}$, while for j's with $|j - i| > L_{k-1}$ we bound the length of γ_j by ρ_0. In all cases, the integrand is bounded by (2.8) and (2.9). Summing the resulting geometric series in j we find (3.5) is bounded in magnitude by

$$2^8|v|\rho_{k+1} + 2^2|v|\rho_0 e^{-(1-\eta_k)mL_{k-1}}, \tag{3.6}$$

(provided m is sufficiently large). Bounding the factor of $\langle \omega^k(I_k), v\rangle^{-1}$ by (3.3), and using the fact that $|v| \leqq M_k$ since $v \in \mathbb{X}_k$, we see that the quantity in $\{\cdots\}$ in (3.4) can be bounded below by (1/2). Then Lemma 3.1 follows. (If $k = 0$ or 1, the second term in (3.6) may be omitted and Lemma 3.1 still follows.)

Lemma 3.1 allows us to bound the derivatives of $S(I', \phi)$. We obtain the following results which we prove in Sect. 7. To simplify our notation let y_i be either I'_i or ϕ_i depending on the context. Then regard $S(I', \phi) = S(y)$.

Proposition 3.2. *On* $W(r_k^1, \xi^k - \delta; \{I_k\})$,

$$\sup\left|\frac{\partial S}{\partial y_i}(y)\right| \leqq \rho_0^n(\lambda(\varepsilon_0))^{-2}\varepsilon_k B^{L_k},$$

and on $W(r_k^2, \xi^k - \delta - 1; \{I_k\})$,

$$\sup\left|\frac{\partial^2 S}{\partial y_i \partial y_j}(y)\right| \leqq \rho_0^n e^{-m(1-\eta_k)(1-\beta_k)|i-j|}, \tag{3.7}$$

for some constant B. As usual the factors of ρ_0^n *are chosen to keep the dimensions of the two sides equal.*

Now define the change of variables by

$$I_i = I'_i + \frac{\partial S}{\partial \phi_i}(I', \phi), \quad \phi'_i = \phi_i + \frac{\partial S}{\partial I'_i}(I', \phi). \tag{3.8}$$

By the implicit function theorem of Appendix A, the first of the pair of Eqs. (3.8) may be inverted in the form:

$$I' = I + \Xi'(I, \phi), \tag{3.9}$$

for $\Xi'(I, \phi)$ analytic on $W(r_k^3, \xi^k - \delta - 1; \{I_k\})$, provided

$$\sup \sum_{i=1}^{N} \left|\frac{\partial^2 S}{\partial I'_j \partial \phi_i}(I', \phi)\right| \leqq (1/2),$$

and (3.10)

$$\sup\left|\frac{\partial S}{\partial \phi_j}\right| < c_1 \rho_{k+1}/k_0,$$

for all j, where the supremum runs over $W(r_k^2, \xi_k - \delta - 1; \{I_k\})$. Similarly the second equation in (3.8) is inverted in the form:

$$\phi = \phi' + \Delta(I', \phi'), \tag{3.11}$$

with $\Delta(I', \phi')$ analytic on $W(r_k^2, \xi_k - \delta - 2; \{I_k\})$, provided

$$\sup \sum_{i=1}^{N} \left|\frac{\partial^2 S}{\partial I'_i \partial \phi_j}(I', \phi)\right| \leqq (1/2)$$

and (3.12)

$$\sup\left|\frac{\partial S}{\partial I'_j}(I', \phi)\right| \leqq (1/2),$$

on $W(r_k^2, \xi_k - \delta - 1; \{I_k\})$ for all j. Both (3.10) and (3.12) follow from (3.7). Note that $\Xi'(I, \phi) = -(\partial S/\partial \phi)\ (I', \phi)$ and $\Delta(I', \phi') = -(\partial S/\partial I')\ (I', \phi)$, so bounds on derivatives of S lead to bounds on Ξ' and Δ. We also define

$$\phi' = \phi + \Delta'(I, \phi), \quad I = I' + \Xi(I', \phi'), \tag{3.13}$$

where $\Delta'(I,\phi) = (\partial S/\partial\phi)(I + \Xi'(I,\phi),\phi)$ is defined and analytic on $W(r_k^3, \xi_k - \delta - 1; \{I_k\})$ and $\Xi(I',\phi') = (\partial S/\partial\phi)(I', \phi' + \Delta(I',\phi'))$ is defined and analytic on $W(r_k^2, \xi_k - \delta - 2; \{I_k\})$. Define the transformations

$$C^k: (I',\phi') \to \begin{cases} I = I' + \Xi(I',\phi') \\ \phi = \phi' + \Delta(I',\phi'), \end{cases}$$

and (3.14)

$$\tilde{C}^k: (I,\phi) \to \begin{cases} I' = I + \Xi'(I,\phi) \\ \phi' = \phi + \Delta'(I,\phi). \end{cases}$$

Both are defined on $W(r_k^3, \xi_k - \delta - 2; \{I_k\})$ and map this set into $W(r_k^2, \xi_k - \delta - 1; \{I_k\})$, and on their common domain of definition $C^k \circ \tilde{C}^k = \tilde{C}^k \circ C^k =$ identity. Also, by construction, C^k and $\tilde{C}^k$ are canonical on $\mathrm{pr}_1(W(r_k^3, \xi_k - \delta - 2; \{I_k\})) \times T^N$. Define $(I_{k+1}, \phi_{k+1}) = \tilde{C}^k(I_k, \phi_k)$. Since $\Xi'(I,\phi) = -(\partial S/\partial\phi)(I'(I,\phi),\phi)$, the bounds of Proposition 3.2 imply

$$|(I_{k+1})_j - (I_k)_j| \leqq \rho_0(\varepsilon_k \rho_k^{-1})^{7/8} \quad \text{for} \quad j = 1, \dots, N. \tag{3.15}$$

Thus, $W(r_k^4, \xi_k - \delta - 2; \{I_{k+1}\}) \subset W(r_k^3, \xi_k - \delta - 2; \{I_k\})$, so C^k and $\tilde{C}^k$ map $W(r_k^4, \xi_k - \delta - 2; \{I_{k+1}\}) \to W(r_k^2, \xi_k - \delta - 1; \{I_k\}) \subset W(\rho^k, \xi_k, \{I_k\})$. Define

$$\begin{aligned} H^{k+1}(I',\phi') = H^k \circ C^k(I',\phi') &= h^k(I' + \Xi(I',\phi')) \\ &+ f^{k,\mathrm{r}}(I' + \Xi(I',\phi'), \phi' + \Delta(I',\phi')) \\ &+ f^{k,\mathrm{nr}}(I' + \Xi(I',\phi'), \phi' + \Delta(I',\phi')). \end{aligned} \tag{3.16}$$

Define $f^{k[\leqq]}(I,\phi) = \sum_{\substack{\nu \in \mathbb{X}_k \cup \{0\} \\ \nu \notin R^k \cup R_p^k}} f_\nu^k(I) e^{i\nu\cdot\phi}$ and define $f^{k[\geqq]}(I,\phi) = \sum_{\substack{\nu \notin \mathbb{X}_k \\ \nu \notin \tilde{R}^k \cup R_p^k \\ \nu \neq 0}} f_\nu^k(I') e^{i\nu\cdot\phi}$. Then applying the fundamental theorem of calculus in a manner similar to [3, 4, 6] we find

$$\begin{aligned} H^{k+1}(I',\phi') = h^k(I') + f_0^k(I') &+ f^{k,\mathrm{r}}(I' + \Xi(I',\phi'), \phi' + \Delta(I',\phi')) \\ &+ f^{\mathrm{I}}(I',\phi') + f^{\mathrm{II}}(I',\phi') + f^{\mathrm{III}}(I',\phi') + f^{\mathrm{IV}}(I',\phi'), \end{aligned} \tag{3.17}$$

with

$$f^{\mathrm{I}}(I',\phi') = \int_0^1 dt \int_0^t ds \frac{\partial^2 h^k}{\partial I \partial I}(I' + s\Xi(I',\phi')) \cdot \Xi(I',\phi') \cdot \Xi(I',\phi'),$$

$$f^{\mathrm{II}}(I',\phi') = \int_0^1 ds \frac{\partial f^{k[\leqq]}}{\partial I}(I' + s\Xi(I',\phi'), \phi' + \Delta(I',\phi')) \cdot \Xi(I',\phi'),$$

$$f^{\mathrm{III}}(I',\phi') = f^{k[\geqq]}(I' + \Xi(I',\phi'), \phi' + \Delta(I',\phi')) \text{ and}$$

$$f^{\mathrm{IV}}(I',\phi') = \sum_{\nu \in R_p^k} f_\nu^k(I' + \Xi(I',\phi')) e^{i\nu\cdot\phi(I',\phi')}.$$

In deriving (3.17) we used the fact that

$$\frac{\partial h^k}{\partial I}(I') \cdot \Xi(I',\phi') + \sum_{\substack{\nu \in \mathbb{X}_k \\ \nu \notin \tilde{R}^k \cup R_p^k}} f_\nu^k(I') e^{i\nu\cdot\phi(I',\phi')} = 0, \tag{3.18}$$

because of our definition of S. Setting $h^{k+1}(I') = h^k(I') + f_0^k(I')$ we see that

$$\frac{\partial^2 h^{k+1}}{\partial I'_i \partial I'_j}(I') = \delta_{ij} + \chi^k(I'; i,j) + \frac{\partial^2 f_0^k}{\partial I'_i \partial I'_j}(I') \equiv \delta_{ij} + \chi^{k+1}(I'; i,j). \tag{3.19}$$

The bound (2.5) and a dimensional estimate implies

$$\left| \frac{\partial^2 f_0^k}{\partial I'_i \partial I'_j}(I') \right| \leqq Bk_0 \varepsilon_k \rho_{k+1}^{-1} \quad \text{on} \quad W(r_k^1, \xi_k; \{I_k\}),$$

while (2.10) implies

$$\left| \frac{\partial^2 f_0^k}{\partial I'_i \partial I'_j}(I') \right| \leqq e^{-(1-\eta_k)m|i-j|} \quad \text{for} \quad i \neq j,$$

on the same domain. Combining these estimates with the bound on $\chi^k(I'; i,j)$ in (2.9) we see that

$$\sup |\chi^{k+1}(I'; i,j)| \leqq \begin{cases} c_1 + Bk_0 \sum_{j=0}^{k-1} \varepsilon_j \rho_{j+1}^{-1} + Bk_0 \varepsilon_k \rho_{k+1}^{-1} & \text{if} \quad i = j \\ 2e^{-(1-\eta_k)m|i-j|} \leqq e^{-(1-\eta_{k-1})m|i-j|} & \text{if} \quad i \neq j. \end{cases} \tag{3.20}$$

Thus, (2.9) can be iterated.

In the next section we begin the task of verifying that the bounds on the interaction terms can also be iterated.

4. Some Preliminary Decay Estimates

We begin iterating the estimates (2.5)–(2.7) and (2.10), which control the interaction terms in H^{k+1}. In this section we prove a series of estimates on various components of f^{k+1}. The first lemma is an application of the chain rule.

Lemma 4.1. *Suppose g is analytic on some domain $\mathscr{D} \subset \mathbb{C}^{2N}$. We let x_j represent either the j^{th} or the $(j+N)^{th}$ coordinate of a point in $\mathbb{C}^{2N}$. (This is in keeping with our notation in (1.14).) Suppose that*

$$\sup_{\mathscr{D}} \left| \frac{\partial g}{\partial x_n}(x) \right| \leqq C_n, \quad \sup_{\mathscr{D}} \left| \frac{\partial^2 g}{\partial x_n \partial x_p}(x) \right| \leqq C_{np}^g e^{-\kappa|n-p|} \tag{4.1}$$

for some non-negative constants, C_n C_{np}^g, and $\kappa(n,p = 1, \ldots, N)$. Suppose $\tilde{x}$ is a holomorphic map from $\mathscr{D}' \to \mathscr{D}$ satisfying

$$\begin{aligned} \sup_{\mathscr{D}'} \left| \frac{\partial \tilde{x}_n}{\partial x'_p}(x') \right| &\leqq C_{np}^1 e^{-\kappa|n-p|}, \\ \sup_{\mathscr{D}'} \left| \frac{\partial^2 \tilde{x}_m}{\partial x'_n \partial x'_p}(x') \right| &\leqq C_{mnp}^2 e^{-\kappa|n-p|}, \end{aligned} \tag{4.2}$$

for some constants C_{np}^1 and C_{mnp}^2. Here, in analogy with our notation above, we let $\tilde{x}_j$ denote either the j^{th} or the $(j+N)^{th}$ component of the map. (See Lemma 4.2 for an *explicit example of this notation.) Then*

$$\sup_{\mathscr{D}'}\left|\frac{\partial}{\partial x_i'}(g\circ\tilde{x}(x'))\right|\leqq D_i, \quad \text{and} \quad \sup_{\mathscr{D}'}\left|\frac{\partial^2}{\partial x_i'\partial x_j'}(g\circ\tilde{x}(x'))\right|\leqq De^{-\kappa|i-j|}. \tag{4.3}$$

Here $D_i=2\sum_{m=1}^{N} C_m C_{mi}^1$ and $D=4\sup_{ij}\left(\sum_{n=1}^{N} C_n C_{nij}^2+\sum_{m,n=1}^{N} C_{mn}^g C_{mi}^1 C_{nj}^1\right)$. The easy proof is omitted. Note that (4.3) can always be satisfied by picking D_i and D sufficiently large. The point of the lemma is that D_i and D may be chosen to have the stated form.

The next lemma tells how the changes of variables $I_i=I_i(I',\phi')$ and $\phi_i=\phi_i(I',\phi')$ depend on the variables I_j' and ϕ_i' as $|i-j|$ becomes large. To state the result concisely, let x_i' be either I_i' or ϕ_i', and let $\tilde{x}_i(x')$ be either $I_i(I',\phi')$ or $\phi_i(I',\phi')$ depending on the context.

Lemma 4.2. *Let* $\mathscr{D}'=W(r_k^4,\xi_k-\delta-3;\{I_k\})$. *Then*

$$\sup_{\mathscr{D}'}\left|\frac{\partial\tilde{x}_i}{\partial x_j'}(x')-\tilde{\delta}_{ij}\right| \leqq \rho_0^n\min\left((\lambda(\varepsilon_0))^{-2}k_0\varepsilon_k\rho_{k+1}^{-1}B^{L_k}, e^{-m(1-\eta_k)(1-3\beta_k)|i-j|}\right). \tag{4.4}$$

Also

$$\sup_{\mathscr{D}'}\left|\frac{\partial^2\tilde{x}_\ell}{\partial x_i'\partial x_j'}(x')\right|\leqq B\rho_0^n[(r_k^2)_\ell-(r_k^3)_\ell]^{-1}\times e^{-m(1-\eta_k)(1-4\beta_k)|i-j|}, \tag{4.5}$$

Here, $\tilde{\delta}_{ij}=1$ *if* $\tilde{x}_i=I_i(I',\phi')$ *and* $x_i'=I_i'$ *or* $\tilde{x}_i=\phi_i(I',\phi')$ *and* $x_i'=\phi_i'$, *and* $\tilde{\delta}_{ij}=0$ *otherwise.* This slightly awkward definition is necessitated by the fact that if $\tilde{x}_i=I_i(I',\phi')$ and $x_i'=\phi_i'$ (or if $\tilde{x}_i=\phi_i(I',\phi')$ and $x_i'=I_i'$), then $\left|\frac{\partial\tilde{x}_i}{\partial x_i'}\right|\ll 1$—not $\mathcal{O}(1)$. However, this notation has proved so convenient otherwise that I think it is worth putting up with this problem. The constant n is chosen as usual to insure that the dimensions of the two sides are equal.

Remark. Note that $(r_k^2)_\ell-(r_k^3)_\ell\sim\mathcal{O}((b^m)_\ell)$ if $\ell\in S^m$ for some $m=0,\dots,k-2$, and $(r_k^2)_\ell-(r_k^3)_\ell\sim\mathcal{O}(\rho_{k+1})$ otherwise, so the "size" of these derivatives depends on the order of the site ℓ.

At numerous points in what follows we will have occasion to bound sums over $v\in\mathbb{Z}^N$, in which the summands obey certain estimates that we wish to show are inherited by the sum. Estimates on the sums are provided by

Proposition 4.3. *Suppose* $g(x)=\sum_{v\in\mathbb{Z}^N} g_v(x)$ *and*

$$\sup_{\mathscr{D}}\left|\frac{\partial g_v}{\partial x_i}(x)\right|\leqq\min\begin{cases}K_1 e(i)e^{-\delta'|v|}\\ K_1 e^{-\kappa_1|i-\ell|}e^{-\delta'|v|},\end{cases} \tag{4.6}$$

for some domain $\mathscr{D}$, *some constant* $e(i)$ *and* ℓ *the point in* supp v *such* $|\ell-i|$ *is* *maximized. Then if δ' and κ_1 are greater than* 6,

$$\sup_{\mathscr{D}} \left| \frac{\partial g}{\partial x_i}(x) \right| \leqq K_1 e(i) B^L, \tag{4.7}$$

where L is any number greater than $(2/\kappa_1)|\ln e(i)|$, *and B is some universal constant.*

Suppose further that

$$\sup_{\mathscr{D}} \left| \frac{\partial^2 g_\nu}{\partial x_i \partial x_j}(x) \right| \leqq \begin{cases} K_2 e^{-\kappa_2 |i-j|} e^{-\delta' |\nu|} \\ K_3 e^{-\kappa_2 |i-\ell|} e^{-\delta' |\nu|} \end{cases} \tag{4.8}$$

and $\kappa_2 > 6$. *Then*

$$\sup_{\mathscr{D}} \left| \frac{\partial^2 g}{\partial x_i' \partial x_j'}(x) \right| \leqq (K_2 + K_3) e^{-\kappa_2 |i-j|} B^{(|i-j|+1)} \tag{4.9}$$

for some constant B.

The proofs of Lemma 4.2 and Proposition 4.3 are presented in Sect. 7. With these two results we are ready to study H^{k+1}, beginning with these corollaries of Proposition 4.3.

Corollary 4.4. *Let* $\mathscr{D} = W(r_k^1, \xi_k - \delta; \{I_k\})$. *Then*

$$\sup_{\mathscr{D}} \left\{ \left| \frac{\partial f^{k[\geqq]}}{\partial I_i} \right| + \rho_{k+1}^{-1} \left| \frac{\partial f^{k[\geqq]}}{\partial \phi_i} \right| \right\} \leqq \rho_0^2 k_0 \rho_k^{-1} B^{L_{k+1}} (\varepsilon_0 \rho_0^{-1})^{(13/8)(3/2)^k},$$

for $i = 1, \ldots, N$. *Let* x_i *be as in (1.14), then*

$$\left| \frac{\partial^2 f^{k[\geqq]}}{\partial x_i \partial x_j}(x) \right| \leqq B \rho_0^n e^{-m(1-\eta_k)(1-3\beta_k)(|i-j|)}, \tag{4.11}$$

on $W(r_k^2, \xi_k - \delta - 1; \{I_k\})$, *where as usual the factors of* ρ_0 *are chosen to keep the dimensions of the two sides the same.*

Proof.

$$\frac{\partial f^{k[\geqq]}}{\partial x_i}(x) = \sum_{\substack{\nu \notin \mathbb{X}_k \\ \nu \neq 0}} \frac{\partial}{\partial x_i}(g_\nu(x)), \tag{4.12}$$

where $g_\nu(x) = f_\nu^k(I) e^{i\nu\cdot\phi}$. Using (2.5), (2.10), the inductive hypotheses, and Cauchy's theorem bound $|(\partial/\partial I_i)(f_\nu^k(I) e^{i\nu\cdot\phi})|$ by $\min(B k_0 \rho_{k+1}^{-1} \rho_0^2 e^{-m(1-\eta_k) d(\operatorname{supp}\nu)} e^{-\delta|\nu|}, \rho_0 e^{-m(1-\eta_k)|i-\ell|} e^{-\delta|\nu|}, \varepsilon_k e^{-\delta|\nu|})$ on $\mathscr{D}$, which is in turn bounded by $\min(\rho_0 (\varepsilon_0 \rho_0^{-1})^{(13/8)(3/2)^k} e^{-\delta|\nu|/4}, \rho_0 e^{-m(1-\eta_k)|i-\ell|} e^{-\delta|\nu|})$, since $\nu \notin \mathbb{X}_k$. (Recall that ℓ is the point in $\operatorname{supp}\nu$ furthest from i). In deriving this estimate we have used a little trick that we will often call on below, and so we mention it here. If j and ℓ are the two most widely separated sites in $\operatorname{supp}\nu$, then by integrating by parts,

$$f_\nu^k(I) = \int d\phi \left(-\frac{1}{\nu_j \nu_\ell} \right) \left[\frac{\partial^2}{\partial \phi_j \partial \phi_\ell} f^k(I, \phi) \right] e^{i\nu\cdot\phi},$$

where $\int d\phi \equiv \left(\prod_{i=1}^{N} \int_0^{2\pi} (d\phi_i / 2\pi) \right)$. Thus $\partial f_\nu^k / \partial I_i$ is the ν^{th} harmonic of the function

$-(1/v_jv_\ell)\,[(\partial^3/\partial I_i\partial\phi_j\partial\phi_\ell)\,f^k(I,\phi)]$, which with the aid of (2.10) and a dimensional estimate we bound in magnitude by $B\rho_0^n k_0 \rho_{k+1}^{-1} e^{-(1-\eta_k)m|\ell-j|}$. But $|\ell-j|=d(\operatorname{supp} v)$, and the first estimate above follows from Cauchy's theorem. Similarly $(\partial/\partial\phi_i)(g_v(x))=f_v(I)(iv_i)e^{iv\cdot\phi}=0$ unless $i\in\operatorname{supp} v$. Using this observation, plus (2.5), (2.10) and Cauchy's theorem, we find $|(\partial/\partial\phi_i)g_v(x)|$ is bounded by

$$\begin{aligned}&\min(\rho_0\varepsilon_k|v|e^{-\delta|v|},\rho_0^2 e^{-m(1-\eta_k)|i-\ell|}|v|e^{-\delta|v|},B\rho_0^2 e^{-m(1-\eta_k)d(\operatorname{supp} v)}|v|e^{-\delta|v|})\\ &\leqq \min(\rho_0^2(\varepsilon_0\rho_0^{-1})^{(13/8)(3/2)^k}e^{-\delta|v|/4},\rho_0^2 e^{-m(1-\eta_k)|i-\ell|}e^{-\delta|v|/2}).\end{aligned}$$

Applying (4.7) yields (4.10). Next note that (2.5), (2.10) and Cauchy's theorem imply

$$\begin{aligned}\left|\frac{\partial^2 g_v}{\partial x_i\partial x_j}(x)\right| &\leqq \rho_0^n \min\begin{cases}\varepsilon_k[(\rho^k)_j-(r_k^1)_j]^{-1}e^{-\delta|v|}\\ e^{-m(1-\eta_k)|i-j|}e^{-\delta|v|}\\ \rho_0[(\rho^k)_j-(r_k^1)_j]^{-1}e^{-m(1-\eta_k)|i-\ell|}e^{-\delta|v|}\end{cases} \qquad (4.13)\\ &\leqq \rho_0^n \min\begin{cases}e^{-m(1-\eta_k)|i-j|}e^{-\delta|v|}\\ e^{-m(1-\eta_k)(1-2\beta_k)|i-j|}e^{-m\beta_k(1-\eta_k)|i-\ell|}e^{-\delta|v|},\end{cases}\end{aligned}$$

where as usual n is chosen to keep the dimensions correct. In addition the last inequality used the fact that if $\chi<\min(c_1,c_2)$, one has $\chi<c_1^\beta c_2^{(1-\beta)}$ for $\beta\in[0,1]$. Inequality (4.11) then follows from the second half of Proposition 4.3. (Note that we can assume $|i-j|>(1/8)L_k$, since otherwise (4.11) follows from (4.10) by a dimensional estimate.)

In like fashion, if we let $f^4(I,\phi)=\sum_{v\in R_k^p} f_v(I)\,e^{iv\cdot\phi}$, we have

Corollary 4.5. *If* $\mathscr{D}=W(r_k^1,\xi_k-\delta;\{I_k\})$, *then*

$$\sup_{\mathscr{D}}\left\{\left|\frac{\partial f^{k[\leqq]}}{\partial I_i}\right|+\rho_0^{-1}\left|\frac{\partial f^{k[\leqq]}}{\partial\phi_i}\right|\right\}\leqq\varepsilon_k B^{L_k},$$

$$\sup_{\mathscr{D}}\left\{\left|\frac{\partial f^4}{\partial I_i}\right|+\rho_0^{-1}\left|\frac{\partial f^4}{\partial\phi_i}\right|\right\}\leqq\varepsilon_k B^{L_k}, \qquad (4.14)$$

and

$$\sup_{\mathscr{D}}\left\{\left|\frac{\partial f^{k,r}}{\partial I_i}\right|+\rho_0^{-1}\left|\frac{\partial f^{k,r}}{\partial\phi_i}\right|\right\}\leqq A(k,i)B^{\tilde L(i)},$$

where $\tilde L(i)=L_m$ *if* $i\in S^m$, $m=0,\dots,k-1$, $\tilde L(i)=L_k$ *otherwise. Also, if* x *is as in (1.14), then on* $W(r_k^2,\xi_k-\delta-1;\{I_k\})$,

$$\sup\left|\frac{\partial^2 f^{k[\leqq]}}{\partial x_i\partial x_j}(x)\right|\leqq\rho_0^n e^{-m(1-\eta_k)(1-2\beta_k)|i-j|},$$

$$\sup\left|\frac{\partial^2 f^4}{\partial x_i\partial x_j}(x)\right|\leqq\rho_0^n e^{-m(1-\eta_k)(1-2\beta_k)|i-j|},$$

and (4.15)

$$\sup\left|\frac{\partial^2 f^{k,r}}{\partial x_i\partial x_j}(x)\right|\leqq\rho_0^n e^{-m(1-\eta_k)(1-2\beta_k)|i-j|}.$$

Once again the factors of ρ_0 are chosen to "fix" the dimensions.

Proof. By (2.5), (2.6), (2.10), and Cauchy's theorem we have

$$\left|\frac{\partial f_v^k}{\partial I_j} e^{iv\cdot\phi}\right| + \rho_0^{-1}\,|f_v^k(v_j)e^{iv\cdot\phi}| \leqq \min\left(A(k,j)|v|e^{-\delta|v|}, \rho_0 e^{-m(1-\eta_k)|j-\ell|}|v|e^{-\delta|v|}\right),$$

while

$$\left|\frac{\partial^2 f_v}{\partial x_i \partial x_j}(x)\right| \leqq \rho_0^n \min\left(e^{-m(1-\eta_k)|i-j|}e^{-\delta|v|}, A(k,i)\right.$$
$$\left.\cdot[(\rho^k)_j - (r_k^1)_j]^{-1}e^{-\delta|v|}, \rho_0[(\rho^k)_j-(r_k^1)_j]^{-1}e^{-|i-\ell|}e^{-\delta|v|}\right),$$

on $W(r_k^1, \xi_k - \delta; \{I_k\})$ with ℓ the point in supp v such that $|i-\ell|$ is maximized. Note that this last inequality implies

$$\left|\frac{\partial^2 f_v^k}{\partial x_i \partial x_j}(x)\right| \leqq \rho_0^n \min\left(e^{-m(1-\eta_k)|i-j|}e^{-\delta|v|},\right.$$
$$\left.\cdot e^{-m(1-\eta_k)(1-2\beta_k)|i-j|}e^{-m(1-\eta_k)\beta_k|i-\ell|}e^{-\delta|v|}\right).$$

(The last step used the fact, easily derived from the definitions of the m^{th} order zones, S^m, that if $i\in S^m$ and $L_{m+n} \leqq |i-j| < L_{m+n+1}$, then $|[(\rho^k)_j-(r_k^1)_j]^{-1}| \leqq Bk_0\rho_{m+n+2}^{-1}$.) Proposition 4.3 then yields (4.15) and the last of the three inequalities in (4.14). (Note that we may assume $|i-j| > (1/8)L_k$, in the first two inequalities in (4.15), and $|i-j| > (1/8)\max(\tilde{L}(i), \tilde{L}(j))$ in the last of these inequalities since otherwise they follow immediately from (4.14) by dimensional estimates.) The first two inequalities of (4.14) follow if we note that all the Fourier coefficients $f_v^k(I)$ in $f^{k[\leqq]}$ and f^4 satisfy $v\notin\tilde{R}$, so by (2.5) and Cauchy's theorem we have

$$\sup\left\{\left|\frac{\partial f_v^k}{\partial I_j}(I)e^{iv\cdot\phi}\right| + \rho_0^{-1}\left|f_v^k(I)(v_j)e^{iv\cdot\phi}\right|\right\} \leqq \varepsilon_k e^{-\delta|v|} \tag{4.16}$$

on $W(\rho^k, \xi_k - \delta; \{I_k\})$ for these harmonics. Combining this estimate with those above, and then applying Proposition 4.3 completes the proof of (4.14).

5. The Short Range Nature of f^{k+1}

We demonstrate in this section that the estimate (2.10) holds for the Hamiltonian H^{k+1}. Let $\tilde{x}(x')$ be as in Lemma 4.2. If $\mathscr{D}' = W(r_k^4, \xi_k - \delta - 3; \{I_k\})$ and $\mathscr{D} = W(r_k^3, \xi_k - \delta - 2; \{I_k\})$, then $\tilde{x}$: $\mathscr{D}' \to \mathscr{D}$. If f^{III} and f^{IV} are as in (3.17) we have $f^{\mathrm{III}}(I', \phi') = f^{k[\geqq]}\circ\tilde{x}(x')$ and $f^{\mathrm{IV}}(I',\phi') = f^4\circ\tilde{x}(x')$, while $f^{k,r}(I' + \Xi(I',\phi'), \phi' + \Delta(I',\phi')) = f^{k,r}\circ\tilde{x}(x')$. We then have

Proposition 5.1.

$$\sup_{\mathscr{D}'}\left|\frac{\partial f^{\mathrm{III}}}{\partial x_i'}(x')\right| \leqq \rho_0^n \rho_{k+1}^{-1}(\varepsilon_0\rho_0^{-1})^{(13/8)(3/8)^k}B^{L_k},$$

$$\sup_{\mathscr{D}'}\left|\frac{\partial f^{\mathrm{IV}}}{\partial x_i'}(x')\right| \leqq \rho_0^n \varepsilon_k B^{L_k}, \tag{5.1}$$

and

$$\sup_{\mathscr{D}'}\left|\frac{\partial f^{k,r}}{\partial x_i'}\circ\tilde{x}(x')\right| \leqq \rho_0^n A(k,i)B^{\bar{L}(i)}.$$

Furthermore on $W(r_k^5, \xi_k-\delta-4; \{I_k\})$

$$\left|\frac{\partial^2 f^{\mathrm{III}}}{\partial x_i'\partial x_j'}(x')\right| \leqq (\rho_0^n/2^3)e^{-m(1-\eta_{k+1})|i-j|},$$

$$\left|\frac{\partial^2 f^{\mathrm{III}}}{\partial x_i'\partial x_j'}(x')\right| \leqq (\rho_0^n/2^3)e^{-m(1-\eta_{k+1})|i-j|},$$

and

$$\left|\frac{\partial^2 f^{k,r}}{\partial x_i'\partial x_j'}\circ\tilde{x}(x')\right| \leqq (\rho_0^n/2^3)e^{-m(1-\eta_{k+1})|i-j|}. \tag{5.2}$$

Proof. Apply Lemma 4.1, taking the function $g(x)$ in that lemma to be $f^{k[\geqq]}, f^4$, and $f^{k,r}$ respectively in each of the inequalities in (5.1) and (5.2). Choose κ (in (4.1) and (4.2)) equal to zero for (5.1). From Lemma 4.2 take $C_{ij}^1=2$ if $i=j$, and

$$C_{ij}^1=\min(\rho_0^n e^{-m(1-\eta_k)(1-4\beta_k)|i-j|}, \rho_0^n(\lambda(\varepsilon_0))^{-2}k_0\varepsilon_k\rho_{k+1}^{-1}B^{L_k}) \text{ if } i\neq j.$$

From Corollaries 4.4 and 4.5 we see that the constants C_ℓ may be chosen to be $\rho_0^n B^{L_{k+1}}(\varepsilon_0\rho_0^{-1})^{(13/8)(3/2)^k}$, $\rho_0^n\varepsilon_k B^{L_k}$ and $\rho_0^n A(k,\ell)B^{\bar{L}(\ell)}$ respectively in the cases g equals $f^{k[\geqq]}$, f^4, and $f^{k,r}$. Inequalities (5.1) then follow from the first inequality in (4.3). In calculating the constant that appears on the right-hand side of the third inequality, we note that $k_0A(k,p)(\varepsilon_k\rho_{k+1}^{-1})B^{L_k}\leqq A(k,i)B^{\bar{L}(i)}$ for all sites p, while $k_0\varepsilon_k\rho_{k+1}^{-1}B^{L_k}<e^{-m(1-\eta_k)(1-4\beta_k)|i-p|/2}$ for at most BL_k sites p, so

$$\begin{aligned}\sum_p C_pC_{ip}^1 &\leqq (2+2^2)A(k,i)B^{\bar{L}(i)} + (\sup_p k_0A(k,p)(\varepsilon_k\rho_k^{-1})B^{L_k})BL_k\\ &\leqq A(k,i)B''^{\bar{L}(i)}.\end{aligned} \tag{5.3}$$

For (5.2) choose $\kappa=m(1-\eta_k)(1-5\beta_k)$. Then $C_{ij}^1=2$ if $i=j$ and $C_{ij}^1=\rho_0^n e^{-m(1-\eta_k)\beta_k|i-j|}$ if $i\neq j$ by Lemma 4.2.

$$C_{\ell ij}^2 = B\rho_0^n[(r_k^2)_\ell-(r_k^3)_\ell]^{-1}e^{-m(1-\eta_k)\beta_k|i-j|} \quad\text{if}\quad |i-\ell|<2|i-j|$$

and

$$C_{\ell ij}^2=\rho_0^n[(r_k^4)_j-(r_k^5)_j]^{-1}e^{-m(1-\eta_k)(1-3\beta_k)|i-\ell|/2}$$

otherwise. This estimate follows by noting that (4.4) and a dimensional estimate on $W(r_k^5, \xi_k-\delta-4; \{I_k\})$ imply

$$\begin{aligned}\left|\frac{\partial^2\tilde{x}_\ell}{\partial x_i'\partial x_j'}\right| &\leqq B\rho_0^n[(r_k^4)_j-(r_k^5)_j]^{-1}e^{-m(1-\eta_k)(1-3\beta_k)|i-\ell|}\\ &\leqq B\rho_0^n[(r_k^4)_j-(r_k^5)_j]^{-1}e^{-m(1-\eta_k)(1-3\beta_k)|i-\ell|/2}e^{-m(1-\eta_k)(1-3\beta_k)|i-j|},\end{aligned}$$

if $|i-\ell|>2|i-j|$.

Choose the constants C_n, as we did in the previous paragraph. Corollaries 4.4 and 4.5 also imply that the constants C_{mn}^g may be chosen to be $\rho_0^n e^{-m(1-\eta_k)|m-n|}$ in each of the three cases. To prove (5.2) first note that we may assume $|i-j| > (1/8)L_k$ in the first two inequalities in (5.2) while in the third inequality we can assume $|i-j| > (1/8)\max(\tilde{L}(i), \tilde{L}(j))$, since (5.2) would follow from (5.1) and a dimensional estimate were this not the case. Given this assumption, (5.2) follows from the second inequality in (4.3), since a straightforward computation of the constant D appearing there shows it may be bounded by $(\rho_0^n/2^3)$ in each case.

We now study the two remaining terms in (3.17), f^{I} and f^{II}. We first prove the following easy consequence of the product rule

Lemma 5.2. *Suppose f_ℓ and $g_\ell (\ell = 1, \dots, N)$ are analytic on some domain $\mathscr{D} \subset \mathbb{C}^{2N}$, and satisfy the bounds*

$$\sup|g_\ell| < C_{g\ell}, \sup|f_\ell| < C_{f\ell},$$

$$\sup\left|\frac{\partial g_\ell}{\partial x_i}\right| < \min(C_{g\ell}^1, C_{g\ell}^2 e^{-\kappa|i-\ell|}),$$

$$\sup\left|\frac{\partial f_\ell}{\partial x_i}\right| < \min(C_{f\ell}^1, C_{f\ell}^2 e^{-\kappa|i-\ell|}), \tag{5.5}$$

$$\sup\left|\frac{\partial^2 g_\ell}{\partial x_i \partial x_j}\right| < \min(C_{g\ell}^3 e^{-\kappa|i-j|}, C_{g\ell}^4 e^{-\kappa|i-\ell|}),$$

and

$$\sup\left|\frac{\partial^2 f_\ell}{\partial x_i \partial x_j}\right| < \min(C_{f\ell}^3 e^{-\kappa|i-j|}, C_{f\ell}^4 e^{-\kappa|i-\ell|}),$$

where we again let x_j represent either the j^{th} or $(j+N)^{th}$ component of $x \in \mathbb{C}^{2N}$. Then if $\kappa > 0$, and L is a nonnegative integer we have

$$\sup_{\mathscr{D}}\left|\frac{\partial}{\partial x_i}\left(\sum_{\ell=1}^N f_\ell(x) g_\ell(x)\right)\right| \leqq 2^2(1-e^{-\kappa})^{-1} \sup_\ell (C_{g\ell} C_{f\ell}^2 + C_{f\ell} C_{g\ell}^2) e^{-\kappa L}$$
$$+ 2^2 \sup_\ell (C_{g\ell} C_{f\ell}^1 + C_{g\ell}^1 C_{f\ell})(L). \tag{5.6}$$

Furthermore,

$$\sup_{\mathscr{D}}\left|\frac{\partial^2}{\partial x_i \partial x_j}\left(\sum_{\ell=1}^N f_\ell(x) g_\ell(x)\right)\right| \leqq [2(|i-j| + 2(1-e^{-\kappa})^{-1}) C_{f\ell}^2 C_{g\ell}^2$$
$$+ \sup_\ell (2^2|i-j| + 2^2)(C_{f\ell} C_{g\ell}^3 + C_{g\ell} C_{f\ell}^3)$$
$$+ 2^2 \sup_\ell (C_{f\ell} C_{g\ell}^4 + C_{g\ell} C_{f\ell}^4)](1-e^{-\kappa})^{-1} e^{-\kappa|i-j|}. \tag{5.7}$$

Proof. By the product rule,

$$\left|\frac{\partial}{\partial x_i}(\sum_\ell f_\ell(x)g_\ell(x))\right| \leqq \sum_\ell \left| g_\ell(x)\frac{\partial f_\ell}{\partial x_i}(x) + f_\ell(x)\frac{\partial g_\ell}{\partial x_i}(x)\right|$$
$$\leqq \sum_{\substack{\ell: \\ |i-\ell|\geqq L}} (C_{g\ell}C^2_{f\ell} + C_{f\ell}C^2_{g\ell})e^{-\kappa|i-\ell|}$$
$$+ \sum_{\substack{\ell: \\ |i-\ell|< L}} (C_{g\ell}C^1_{f\ell} + C_{f\ell}C^1_{g\ell}), \tag{5.8}$$

and (5.6) follows immediately.

Applying this product rule a second time we find

$$\left|\frac{\partial^2}{\partial x_i\partial x_j}(\sum_\ell f_\ell(x)g_\ell(x))\right| \leqq \sum_\ell \left| g_\ell(x)\frac{\partial^2 f_\ell}{\partial x_i\partial x_j}(x) + f_\ell(x)\frac{\partial^2 g_\ell}{\partial x_i\partial x_j}(x)\right.$$
$$\left. + \frac{\partial g_\ell}{\partial x_i}(x)\frac{\partial f_\ell}{\partial x_j}(x) + \frac{\partial f_\ell}{\partial x_i}(x)\frac{\partial g_\ell}{\partial x_j}(x)\right|. \tag{5.9}$$

Bound the last two terms by bounding the derivatives by $C^2_{g\ell}C^2_{f\ell}e^{-\kappa[|i-\ell|+|j-\ell|]}$ and then summing over ℓ to obtain the term $\sup_\ell(2(|i-j|)+2(1-e^{-\kappa})^{-1})C^2_{g\ell}C^2_{f\ell}e^{-\kappa|i-j|}$. Bound the second term by first summing over all ℓ such that $|i-\ell|\leqq 2|i-j|$. If we bound f_ℓ by $C_{f\ell}$ and bound $|\partial^2 g_\ell/\partial x_i\partial x_j|$ by $C^3_{g\ell}e^{-\kappa|i-j|}$, these terms give a contribution of $\sup 2^2(|i-j|+2)C_{f\ell}C^3_{g\ell}e^{-\kappa|i-j|}$, while the remaining terms are bounded by

$$\sum_{\substack{\ell: \\ |i-\ell|>2|i-j|}} C_{f\ell}C^4_{g\ell}e^{-\kappa|i-\ell|} \leqq \sup_\ell 2^2 C_{f\ell}C^4_{g\ell}(1-e^{-\kappa})^{-1}e^{-\kappa|i-j|}.$$

The first term in (5.9) is bounded exactly as the second, interchanging the roles of f_ℓ and g_ℓ, and (5.7) follows.

Lemma 5.3. *Let $\mathscr{D}' = W(r^5_k, \xi_k - \delta - 4; \{I_k\})$, and let x' be as in Lemma 4.2. Then*

$$\sup_{\mathscr{D}'}\left|\frac{\partial f^{\mathrm{II}}}{\partial x'_i}(x')\right| \leqq \rho^n_0 C_1 \varepsilon_{k+1},$$

and (5.10)

$$\sup\left|\frac{\partial^2 f^{\mathrm{II}}}{\partial x'_i\partial x'_j}(x')\right| \leqq (\rho^n_0/2^3)e^{-m(1-\eta_k)(1-6\beta_k)|i-j|},$$

on $W(r^6_k, \xi_k - \delta - 5; \{I_k\})$, where we can choose $C_1 \sim \mathcal{O}((\varepsilon_k\rho_k^{-1})^\alpha)$, for some $\alpha>0$.

Proof. Define $\tilde{x}^s_i(x')$ to be either $I'_i + s\Xi_i(x')$ or $\phi'_i + \Delta_i(x')$ depending on the context. Since

$$|s|\leqq 1, \left|\frac{\partial \tilde{x}^s_i}{\partial x'_j}(x') - \delta_{ij}\right| \leqq \left|\frac{\partial \tilde{x}_i}{\partial x'_j}(x') - \delta_{ij}\right|,$$

so that Lemma 4.2 bounds derivatives of $\tilde{x}^s$. Let $g_\ell(x) = \partial f^{k[\leqq]}/\partial I_\ell(x)$, $(\ell = 1,\dots,N)$.

Then Corollary 4.5 implies $|g_\ell(x)| \leqq \varepsilon_k B^{L_k}$ on $W(r_k^1, \xi_k - \delta; \{I_k\})$. On the domain $W(r_k^2, \xi_k - \delta - 1; \{I_k\})$, (4.14), (4.15) and a dimensional estimate imply

$$\left|\frac{\partial g_\ell}{\partial x_i}(x)\right| \leqq \min(\rho_0^n k_0 \rho_{k+1}^{-1} \varepsilon_k B^{L_k}, \rho_0^n e^{-m(1-\eta_k)(1-2\beta_k)|i-\ell|}),$$

while (4.15) and a second dimensional estimate bounds $|(\partial^2 g_\ell/\partial x_i \partial x_j)(x)|$ by

$$\min(\rho_0^n k_0^2 \rho_{k+1}^{-2} \varepsilon_k B^{L_k}, B\rho_0^n k_0 \rho_{k+1}^{-1} e^{-m(1-\eta_k)(1-2\beta_k)|i-\ell|}, B\rho_0^n k_0 \rho_{k+1}^{-1} e^{-m(1-\eta_k)(1-2\beta_k)|i-j|})$$

on $W(r_k^3, \xi_k - \delta - 2; \{I_k\})$. Combining these estimates with the bounds on derivatives of x^s that come from Lemma 4.2, and applying Lemma 4.1 yields

$$\sup\left|\frac{\partial}{\partial x_i'}\left(\frac{\partial f^{k[\leqq]}}{\partial I_\ell} \circ \tilde{x}^s(x')\right)\right| \leqq B\rho_0^n(|i-\ell|+1)e^{-m(1-\eta_k)(1-4\beta_k)|i-\ell|}$$
$$\leqq B\rho_0^n e^{-m(1-\eta_k)(1-5\beta_k)|i-\ell|}, \tag{5.11}$$

on $W(r_k^4, \xi_k - \delta - 3; \{I_k\})$. Note also that (4.14) and a dimensional estimate imply

$$\sup_{\mathscr{D}'}\left|\frac{\partial f^{k[\leqq]}}{\partial I_\ell} \circ \tilde{x}^s(x')\right| \leqq \varepsilon_k B^{L_k}$$

and (5.12)

$$\sup_{\mathscr{D}'}\left|\frac{\partial}{\partial x_i'}\left(\frac{\partial f^{k[\leqq]}}{\partial I_\ell} \circ \tilde{x}^s(x')\right)\right| \leqq \rho_0^n k_0 \rho_{k+1}^{-1} \varepsilon_k B^{L_k}.$$

The second half of (4.3), combined with Lemma 4.2 and the estimates above imply

$$\sup_{\mathscr{D}'}\left|\frac{\partial^2}{\partial x_i' \partial x_j'}\left(\frac{\partial f^{k[\leqq]}}{\partial I_\ell} \circ \tilde{x}^s(x')\right)\right| \leqq B\rho_0^n k_0 \rho_{k+1}^{-1} e^{-m(1-\eta_k)(1-5\beta_k)|i-j|}. \tag{5.13}$$

This follows by noting that the estimates above imply we can pick the constants κ, C_n, and C_{np}^g of (4.1) equal to $m(1-\eta_k)(1-5\beta_k)$, $\rho_0^n k_0 \rho_{k+1}^{-1} \varepsilon_k B^{L_k}$, and $2^3 \rho_0^n k_0 \rho_{k+1}^{-1} e^{-m(1-\eta_k)\beta_k|n-p|}$, respectively, on the domain $\mathscr{D} = W(r_k^4, \xi_k - \delta - 3; \{I_k\})$, while Lemma 4.2 implies we can pick the constants $C_{np}^1 = B\rho_0^n e^{-m\beta_k(1-\eta_k)|n-p|}$, and $C_{\ell ij}^2 = B\rho_0^n k_0 \rho_{k+1}^{-1} e^{-m\beta_k(1-\eta_k)|i-j|}$ if $|\ell - j| \leqq 2|i-j|$ and $C_{\ell ij}^2 = B\rho_0^n k_0 \rho_{k+1}^{-1} e^{-m(1-\eta_k)(1-4\beta_k)|\ell-j|/2}$ otherwise. (This last estimate follows from a dimensional estimate similar to (5.4).) Estimate (5.13) then follows by inserting these bounds in the definition of the constant D in (4.3).

Note also, that (5.11) and a dimensional estimate yields

$$\sup_{\mathscr{D}'}\left|\frac{\partial^2}{\partial x_i' \partial x_j'}\left(\frac{\partial f^{k[\leqq]}}{\partial I_\ell} \circ \tilde{x}^s(x')\right)\right| \leqq B\rho_0^n k_0 \rho_{k+1}^{-1} e^{-m(1-\eta_k)(1-5\beta_k)|i-\ell|}. \tag{5.14}$$

The factors of $\Xi_\ell(x')$, in f^{II} are bounded by $\rho_0(\varepsilon_k \rho_k^{-1})^{7/8}$ on $W(r_k^3, \xi_k - \delta - 2; \{I_k\})$, using (3.7) and the fact that $\Xi_\ell(I', \phi') = (\partial S/\partial \phi_\ell)(I', \phi(I', \phi'))$. Factors of $\partial \Xi_\ell/\partial x_i'$ and $\partial^2 \Xi_\ell/\partial x_i' \partial x_j'$ are bounded by Lemma 4.2 using the observation that

$$\frac{\partial \Xi_\ell}{\partial x_i'} = \frac{\partial \tilde{x}_\ell}{\partial x_i'} - \delta_{i\ell} \quad \text{and} \quad \frac{\partial^2 \Xi_\ell}{\partial x_i' \partial x_j'} = \frac{\partial^2 \tilde{x}_\ell}{\partial x_i' \partial x_j'}.$$

Define $\tilde{g}_\ell(x') = \partial f^{k[\leqq]}/\partial I_\ell \circ \tilde{x}^s(x')$ and $f_\ell(x') = \Xi_\ell(x')$. Then inequalities (5.11)–(5.14) and the remarks in the preceding paragraph are just what we need to apply Lemma 5.2 to bound derivatives of $f_s^2(x') \equiv \sum_{\ell=1}^{N} \tilde{g}_\ell(x') f_\ell(x')$. Take κ in that lemma equal to $m(1-\eta_k)(1-5\beta_k)$, set the constants $C_{\tilde{g}\ell} = \varepsilon_k B^{L_k}$, $C^1_{\tilde{g}\ell} = \rho_0^n k_0 \rho_{k+1}^{-1} \varepsilon_k B^{L_k}$, $C^2_{\tilde{g}\ell} = C^2_{f\ell} = B\rho_0^n$, $C^3_{\tilde{g}\ell} = C^4_{\tilde{g}\ell} = C^3_{f\ell} = C^4_{f\ell} = B\rho_0^n k_0 \rho_{k+1}^{-1}$, and $C_{f\ell} = \rho_0(\varepsilon_k \rho_k^{-1})^{7/8}$ and $C^1_{f\ell} = \rho_0^n k_0 \rho_{k+1}^{-1} \lambda(\varepsilon_0)^{-2} \varepsilon_k B^{L_k}$, and let the constant $L = L_k$ in (5.6). We find

$$\sup_{\mathscr{D}'} \left| \frac{\partial f_s^2}{\partial x_i'}(x') \right| \leqq \rho_0^n C_1 \varepsilon_{k+1}, \tag{5.15}$$

where C_1 may be chosen $\mathcal{O}((\varepsilon_k \rho_k^{-1})^\alpha)$ for α some small positive constant. On $W(r_k^6, \xi_k - \delta - 5; \{I_k\})$,

$$\sup \left| \frac{\partial^2 f_s^2}{\partial x_i' \partial x_j'}(x') \right| \leqq B\rho_0^n [|i-j| + {}^{\cdot}] e^{-\kappa|i-j|} \leqq (\rho_0^n/2^3) e^{-m(1-\eta_k)(1-6\beta_k)|i-j|}. \tag{5.16}$$

In the last step in (5.16) we used the fact that we can assume $|i-j| > (1/8)L_k$, since (5.16) follows from (5.15) and a dimensional estimate otherwise. But if we now note that

$$\left| \frac{\partial f^{\mathrm{II}}}{\partial x_i'}(x') \right| \leqq \sup_s \left| \frac{\partial}{\partial x_i'} f_s^2(x') \right| \quad \text{and} \quad \left| \frac{\partial^2 f^{\mathrm{II}}}{\partial x_i' \partial x_j'}(x') \right| \leqq \sup_s \left| \frac{\partial^2 f_s^2}{\partial x_i' \partial x_j'}(x') \right|,$$

we see that (5.10) follows from (5.15) and (5.16).

Finally we address derivatives of f^{I}.

Lemma 5.4. *Let $\tilde{x}^s(x')$ be as defined in the proof of Lemma 5.3. Define*

$$h_n(x') = \sum_{l=1}^{N} \frac{\partial^2 h^k}{\partial I_\ell \partial I_n} \circ \tilde{x}^s(x') \cdot \Xi_\ell(x').$$

Then on $W(r_k^6, \xi_k - \delta - 5; \{I_k\})$,

$$\sup |h_n(x')| \leqq B\rho_0 (\varepsilon_k \rho_k^{-1})^{7/8}, \tag{5.17}$$

$$\sup \left| \frac{\partial h_n}{\partial x_i'}(x') \right| \leqq \min \begin{cases} B\rho_0^n k_0 \rho_{k+1}^{-1} (\varepsilon_k \rho_k^{-1})^{7/8} \\ \rho_0^n e^{-m(1-\eta_k)(1-7\beta_k)|i-n|} \end{cases}, \tag{5.18}$$

and

$$\sup \left| \frac{\partial^2 h_n}{\partial x_i' \partial x_j'}(x') \right| \leqq \min \begin{cases} B\rho_0^n k_0 \rho_{k+1}^{-1} e^{-m(1-\eta_k)(1-7\beta_k)|i-n|} \\ B\rho_0^n k_0^2 \rho_{k+1}^{-2} e^{-m(1-\eta_k)(1-7\beta_k)|i-j|} \end{cases}. \tag{5.19}$$

Assuming Lemma 5.4 holds for the moment we prove

Lemma 5.5. *Let $\mathscr{D} = W(r_k^7, \xi_k - \delta - 6; \{I_k\})$. Then*

$$\sup_{\mathscr{D}} \left| \frac{\partial f^{\mathrm{I}}}{\partial x_i'}(x') \right| \leqq \rho_0^n C_1 \varepsilon_{k+1}$$

and

$$\sup_{\mathscr{D}}\left|\frac{\partial^2 f^1}{\partial x_i'\partial x_j'}(x')\right| \leqq (\rho_0^n/2^3)e^{-m(1-\eta_k)(1-8\beta_k)|i-j|}, \tag{5.20}$$

where we can choose $C_1 \sim \mathcal{O}((\varepsilon_k\rho_k^{-1})^\alpha)$ *for* α *some small positive constant.*

Proof. Note that

$$\left|\frac{\partial}{\partial x_i'}f^1(x')\right| \leqq \sup_s\left|\frac{\partial}{\partial x_i'}\left(\sum_{n=1}^N h_n(x')\Xi_n(x')\right)\right| \tag{5.21}$$

and

$$\left|\frac{\partial^2}{\partial x_i'\partial x_j'}f^1(x')\right| \leqq \sup_s\left|\frac{\partial^2}{\partial x_i'\partial x_j'}\left(\sum_{n=1}^N h_n(x')\Xi_n(x')\right)\right|.$$

We apply Lemma 5.2, with $f_\ell = h_\ell$ (the function defined in Lemma 5.4), and $g_\ell = \Xi_\ell$, to derive bounds on the right hand side of these inequalities. (Our bounds are independent of s.) Bounds on f_ℓ are given in Lemma 5.4, while bounds on Ξ_ℓ and its derivatives were discussed in the proof of Lemma 5.3. Pick the domain $\mathscr{D}$ in Lemma 5.2 to be $W(r_k^6, \xi_k-\delta-5; \{I_k\})$, and choose $\kappa = m(1-\eta_k)(1-7\beta_k)$, $C_{g\ell} = C_{f\ell} = B\rho_0^n(\varepsilon_k\rho_k^{-1})^{7/8}$, $C_{g\ell}^1 = C_{f\ell}^1 = B\rho_0^n k_0\rho_{k+1}^{-1}(\varepsilon_k\rho_k^{-1})^{7/8}$, $C_{f\ell}^2 = C_{g\ell}^2 = B\rho_0^n$, $C_{f\ell}^3 = C_{f\ell}^4 = C_{g\ell}^3 = C_{g\ell}^4 = B\rho_0^n\kappa_0^2\rho_{k+1}^{-2}$, and the constant L in (5.6) to be L_k. Then (5.20) follows easily from (5.6) and (5.7).

Combine Proposition 5.1 with Lemmas 5.3 and 5.5., and take the definition of $f^{k+1}(I', \phi')$ in (3.17) to obtain

Corollary 5.6. *Let* $\mathscr{D} = W(r_k^7, \xi_k-\delta-6; \{I_k\}) \supset W(\rho^{k+1}, \xi_{k+1}; \{I_{k+1}\})$. *Then*

$$\sup_{\mathscr{D}}\left|\frac{\partial^2 f}{\partial x_i'\partial x_j'}(x')\right| \leqq \rho_0^n e^{-m(1-\eta_{k+1})|i-j|}. \tag{5.22}$$

This implies that (2.10) holds for the Hamiltonian H^{k+1}. All that remains is to verify that (2.5) and (2.6) also hold for f^{k+1}, which we do in the next section.

We now prove Lemma 5.4. By (2.8) and (2.9),

$$\left|\frac{\partial^2 h^k}{\partial I_\ell \partial I_n}\circ\tilde{x}^s(x') - \delta_{\ell n}\right| \leqq e^{-m(1-\eta_k)|\ell-n|} \tag{5.23}$$

on $W(r_k^3, \xi_k-\delta-2; \{I_k\})$. Since $|\Xi_\ell(x')| \leqq \rho_0(\varepsilon_k\rho_k^{-1})^{7/8}$, on this same domain, we see that $|h_n(x')| \leqq B\rho_0(\varepsilon_k\rho_k^{-1})^{7/8}$ as claimed in (5.17). A dimensional estimate immediately yields the first estimate in (5.18) on the smaller domain $W(r_k^4, \xi_k-\delta-3; \{I_k\})$.

Note that (2.8) and (2.9) coupled with a pair of dimensional estimates imply

$$\sup\left|\frac{\partial}{\partial x_i}\left(\frac{\partial^2 h^k}{\partial I_\ell \partial I_n}(x)\right)\right| \leqq \min\begin{cases} B\rho_0^n k_0\rho_{k+1}^{-1}e^{-m(1-\eta_k)|i-n|} \\ B\rho_0^n k_0\rho_{k+1}^{-1}e^{-m(1-\eta_k)|\ell-n|}\end{cases} \tag{5.24}$$

and

$$\sup\left|\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{\partial^2 h^k}{\partial I_\ell \partial I_n}(x)\right)\right| \leqq \min\begin{cases} B\rho_0 k_0^2\rho_{k+1}^{-2}e^{-m(1-\eta_k)|i-\ell|} \\ B\rho_0 k_0^2\rho_{k+1}^{-2}e^{-m(1-\eta_k)|i-j|},\end{cases} \tag{5.25}$$

where the suprema run over $W(r_k^1, \xi_k - \delta; \{I_k\})$ and x is as in (1.14). Now apply Lemma 4.1, with the domains $\mathscr{D} = W(r_k^3, \xi_k - \delta - 2; \{I_k\})$ and $\mathscr{D}' = W(r_k^4, \xi_k - \delta - 3; \{I_k\})$ and the function $g = \partial^2 h^k/\partial I_\ell \partial I_n$. The bounds on g and its derivatives coming from (5.23)–(5.25) and the bounds on $\tilde{x}^s$ from Lemma 4.2 combined with the second estimate of (4.3) imply

$$\sup_{\mathscr{D}'}\left|\frac{\partial^2}{\partial x_i'\partial x_j'}\left(\frac{\partial^2 h^k}{\partial I_\ell \partial I_n}\circ \tilde{x}^s(x')\right)\right| \leqq B\rho_0^n k_0^2 \rho_{k+1}^{-2} e^{-m(1-\eta_k)(1-5\beta_k)|i-j|},$$

while the first estimate in (4.3) implies

$$\sup_{\mathscr{D}'}\left|\frac{\partial}{\partial x_i'}\left(\frac{\partial^2 h^k}{\partial I_\ell \partial I_n}\circ \tilde{x}^s(x')\right)\right| \leqq B\rho_0^n k_0 \rho_{k+1}^{-1} e^{-m(1-\eta_k)(1-5\beta_k)|i-\ell|}. \tag{5.27}$$

Because of the symmetry between ℓ and n, (5.27) also holds with ℓ and n interchanged. Furthermore, (5.23) and (5.27) plus a pair of dimensional estimates on $W(r_k^5, \xi_k - \delta - 4; \{I_k\})$ imply

$$\left|\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{\partial^2 h^k}{\partial I_\ell \partial I_n}\circ \tilde{x}^s(x')\right)\right| \leqq \min\begin{cases} B\rho_0^n k_0^2 \rho_{k+1}^{-2} e^{-m(1-\eta_k)(1-5\beta_k)|i-\ell|} \\ B\rho_0^n k_0^2 \rho_{k+1}^{-2} e^{-m(1-\eta_k)(1-5\beta_k)|i-n|}, \end{cases}$$

and

$$\left|\frac{\partial}{\partial x_i}\left(\frac{\partial^2 h^k}{\partial I_\ell \partial I_n}\circ \tilde{x}^s(x')\right)\right| \leqq B\rho_0 k_0 \rho_{k+1}^{-1} e^{-m(1-\eta_k)|\ell-n|}. \tag{5.28}$$

Now apply Lemma 5.2, with $f_\ell(x') = (\partial^2 h^k/\partial I_\ell \partial I_n)\circ \tilde{x}^s(x')$, $g_\ell(x') = \Xi_\ell(x')$ and $\mathscr{D} = W(r_k^5, \xi_k - \delta - 4; \{I_k\})$. Then (5.23) and (5.26)–(5.28) bound f_ℓ and its derivatives while (3.7) and Lemma 4.2, coupled with the observations in the proof of Lemma 5.3 bound g_ℓ and its derivatives. Estimate (5.6) implies

$$\sup_{\mathscr{D}}\left|\frac{\partial}{\partial x_i'} h_n(x')\right| \leqq B\rho_0^n e^{-m(1-\eta_k)(1-6\beta_k)|i-n|}, \tag{5.29}$$

which implies the second estimate in (5.18). The first of the bounds in (5.19) follows from (5.29) by a dimensional estimate. Applying Lemma 5.2 a second time we find (5.7) implies

$$\sup_{\mathscr{D}}\left|\frac{\partial^2}{\partial x_i'\partial x_j'} h_n(x')\right| \leqq B\rho_0^n k_0^2 \rho_{k+1}^{-2} e^{-m(1-\eta_k)(1-7\beta_k)|i-j|}, \tag{5.30}$$

which completes the proof of Lemma 5.4. We note that in applying Lemma 5.2 to derive (5.29) and (5.30) it is necessary to make different choices of the constant κ, and hence of the other constants, in the two cases but the details are not difficult to work out so we omit them.

6. The Strength of the "Renormalized" Interactions

In the present section we verify that (2.5) and (2.6) hold for the Hamiltonian H^{k+1}. First note that the fundamental theorem of calculus allows us to write

$$\begin{aligned}
&f^{k,r}(I'+\Xi(I',\phi'),\phi'+\Delta(I',\phi'))\\
&\quad=f^{k,r}(I',\phi')+\sum_{\ell=1}^{N}\int_0^1 ds\frac{\partial f^{k,r}}{\partial I_\ell}(I'+s\Xi,\phi'+s\Delta)\cdot\Xi_\ell(I',\phi')\\
&\qquad+\sum_{\ell=1}^{N}\int_0^1 ds\frac{\partial f^{k,r}}{\partial\phi_\ell}(I'+s\Xi,\phi'+s\Delta)\cdot\Delta_\ell(I',\phi')\\
&\quad\equiv f^{k,r}(I',\phi')+\delta f^{k,r}(I',\phi')
\end{aligned}\tag{6.1}$$

and

$$\begin{aligned}
f^{\mathrm{IV}}(I',\phi')&=f^4(I',\phi')+\sum_{\ell=1}^{N}\int_0^1 ds\frac{\partial f^4}{\partial I_\ell}(I'+s\Xi,\phi'+s\Delta)\cdot\Xi_\ell(I',\phi')\\
&\quad+\sum_{\ell=1}^{N}\int_0^1 ds\frac{\partial f^4}{\partial\phi_\ell}(I'+s\Xi,\phi'+s\Delta)\cdot\Delta_\ell(I',\phi')\\
&\equiv f^4(I',\phi')+\delta f^4(I',\phi').
\end{aligned}\tag{6.2}$$

Proposition 6.1. *On* $W(r_k^6,\xi_k-\delta-5;\{I_k\})$

$$\sup\left|\frac{\partial}{\partial x_i'}(\delta f^4(x'))\right|\leqq C_1\rho_0^n\varepsilon_{k+1}$$

and

$$\sup\left|\frac{\partial}{\partial x_i'}(\delta f^{k,r}(x'))\right|\leqq\begin{cases}\rho_0^n k_0\varepsilon_k\rho_{k+1}^{-1}B^{L_k} & \text{if } i\in S^m, m=0,\dots,k\\ \rho_0^n C_1\varepsilon_{k+1} & \text{if } i\notin S^m, m=0,\dots,k,\end{cases}\tag{6.3}$$

where $C_1\sim\mathcal{O}((\varepsilon_k\rho_k^{-1})^\alpha)$, for α some small positive constant.

Now note that

$$\begin{aligned}
f_\nu^{k+1}(I')&=\int d\phi' f^{k+1}(I',\phi')e^{i\nu\cdot\phi'}\\
&=\int d\phi'\{f^{k,r}(I',\phi')+f^4(I',\phi')+\delta f^{k,r}(I',\phi')\\
&\quad+\delta f^4(I',\phi')+f^{\mathrm{I}}(I',\phi')+f^{\mathrm{II}}(I',\phi')+f^{\mathrm{III}}(I',\phi')\}e^{i\nu\cdot\phi'}.
\end{aligned}\tag{6.4}$$

(Here, $\int d\phi'\equiv\prod_{i=1}^{N}\left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi}d\phi_i'$.) But

$$\int d\phi'\{f^{k,r}(I',\phi')+f^4(I',\phi)\}e^{i\nu\cdot\phi'}=f_\nu^k(I')\tag{6.5}$$

if $\nu\in\left[\bigcup_{m=0}^{k-1}(R_p^m\cup R_s^m)\right]\cup R_p^k$, and zero otherwise. Note that if $\nu\notin\bigcup_{m=0}^{k}[R_p^m\cup R_s^m]$, then either there exists $\ell\in\operatorname{supp}\nu$ such that $\ell\notin S^m$, $m=0,\dots,k$, or $\nu\notin\mathbb{X}_{k+1}$. In the first of these cases, notice that Proposition 6.1, Proposition 5.1, and Lemmas 5.4 and 5.5. when combined with Cauchy's theorem, and the observation that for such a ν, $\int d\phi'\{f^{k,r}(I',\phi')+f^4(I',\phi')\}e^{i\nu\cdot\phi'}=0$, imply that (on $W(r_k^7,\xi_k-\delta-6;\{I_k\})$)

$$\left|\frac{\partial}{\partial I_j'}f_\nu^{k+1}(I')\right|+\rho_0^{-1}|\nu_j f_\nu^{k+1}(I')|\leqq C_1\rho_0k_0\varepsilon_{k+1}\rho_{k+1}^{-1}e^{-(\xi_k-\delta-6)|\nu|}.\tag{6.6}$$

In the second case, Corollary 5.6. combined with the estimates above, implies

$$\begin{aligned}
&\left|\frac{\partial}{\partial I_j'}f_\nu^{k+1}(I')\right|+\rho_0^{-1}|\nu_j f_\nu^{k+1}(I')|\\
&\leqq\min(B\rho_0^2k_0\rho_{k+1}^{-1}e^{-m(1-\eta_{k+1})d(\operatorname{supp}\nu)}e^{-(\xi_k-\delta-6)|\nu|},\rho_0k_0\varepsilon_k\rho_{k+1}^{-1}B^{L_k}e^{-(\xi_k-\delta-6)|\nu|}),
\end{aligned}\tag{6.7}$$

on the same domain, from which it follows that if $\nu \notin \bigcup_{m=0}^{k} [R_p^m \cup R_s^m]$,

$$\left|\frac{\partial}{\partial I_j'} f_\nu^{k+1}(I')\right| + \rho_0^{-1} |\nu_j f_\nu^{k+1}(I')| \leqq C_2 \varepsilon_{k+1} e^{-(\xi_k - 2\delta - 6)|\nu|}, \tag{6.8}$$

with both C_1 and $C_2 \sim \mathcal{O}((\varepsilon_k \rho_k^{-1})^\alpha)$.

Remark. Inequality (6.7) does not imply (6.8) in the case $\nu = 0$—we treat this special case at the end of this section.

Next remark that on $W(\rho^{k+1}, \xi_{k+1}; \{I_{k+1}\})$,

$$\left|\frac{\partial f^{k+1,\mathrm{nr}}}{\partial I_j'}\right| + \rho_0^{-1} \left|\frac{\partial f^{k+1,\mathrm{nr}}}{\partial \phi_j'}\right| \leqq \sum_{\nu \notin R'} \left(\left|\frac{\partial f_\nu^{k+1}}{\partial I_j'}\right| + \rho_0^{-1} |\nu_j f_\nu^{k+1}| \right) e^{\xi_{k+1}|\nu|}, \tag{6.9}$$

with $\tilde{R}' = \bigcup_{m=0}^{k} (R_p^m \cup R_s^m)$. Corollary 5.6, (6.8), and Cauchy's theorem imply the summand in (6.9) is bounded by $\min(C_2 \varepsilon_{k+1} e^{-\delta|\nu|}, \rho_0 e^{-m(1-\eta_{k+1})|j-\ell|} e^{-\delta|\nu|})$, where ℓ is the point is supp ν farthest from j. If we now apply Proposition 4.3 we see that on $W(\rho^{k+1}, \xi_{k+1}; \{I_{k+1}\})$,

$$\sup\left\{ \left|\frac{\partial f^{k+1,\mathrm{nr}}}{\partial I_j'}\right| + \rho_0^{-1} \left|\frac{\partial f^{k+1,\mathrm{nr}}}{\partial \phi_j}\right| \right\} \leqq \varepsilon_{k+1}, \tag{6.10}$$

so (2.5) applies to the Hamiltonian H^{k+1}.

Next assume that $j \in S^m$ for some $m = 0, \ldots, k$. Then on $W(\rho^{k+1}, \xi_{k+1}, \{I_{k+1}\})$,

$$\begin{aligned} \left|\frac{\partial f^{k+1,\mathrm{r}}}{\partial I_j'}\right| + \rho_0^{-1} \left|\frac{\partial f^{k+1,\mathrm{r}}}{\partial \phi_j'}\right| &\leqq \left|\frac{\partial f^{k,\mathrm{r}}}{\partial I_j'}\right| + \rho_0^{-1} \left|\frac{\partial f^{k,\mathrm{r}}}{\partial \phi_j'}\right| \\ &+ \sum_{\nu \in \tilde{R}} \left\{ \left|\frac{\partial}{\partial I_j'}(f_\nu^{k+1}(I') - f_\nu^k(I'))\right| + \rho_0^{-1} |\nu_j (f_\nu^{k+1}(I') - f_\nu^k(I'))| \right\} e^{\xi_{k+1}|\nu|}. \end{aligned} \tag{6.11}$$

By Proposition 6.1, Corollary 5.6, (2.10), (6.5) and Cauchy's theorem, the summand on the right hand side of (6.11) can be bounded by

$$\min(\rho_0^n k_0 \varepsilon_k \rho_{k+1}^{-1} B^{L_k} e^{-\delta|\nu|}, 2\rho_0 e^{-m(1-\eta_{k+1})|j-\ell|} e^{-\delta|\nu|}),$$

where ℓ is the point in supp ν farthest from j. Apply Proposition 4.3 and bound this sum by $\rho_0 k_0 \varepsilon_k \rho_{k+1}^{-1} B^{L_{k+1}}$. By (2.6)

$$\left|\frac{\partial f^{k,\mathrm{r}}}{\partial I_j'}\right| + \rho_0^{-1} \left|\frac{\partial f^{k,\mathrm{r}}}{\partial \phi_j'}\right| \leqq A(k,j),$$

and since $j \in S^m$, $A(k,j) + \rho_0 k_0 \varepsilon_k \rho_{k+1}^{-1} B^{L_{k+1}} \leqq A(k+1,j)$, so (2.6) holds for the Hamiltonian H^{k+1} if $j \in \bigcup_{m=0}^{k} S^m$ and it remains only to check that it holds for $j \notin \bigcup_{m=0}^{k} S^m$.

If $j \notin \bigcup_{m=0}^{k} S^m$, and $\nu \in \bigcup_{m=0}^{k-1} (R_p^m \cup R_s^m)$, the definition of R_s^{k-1} implies there must be some $\ell \in \mathrm{supp}\,\nu$ such that $\mathrm{dist}(j,\ell) > L_k$. Thus applying (2.10) and Cauchy's theorem we see that if $\nu \in \bigcup_{m=0}^{k-1} [R_p^m \cup R_s^m]$,

$$\left\{\left|\frac{\partial f_\nu^k}{\partial I_j'}(I')\right|+\rho_0^{-1}|\nu_j f_\nu^k(I')|\right\}\leqq 2\rho_0 e^{-m(1-\eta_k)|j-\ell|}e^{-\xi_k|\nu|}$$
$$\leqq \min(B\rho_0(\varepsilon_0\rho_0^{-1})^{(8/5)(3/2)^k}e^{-\xi_k|\nu|},\ B\rho_0 e^{-m(1-\eta_k)|j-\ell|}e^{-\xi_k|\nu|}), \tag{6.12}$$

on $W(\rho^k,\xi_k;\{I_k\})$. Here ℓ is the point in supp ν farthest from j, and the second inequality used our observation that $|j-\ell|>L_k$. Hence, applying Proposition 4.3 we find

$$\sup\left\{\left|\frac{\partial f^{k,r}}{\partial I_j'}\right|+\rho_0^{-1}\left|\frac{\partial f^{k,r}}{\partial \phi_j'}\right|\right\}\leqq\sum_{\nu\in R'}\left\{\left|\frac{\partial f_\nu^k}{\partial I_j'}\right|+\rho_0^{-1}|\nu_j f_\nu^k|\right\}e^{\xi_{k+1}|\nu|}$$
$$\leqq \rho_0(\varepsilon_0\rho_0^{-1})^{(8/5)(3/2)^k}B^{L_{k+1}}, \tag{6.13}$$

on $W(\rho^{k+1},\ \xi_{k+1};\ \{I_{k+1}\})$. Combining this with (6.11) we see that if $j\notin\bigcup_{m=0}^{k}S^m$,

$$\left\{\left|\frac{\partial f^{k+1,r}}{\partial I_j'}\right|+\rho_0^{-1}\left|\frac{\partial f^{k+1,r}}{\partial \phi_j'}\right|\right\}$$
$$\leqq \rho_0(\varepsilon_0\rho_0^{-1})^{(8/5)(3/2)^k}B^{L_{k+1}}+C_1\varepsilon_{k+1}B^{L_{k+1}},\ \leqq\varepsilon_{k+1}, \tag{6.14}$$

so that (2.6) holds for all sites j, and the Hamiltonian satisfies the estimates (2.5)–(2.10). Note that in (6.14) we used the fact that on $W(r_k^7,\ \xi_k-\delta-6;\ \{I_k\})$,

$$\left|\frac{\partial}{\partial I_j'}(f_\nu^{k+1}(I')-f_\nu^k(I'))\right|=|\textstyle\int d\phi'\frac{\partial}{\partial I_j'}(\delta f^{k,r}(I',\phi')+\delta f^4(I',\phi')$$
$$+f^{\mathrm{I}}(I',\phi')+f^{\mathrm{II}}(I',\phi')+f^{\mathrm{III}}(I',\phi'))e^{i\nu\cdot\phi'}|$$
$$\leqq C_1\varepsilon_{k+1}e^{-(\xi_k-\delta-5)|\nu|} \tag{6.15}$$

if $j\notin S^m$, $m=0,\dots,k$, and analogously for $\rho_0^{-1}|\nu_j(f_\nu^{k+1}(I')-f_\nu^k(I'))|$, and then applied Proposition 4.3 to bound the sum in (6.11) by $C_1\varepsilon_{k+1}B^{L_{k+1}}$.

We complete this section with the proof of Proposition 6.1. The first inequality in (6.3) is an easy application of Lemma 5.2. Let x be as in (1.14), x' as in Lemma 4.2, and redefine $\tilde{x}_i^s(x')$ as either $I_i'+s\Xi_i(x')$ or $\varphi_i'+s\Delta_i(x')$ depending on context. Let the functions $f_\ell(x')$ and $g_\ell(x')$ of Lemma 5.2 be defined by $f_\ell(x')=(\partial f^4/\partial x_\ell)\circ\tilde{x}^s(x'), g_\ell(x')=\Xi_\ell(x')$ in the second term on the right hand side of (6.2), and $g_\ell(x')=\Delta_\ell(x')$ in the third term on the right hand side of (6.2). Let $\mathscr{D}$, the domain in Lemma 5.2 be $W(r_k^5,\xi_k-\delta-4;\{I_k\})$. We will give estimates on $|(\partial/\partial x_i')(\sum_\ell f_\ell(x')g_\ell(x'))|$ which are independent of $s\in[0,1]$, and which therefore bound $|(\partial/\partial x_i')(\delta f^4(x'))|$. By Corollary 4.5 and a dimensional estimate we choose the constants $C_{f\ell}=\rho_0^n\varepsilon_k B^{L_k}$ and $C_{f\ell}^1=\rho_0^n\varepsilon_k k_0\rho_{k+1}^{-1}B^{L_k}$. Applying Lemma 4.1, with the bounds on derivatives of the map $\tilde{x}^s$ that come from Lemma 4.2 (since $(\partial\tilde{x}_\ell^s/\partial x_j')-\tilde{\delta}_{\ell j}=s((\partial\tilde{x}_\ell/\partial x_j')-\tilde{\delta}_{\ell j})$) we see that

$$\left|\frac{\partial f_\ell}{\partial x_i'}(x')\right|=\left|\frac{\partial}{\partial x_i'}\left(\frac{\partial f^4}{\partial x_\ell}\circ\tilde{x}^s(x')\right)\right|\leqq B\rho_0^n e^{-m(1-\eta_k)(1-5\beta_k)|i-\ell|},$$

on $W(r_k^4,\xi_k-\delta-3;\{I_k\})$, so we can take $C_{f\ell}^2=B\rho_0^n$ and $\kappa=m(1-\eta_k)(1-5\beta_k)$. Since $\Xi_\ell(x')=(\partial S/\partial\phi_\ell)\ (I',\phi(I',\phi'))$ and $\Delta_\ell(x')=(\partial S/\partial I_\ell')\ (I',\phi(I',\phi'))$, (3.7) and a dimensional estimate imply we can pick $C_{g\ell}=\rho_0^n(\lambda(\varepsilon_0))^{-2}\varepsilon_k B^{L_k}$, $C_{g\ell}^1=\rho_0^n(\lambda(\varepsilon_0))^{-2}\times k_0\varepsilon_k\rho_{k+1}^{-1}B^{L_k}$, while Lemma 4.2 implies that $C_{g\ell}^2=\rho_0^n$ with the same choice of κ as above, (since $\partial g_\ell/\partial x_i' = (\partial \tilde{x}_\ell/\partial x_i') - \delta_{\ell i}$, with $\tilde{x}$ as in Lemma 4.2). Combining these observations with (5.6) and setting $L = L_k$ in that expression we obtain the first inequality in (6.3).

To prove the second estimate in (6.3), let $f_\ell(x') = (\partial f^{k,r}/\partial x_\ell) \circ \tilde{x}^s(x')$, $g_\ell(x')$ be just as above, and set $\mathscr{D} = W(r_k^6, \xi_k - \delta - 5; \{I_k\})$. Once again we will prove estimates on $|(\partial/\partial x_i')(\sum_\ell f_\ell(x') g_\ell(x'))|$ which are independent of $s \in [0,1]$, and thus imply the second inequality in (6.3). Combining Lemma 4.2 with Corollary 4.5 and Lemma 4.1 we see that we can set $C_{f\ell}^2 = C_{g\ell}^2 = B\rho_0^n$, and $\kappa = m(1-\eta_k)(1-6\beta_k)$, in Lemma 5.2.

We now consider two cases.

Case 1. $\ell \notin \bigcup_{m=0}^{k-1} S^m$. We can set $C_{f\ell} = \rho_0(\varepsilon_k \rho_k^{-1})^{(1-\eta_k)}$ and $C_{f\ell}^1 = B\rho_0 k_0 \rho_{k+1}^{-1} \times (\varepsilon_k \rho_k^{-1})^{(1-\eta_k)}$, by combining (2.6), (2.7), and a dimensional estimate. Then just as above set $C_{g\ell} = \rho_0^n (\lambda(\varepsilon_0))^{-2} \varepsilon_k B^{L_k}$ and $C_{g\ell}^1 = \rho_0^n (\lambda(\varepsilon_0))^{-2} k_0 \varepsilon_k \rho_{k+1}^{-1} B^{L_k}$.

Case 2. $\ell \in S^n$ for some $n = 0, \ldots, k-1$. We must now consider two subcases within this case:

(a) $i \in S^p$, $p = 0, \ldots, k$. Set $C_{f\ell} = \rho_0(\varepsilon_n \rho_n^{-1})^{(1-\eta_k)}$, $C_{f\ell}^1 = B\rho_0 k_0 \rho_{k+1}^{-1}(\varepsilon_n \rho_n^{-1})^{(1-\eta_k)}$, $C_{g\ell} = \rho_0^n (\lambda(\varepsilon_0))^{-2} \varepsilon_k B^{L_k}$, and $C_{g\ell}^1 = B\rho_0^n (\lambda(\varepsilon_0))^{-2} k_0 \rho_{k+1}^{-1} \varepsilon_k B^{L_k}$.

(b) $i \notin \bigcup_{m=0}^{k} S^m$. Set $C_{f\ell} = \rho_0(\varepsilon_n \rho_n^{-1})^{(1-\eta_k)}$, and $C_{g\ell} = \rho_0^n (\lambda(\varepsilon_0))^{-2} \varepsilon_k B^{L_k}$ as before. Since $\ell \in S^n$, and $i \notin \bigcup_{m=0} S^m$, we have $|i - \ell| > L_k$. Thus by combining (4.15) with Lemmas 4.1 and 4.2 and our estimates on $\partial g_\ell/\partial x_i$ above we can pick $C_{f\ell}^1 = C_{g\ell}^1 = B\rho_0^n e^{-m(1-\eta_k)(1-6\beta_k)L_k} \leqq BC_1 \rho_0^n \varepsilon_{k+1}$, with $C_1 \sim \mathcal{O}((\varepsilon_k \rho_k^{-1})^\alpha)$.

With these choices for the constants, the second estimate of (6.3) follows from (5.6) if we set $L = L_k$.

Finally we show how (6.8) follows in the case $v = 0$. First note that by another application of the fundamental theorem of calculus, one can write

$$\delta f^{k,r}(I', \phi') = \sum_{\ell=1}^{N} \frac{\partial f^{k,r}}{\partial I_\ell}(I', \phi') \Xi_\ell(I', \phi') + \sum_{\ell=1}^{N} \frac{\partial f^{k,r}}{\partial \phi_\ell}(I', \phi') \Delta_\ell(I', \phi')$$
$$+ \sum_{\ell,p} \int_0^1 ds \int_0^s dt \frac{\partial^2 f^{k,r}}{\partial x_\ell \partial x_p}(x' + tg) g_\ell(x') g_p(x'). \tag{6.16}$$

Here, x is as in (1.14), x' is as in Lemma 4.2, and $g_\ell(x')$ is either $\Xi_\ell(x')$ or $\Delta_\ell(x')$. We note that the summation over ℓ, p here is slightly ambiguous. We emphasize that it should run over both meanings of x_ℓ, and x_p, i.e. both I_ℓ and ϕ_ℓ, and I_p and ϕ_p.

It is straightforward to show that the derivative of the last term with respect to I_j' is bounded by $\rho_0^n C_1 \varepsilon_{k+1}$ on $W(r_k^7, \xi_k - \delta - 6; \{I_k\})$, using Lemma 5.2 and the bounds on $g(x')$ that we used above. (As before $C_1 \sim \mathcal{O}((\varepsilon_k \rho_k^{-1})^\alpha)$.) Since $\Xi_\ell(I', \phi') = (\partial S/\partial \phi_\ell)(I', \phi + \Delta)$, use the fundamental theorem of calculus to write

$$\sum_{\ell=1}^{N} \frac{\partial f^{k,r}}{\partial I_\ell}(I', \phi') \cdot \Xi_\ell(I', \phi') = \sum_{\ell=1}^{N} \frac{\partial f^{k,r}}{\partial I_\ell}(I', \phi') \cdot \frac{\partial S}{\partial \phi_\ell}(I', \phi')$$
$$+ \int_0^1 ds \sum_{\substack{\ell=1 \\ p=1}}^{N} \frac{\partial f^{k,r}}{\partial I_\ell}(I', \phi') \cdot \frac{\partial^2 S}{\partial \phi_p \partial \phi_\ell}(I', \phi' + s\Delta) \cdot \Delta_p(I', \phi'). \tag{6.17}$$

Because $f^{k,r}$ and S have no harmonics in common and no contributions from the $v=0$ harmonic, the orthogonality of the trigonometric functions insures that

$$\int d\phi' \left(\sum_{\ell=1}^{N} \frac{\partial f^{k,r}}{\partial I_\ell}(I',\phi') \cdot \frac{\partial S}{\partial \phi_\ell}(I',\phi') \right) = 0. \tag{6.18}$$

Using Lemma 5.2 it is once again easy to show that the derivative of the second term on the right hand side of (6.17), with respect to I'_j is bounded by $C_1 \varepsilon_{k+1}$ on $W(r_k^7, \xi_k - \delta - 6; \{I_k\})$. By an exactly analogous procedure we obtain

$$\left| \sum_{\ell=1}^{N} \int d\phi' \frac{\partial}{\partial I'_j} \left\{ \frac{\partial f^{k,r}}{\partial I_\ell}(I',\phi') \cdot \Delta_\ell(I',\phi') \right\} \right| \leqq C_1 \varepsilon_{k+1}$$

on $W(r_k^7, \xi_k - \delta - 6; \{I_k\})$. Putting these observations together we see that

$$\left| \int d\phi' \frac{\partial}{\partial I'_j} \{\delta f^{k,r}(I')\} \right| \leqq C_1 \varepsilon_{k+1} \tag{6.19}$$

on $W(r_k^7, \xi_k - \delta - 6; \{I_k\})$, and combining this with (6.4), the fact that $0 \notin \left[\bigcup_{m=0}^{k-1} (R_p^m \cup R_s^m) \right] \cup R_p^k$, and the bounds on δf^4, f^{I}, f^{II}, and f^{III} coming from Proposition 6.1, Proposition 5.1, and Lemmas 5.4 and 5.5 we obtain inequality (6.8) in the case $v=0$.

7. Some Final Estimates

We now complete the results that were stated but not proved in Sects. 3 and 4.

We first note that Proposition 3.2 follows from Proposition 4.3. Using (2.5) and (3.2) we readily verify that on $W(r_k^1, \xi_k - \delta; \{I_k\})$,

$$\left| \frac{\partial}{\partial \phi_j} \left[\frac{f_v^k(I') e^{iv\cdot\phi}}{i \langle \omega^k(I'), v \rangle} \right] \right| \leqq \min \left(B(\varepsilon_k/\lambda(\varepsilon_0)) e^{L_k} e^{-(\delta - 3/2)|v|}, \right.$$
$$\left. B(\rho_0/\lambda(\varepsilon_0)) e^{L_k} e^{-m(1-\eta_k)|j-\ell|} e^{-(\delta-3/2)|v|} \right), \tag{7.1}$$

where ℓ is the point in supp v farthest from j. This follows by using integration by parts to write the expression for $f_v^k(I') v_j$ as $-\int d\phi \, (\partial^2/\partial\phi_j \partial\phi_\ell) \{f^k(I',\phi)\} \, e^{iv\cdot\phi}/v_\ell$, and the using (2.5) or (2.10), and Cauchy's theorem.

Similarly, if we note that (2.8) and (2.9) imply

$$\left| i \left\langle \frac{\partial \omega^k}{\partial I'_j}(I'), v \right\rangle \right| \leqq 2|v| e^{-m(1-\eta_k)|j-n|}, \tag{7.2}$$

where n is the point in supp v closest to j, we find

$$\left| \frac{\partial}{\partial I'_j} \left[\frac{f_v(I') e^{iv\cdot\phi}}{i \langle \omega^k(I'), v \rangle} \right] \right| \leqq \min \left(B(\varepsilon_k/\lambda^2(\varepsilon_0)) \rho_0^{-1} e^{2L_k} |v| e^{(3-\delta)|v|}, \right.$$
$$\left. (B/\lambda^2(\varepsilon_0)) e^{-m(1-\eta_k)|j-\ell|} e^{2L_k} |v| e^{(3-\delta)|v|} \right), \tag{7.3}$$

again on $W(r_k^1, \xi_k - \delta; \{I_k\})$. If we now apply Proposition 4.3 with $K_1 = \rho_0^n (B/\lambda^2(\varepsilon_0)) e^{2L_k}$, $e(i) = \varepsilon_k \rho_0^{-1}$, $\kappa_1 = m(1-\eta_k)$, and $\delta' = \delta/2$, (which implies $\delta \geqq 12$)

we have

$$\sup\left|\frac{\partial S}{\partial y_j}\right| \leqq \rho_0^n(B/\lambda^2(\varepsilon_0))e^{2L_k}\varepsilon_k\rho_0^{-1}B'^{2L_k} \leqq (\rho_0^n/\lambda^2(\varepsilon_0))\varepsilon_k B^{L_k}, \tag{7.4}$$

on $W(r_k^1, \xi_k - \delta; \{I_k\})$, by (4.7), where $y \in \mathbb{C}^{2N}$ is defined in Proposition 3.2. This verifies the first inequality in (3.7).

Next note that using (2.10), Cauchy's theorem, and a dimensional estimate we have

$$\left|\frac{\partial^2}{\partial\phi_i\partial\phi_j}\left[\frac{f_v^k(I')e^{iv\cdot\phi}}{i\langle\omega^k(I'), v\rangle}\right]\right| \leqq \min\left((B\rho_0/\lambda(\varepsilon_0))e^{-m(1-\eta_k)|i-j|}e^{L_k}\cdot|v|^2 e^{(3/2-\delta)|v|},\right.$$
$$\left.(B\rho_0/\lambda(\varepsilon_0))e^{-m(1-\eta_k)|i-l|}e^{L_k}|v|^2 e^{(3/2-\delta)|v|}\right) \tag{7.5}$$

on $W(r_k^1, \xi_k - \delta - 1; \{I_k\})$, where as usual, ℓ is the site in supp v most distant from i. Similarly

$$\left|\frac{\partial^2}{\partial\phi_i\partial I'_j}\left(\frac{f_v(I')e^{iv\cdot\phi}}{i\langle\omega^k(I'), v\rangle}\right)\right| \leqq \min\left((B/\lambda^2(\varepsilon_0))e^{-m(1-\eta_k)|i-j|}|v|^2 e^{2L_k}e^{(3-\delta)|v|},\right.$$
$$\left.B[(\rho^k)_j - (r_k^1)_j]^{-1}(\rho_0/\lambda^2(\varepsilon_0))e^{-m(1-\eta_k)|i-l|}|v|^2 e^{2L_k}e^{(3-\delta)|v|}\right), \tag{7.6}$$

and

$$\left|\frac{\partial^2}{\partial I'_i\partial I'_j}\left(\frac{f_v(I')e^{iv\cdot\phi}}{i\langle\omega^k(I'), v\rangle}\right)\right| \leqq \min\left((B/\rho_0\lambda^3(\varepsilon_0))e^{-m(1-\eta_k)|i-j|}|v|^2 e^{3L_k}e^{(5-\delta)|v|},\right.$$
$$\left.\cdot(B/\lambda^3(\varepsilon_0))[(\rho^k)_j - (r_k^1)_j]^{-1}\rho_0 e^{-m(1-\eta_k)|i-\ell|}|v|^2 e^{3L_k}e^{(5-\delta)|v|}\right), \tag{7.7}$$

on $W(r_k^1, \xi_k - \delta; \{I_k\})$. If we combine (7.5)–(7.7) with the observation that if $\chi < \min(C_1, C_2)$, then $\chi < C_1^\beta C_2^{(1-\beta)}$ for $\beta \in [0, 1]$, we see that we can apply the second half of Proposition 4.5, choosing

$$K_2 = K_3 = B\{\rho_0^{-1}[(\rho^k)_j - (r_k^1)_j]\}^{-\beta_k}\cdot(\lambda(\varepsilon_0))^{-3}e^{-m(1-\eta_k)(1-\beta_k)|i-j|}e^{3L_k},$$

$$\kappa_2 = m(1-\eta_k)\beta_k, \text{ and } \delta' = \delta/2.$$

Then (4.9) implies that

$$\sup\left|\frac{\partial^2 S}{\partial y_i\partial y_j}(y)\right| \leqq B\rho_0^n\{\rho_0^{-1}[(\rho^k)_j - (r_k^1)_j]\}^{-\beta_k}(\lambda(\varepsilon_0))^{-3}e^{3L_k}B''^{|i-j|}e^{-m(1-\eta_k)|i-j|},$$

with the supremum running over $W(r_k^1, \xi_k - \delta; \{I_k\})$. Note that we can assume $|i-j| > (L_k/4)$, since otherwise the second estimate in (3.7) follows from the first by a dimensional estimate. But then, using the fact that

$$B\{\rho_0^{-1}[(\rho^k)_j - (r_k^1)_j]\}^{-\beta_k}(\lambda(\varepsilon_0))^{-3}e^{3L_k}B''^{|i-j|}e^{-m(1-\eta_k)\beta_k|i-j|} \leqq 1,$$

when $|i-j| > (L_k/4)$, we obtain the second estimate of (3.7).

We next prove Proposition 4.3. To prove (4.7) note that

$$\left|\frac{\partial g}{\partial x_1}(x)\right| \leqq \sum_{\substack{v: \\ |i-\ell| \leqq L}}\left|\frac{\partial g_v}{\partial x_i}(x')\right| + \sum_{\substack{v: \\ |i-\ell| > L}}\left|\frac{\partial g_v}{\partial x_i}(x)\right|, \tag{7.9}$$

where L is any positive integer and ℓ is the point in supp v such that $|i-\ell|$ is maximized. Bound the summand in the first term on the right hand side of (7.10) by $K_1 e(i) e^{-\delta'|v|}$, and in the second term bound it by $K_1 e^{-\kappa_1|i-\ell|} e^{-\delta'|v|}$. Next note that the number of vectors with $|v|=M$ and $|i-\ell|=\tilde{L}$ is bounded by $2(2\tilde{L}+1)2^{\tilde{L}}2^{2M}$. This estimate follows from the observation that the number of vectors with $|v|=M$, $d(\operatorname{supp} v)=L'$, and the rightmost point in supp v fixed is bounded by $2^{L'}2^{2M}$ (a discussion of this bound is contained in [6]) and then noting that if $|i-\ell|=\tilde{L}$ one must have $0 \leqq L' \leqq 2\tilde{L}$, and that there are at most $2\tilde{L}+1$ choices for the rightmost point in supp v. Thus (7.9) is bounded by

$$\sum_{M=0}^{\infty} \sum_{\tilde{L}=0}^{L} 2(2\tilde{L}+1)2^{\tilde{L}}2^{2M} K_1 e(i) e^{-\delta' M} + \sum_{M=0}^{\infty} \sum_{\tilde{L}=L+1}^{\infty} 2(2\tilde{L}+1)2^{\tilde{L}}2^{2M} K_1 e^{-\kappa_1 \tilde{L}} e^{-\delta' M}$$
$$\leqq 2^2(2L+1)2^L K_1 e(i) + 2^4 K_1 e^{-(\kappa_1 - 2\ln 2)L}, \tag{7.10}$$

from which (4.7) follows immediately if $L > (2/\kappa_1)|\ln e(i)|$.

Inequality (4.9) is proved in much the same fashion. Note that

$$\left|\frac{\partial^2 g}{\partial x_i \partial x_j}(x)\right| \leqq \sum_{\substack{v: \\ |i-\ell| \leqq |i-j|}} \left|\frac{\partial^2 g_v}{\partial x_i \partial x_j}\right| + \sum_{\substack{v: \\ |i-\ell| > |i-j|}} \left|\frac{\partial^2 g_v}{\partial x_i \partial x_j}\right|. \tag{7.11}$$

Bound the first sum by

$$\sum_{M=0}^{\infty} \sum_{\tilde{L}=0}^{|i-j|} 2(2\tilde{L}+1)2^{\tilde{L}}2^{2M} K_2 e^{-\kappa_2|i-j|} e^{-\delta' M} \leqq K_2 B^{(|i-j|+1)} e^{-\kappa_2|i-j|}. \tag{7.12}$$

Bound by the second term by

$$\sum_{M=0}^{\infty} \sum_{\tilde{L}=|i-j|+1}^{\infty} 2(2\tilde{L}+1)2^{\tilde{L}}2^{2M} K_3 e^{-\kappa_2 \tilde{L}} e^{-\delta' M} \leqq K_3 B^{(|i-j|+1)} e^{-\kappa_2|i-j|}, \tag{7.13}$$

and (4.9) follows immediately by combining (7.12) and (7.13).

We finish up by proving Lemma 4.2. As in [6] the proof turns on the following lemma:

Lemma 7.1. *Let M be an $n \times n$ matrix whose elements satisfy $|M_{ij}| \leqq \min(c, c_1 e^{-\kappa|i-j|})$. If c and c_1 are less than $(1/4)$, and $\kappa \geqq 2\ln 2$ then*

$$|(\mathbb{I} - M)^{-1}_{ij} - \delta_{ij}| \leqq \min(cB^{\bar{L}}, 2^4 2^{|i-j|} e^{-\kappa|i-j|}), \tag{7.14}$$

where B is some universal constant, and $\bar{L}$ is any number such that $\bar{L} > (2/\kappa)|\ln c|$.

The proof of this lemma is in Appendix B.

Let $\tilde{y}$ be the holomorphic map taking $W(r_k^4, \xi_k - \delta - 3; \{I_k\})$ into $W(r_k^3, \xi_k - \delta - 2; \{I_k\})$ defined by $\tilde{y}_i(x') = x'_i$ if $1 \leqq i \leqq N$, and $y_i(x') = \tilde{x}_i(x')$ if $N < i \leqq 2N$, where $\tilde{x}_i(x') = \{I_i(I', \phi')$ if $1 \leqq i \leqq N$ and $\phi_{i-N}(I', \phi')$ if $N < i \leqq 2N\}$ and $x'_i = \{I'_i$ if $1 \leqq i \leqq N$ and ϕ'_{i-N} if $N < i \leqq 2N\}$. Finally let $y_i = \{I'_i$ if $1 \leqq i \leqq N$, ϕ_{i-N} if $N < I \leqq 2N\}$. This notation saves writing out many special cases, but it has the following disadvantage. If we consider $\partial^2 S/\partial I'_j \partial \phi_k\ (I', \phi') = \partial^2 S/\partial y_j \partial y_\ell\ (y)$, where $\ell = k+N$, we expect it to decay only as $\exp[-m(1-\eta_k)|j-k|]$, not as $\exp[-m(1-\eta_k)|j-\ell|]$. Thus we define $\delta(j,\ell) = |(j-\ell)(\operatorname{mod} N)|$, and then we see that the derivatives above decay as $\exp[-m(1-\eta_k)\delta(j,\ell)]$.

If $N < i \leqq 2N$, $\tilde{x}_i(x') = x'_i - \partial S/\partial y_{i-N} \circ \tilde{y}(x')$, by (3.14), and (3.11), where y is as in

Proposition 3.2. Thus,

$$\begin{aligned}\frac{\partial \tilde{x}_i}{\partial x'_j}(x') &= \delta_{ij} - \sum_{\ell=1}^{2N} \frac{\partial^2 S}{\partial y_{i-N} \partial y_\ell} \circ \tilde{y}(x') \cdot \frac{\partial \tilde{y}_\ell}{\partial x'_j}(x') \\ &= \delta_{ij} - \sum_{\ell=1}^{N} \left(\frac{\partial^2 S}{\partial y_{i-N} \partial y_\ell} \circ \tilde{y}(x') \right) \cdot \delta_{\ell j} \\ &\quad - \sum_{\ell=N+1}^{2N} \left(\frac{\partial^2 S}{\partial y_{i-N} \partial y_\ell} \circ \tilde{y}(x') \right) \cdot \left(\frac{\partial \tilde{x}_\ell}{\partial x'_j}(x') \right), \end{aligned} \tag{7.15}$$

where the first of these steps used the chain rule, and the second the definition of $\tilde{y}$. Let M be the $N \times N$ matrix with elements $M_{ij} = (-\partial^2 S/\partial y_i \partial y_{j+N}) \circ \tilde{y}(x')$. We now consider two cases. If the j in (7.15) is less than or equal to N, define D to be the $N \times N$ matrix with elements $D_{ij} = (\partial \tilde{x}_{i+N}/\partial x'_j)(x')$, and Λ the $N \times N$ matrix with elements $\Lambda_{ij} = -(\partial^2 S/\partial y_i \partial y_j) \circ \tilde{y}(x')$. If the j in (7.15) is greater than N, let D' be the matrix with elements $D'_{ij} = (\partial \tilde{x}_{i+N}/\partial x'_{j+N})$ (x'), and $\Lambda' = N \times N$ identity matrix. Then, (7.15) becomes the pair of matrix equations

$$D = \Lambda + MD \quad \text{and} \quad D' = \Lambda' + MD', \tag{7.16}$$

or

$$D = (\mathbb{I} - M)^{-1} \Lambda \quad \text{and} \quad D' = (\mathbb{I} - M)^{-1} \Lambda'. \tag{7.17}$$

By Proposition 3.2 and a dimensional estimate,

$$|M_{ij}| \leqq \min(\rho_0^n(\lambda(\varepsilon_0))^{-2} k_0 \varepsilon_k \rho_{k+1}^{-1} B^{L_k}, \rho_0^n e^{-m(1-\eta_k)(1-\beta_k)|i-j|})$$

for all x' in $W(r_k^4, \xi_k - \delta - 3; \{I_k\})$. Note that if i and j are both greater than N, the second equation in (7.17) implies

$$\frac{\partial \tilde{x}_i}{\partial x'_j}(x') = [(\mathbb{I} - M)^{-1}]_{i-N, j-N}, \tag{7.18}$$

and (4.4) follows (in the case $i, j > N$) from Lemma 7.1. If i is greater than N and j is less than or equal to N, (7.17) implies

$$\frac{\partial \tilde{x}_i}{\partial x'_j}(x') = \sum_{l=1}^{N} (\mathbb{I} - M)^{-1}_{i-N,\ell} \Lambda_{\ell j}. \tag{7.19}$$

Using the bounds on $(\mathbb{I} - M)^{-1}$ that come from Lemma 7.1, and the fact that Proposition 3.2 implies

$$|\Lambda_{\ell j}| \leqq \min(\rho_0^n(\lambda(\varepsilon_0))^{-2} k_0 \varepsilon_k \rho_{k+1}^{-1} B^{L_k}, \rho_0^n e^{-m(1-\eta_k)(1-\beta_k)\delta(\ell, j)})$$

for all x' in $W(r_k^4, \xi_k - \delta - 3; \{I_k\})$, it is easy to bound (7.19) by

$$\left| \frac{\partial \tilde{x}_i}{\partial x'_j}(x') \right| \leqq \min(B' \rho_0^n(\lambda(\varepsilon_0))^{-2} k_0 \varepsilon_k \rho_{k+1}^{-1} B^{L_k}, B \rho_0^n (\delta(i,j) + 1) 2^{\delta(i,j)} \cdot e^{-m(1-\eta_k)(1-\beta_k)\delta(i,j)}). \tag{7.20}$$

from which (4.4) (in the case $j \leqq N$, $i > N$) follows.

If $1 \leqq i \leqq N$, $\tilde{x}_i(x') = x_i + (\partial S/\partial y_{i+N}) \circ \tilde{y}(x')$, so applying the chain rule we obtain

$$\frac{\partial \tilde{x}_i}{\partial x'_j}(x') - \delta_{ij} = \sum_{\ell=1}^{N} \left[\frac{\partial^2 S}{\partial y_{i+N} \partial y_\ell} \circ \tilde{y}(x') \right] \cdot \delta_{\ell j} + \sum_{\ell=N+1}^{2N} \left[\frac{\partial^2 S}{\partial y_{i+N} \partial y_\ell} \circ \tilde{y}(x') \right] \cdot \frac{\partial \tilde{x}_\ell}{\partial x'_j}(x'). \tag{7.21}$$

Use Proposition 3.2 to bound derivatives of S, and use (7.18) and (7.20) to bound $\partial \tilde{x}_\ell / \partial x'_j$, for $N < \ell \leqq 2N$. If we do so we find (on $W(r_k^4, \xi_k - \delta - 3; \{I_k\})$).

$$\left| \frac{\partial \tilde{x}_i}{\partial x'_j}(x') - \delta_{ij} \right| \leqq \min \left(\rho_0^n k_0 \varepsilon_k \rho_{k+1}^{-1} B^{L_k} / (\lambda(\varepsilon_0))^2, B \rho_0^n (\delta(i,j) + 1) \cdot 2^{|i-j|} e^{-m(1-\eta_k)(1-\beta_k)\delta(i,j)} \right). \tag{7.22}$$

The remaining cases of (4.4) follow from (7.22).

Finally, we turn to the proof of (4.5). If $N < \ell \leqq 2N$, $\tilde{x}_\ell(x') = x'_\ell - (\partial S / \partial y_{\ell - N}) \circ \tilde{y}(x')$, and the chain rule and the definition of $\tilde{y}$ imply

$$\frac{\partial^2 \tilde{x}_\ell}{\partial x'_i \partial x'_j}(x') = - \sum_{p,n=1}^{2N} \left[\frac{\partial^3 S}{\partial y_{\ell-N} \partial y_p \partial y_n} \circ \tilde{y}(x') \right] \left[\frac{\partial \tilde{y}_p}{\partial x'_j}(x') \right] \left[\frac{\partial \tilde{y}_n}{\partial x'_i}(x') \right] - \sum_{n=N+1}^{2N} \left[\frac{\partial^2 S}{\partial y_{\ell-N} \partial y_n} \circ \tilde{y}(x') \right] \left[\frac{\partial^2 \tilde{x}_n}{\partial x'_i \partial x'_j}(x') \right]. \tag{7.23}$$

Assume for the moment that j is less than or equal to N. Define D^2 to be the $N \times N$ matrix with elements $D^2_{\ell j} = (\partial^2 \tilde{x}_{\ell+N} / \partial x'_i \partial x'_j)\ (x')$, and Λ^2 the $N \times N$ matrix with elements

$$\Lambda^2_{pj} = \sum_{q,n=1}^{2N} \left[\frac{\partial^3 S}{\partial y_p \partial y_q \partial y_n} \circ \tilde{y}(x') \right] \left[\frac{\partial \tilde{y}_q}{\partial x'_j}(x') \right] \left[\frac{\partial \tilde{y}_n}{\partial x'_i}(x') \right].$$

If M is the same matrix as in (7.16), then (7.23) may be rewritten as

$$D^2 = \Lambda^2 + M D^2 \quad \text{or} \quad D^2 = (\mathbb{I} - M)^{-1} \Lambda^2. \tag{7.24}$$

Using Proposition 3.2, a dimensional estimate, and (4.4) it is easy to show that for x' in $W(r_k^4, \xi_k - \delta - 3; \{I_k\})$,

$$|\Lambda^2_{pj}| \leqq B \rho_0^n [(r_k^2)_p - (r_k^3)_p]^{-1} e^{-m(1-\eta_k)(1-4\beta_k)\delta(i,j)}.$$

Using this bound, and the bound on $(\mathbb{I} - M)^{-1}$ that comes from Lemma 7.1 (and which we used previously in (7.18)), we find

$$\left| \frac{\partial^2 \tilde{x}_\ell}{\partial x'_i \partial x'_j}(x') \right| \leqq \sum_n |(\mathbb{I} - M)^{-1}_{\hat{\ell} n}| \, |\Lambda^2_{nj}| \leqq B \rho_0^n [(r_k^2)_{\hat{\ell}} - (r_k^3)_{\hat{\ell}}]^{-1} e^{-m(1-\eta_k)(1-4\rho_k)\delta(i,j)}, \tag{7.25}$$

where $\hat{\ell} = \ell (\text{mod}\, N)$. The bound in the case $j > N$ follows in analogous fashion, and we don't write out the details. This completes the proof of (4.5) for $\ell > N$.

If $1 \leqq \ell \leqq N$, then the definition of $\tilde{x}_\ell(x')$, and the chain rule imply

$$\frac{\partial^2 \tilde{x}_\ell}{\partial x_i' \partial x_j'}(x') = \sum_{p,q=1}^{2N} \left[\frac{\partial^2 S}{\partial y_{\ell+N} \partial y_p \partial y_q} \circ \tilde{y}(x') \right] \left[\frac{\partial \tilde{y}_p}{\partial x_j'}(x') \right] \left[\frac{\partial \tilde{y}_q}{\partial x_i'}(x') \right] + \sum_{p=N+1}^{2N} \left[\frac{\partial^2 S}{\partial y_{\ell+N} \partial y_p} \circ \tilde{y}(x') \right] \left[\frac{\partial^2 \tilde{x}_p}{\partial x_i' \partial x_j'}(x') \right]. \tag{7.26}$$

Bound derivatives of S by Proposition 3.2, bound $\partial^2 \tilde{x}_p / \partial x_i' \partial x_j'$ for $p > N$ by (7.25), and bound derivatives of $\tilde{y}$ by noting that $\partial \tilde{y}_p / \partial x_j' = \delta_{pj}$ if $p \leqq N$ and $\partial \tilde{y}_p / \partial x_j' = \partial \tilde{x}_p / \partial x_j'$ if $p > N$, and then apply (4.4). Combining these observations yields the remaining cases of (4.5).

Appendix A. The Inverse Function Theorem

We wish to invert the equations

$$I = I' + \frac{\partial S}{\partial \phi}(I', \phi),$$
$$\phi' = \phi + \frac{\partial S}{\partial I'}(I', \phi). \tag{A.1}$$

Standard analytic inverse function theorems guarantee that these maps can be inverted if they are $1-1$. Assume there exists I^1 and I^2 in $W(r_k^2, \xi_k - \delta - 1; \{I_k\})$, with

$$I^1 + \frac{\partial S}{\partial \phi}(I^1, \phi) = I^2 + \frac{\partial S}{\partial \phi}(I^2, \phi). \tag{A.2}$$

There is a path γ, contained in $W(r_k^2, \xi_k - \delta - 1; \{I_k\})$, joining I^1 to I^2 consisting of N pieces, γ_j, along which only the j^{th} coordinate of I varies, and the length of γ_j is $|I_j^1 - I_j^2|$. Then the fundamental theorem of calculus implies

$$\sum_{i=1}^{N} |I_i^1 - I_i^2| \leqq \sum_{i,j} \left| \int_{\gamma_j} dI_j'' \frac{\partial^2 S}{\partial \phi_i \partial I_j'}(I'', \phi) \right|. \tag{A.3}$$

But if $\sup \sum_i |(\partial^2 S / \partial \phi_i \partial I_j')| \leqq (1/2)$ on $W(r_k^2, \xi_k - \delta - 1; \{I_k\})$, for all j, this last sum is bounded by $(1/2) \sum_j |I_j^1 - I_j^2|$, implying $I^1 = I^2$. Thus, the first equation in (A.1) is invertible as $I' = I + \Xi(I, \phi)$ on the image of $W(r_k^2, \xi_k - \delta - 1; \{I_k\})$. If $\sup |(\partial S / \partial \phi_i)(I', \phi)| < c_1 \rho_{k+1} / k_0$ for all i, the range of the map must contain $W(r_k^3, \xi_k - \delta - 1; \{I_k\})$, and hence $\Xi'(I, \phi)$ is analytic on this domain.

To invert the second equation in (A.1) assume there are ϕ^1 and ϕ^2 in $W(r_k^2, \xi_k - \delta - 1; \{I_k\})$ such that

$$\phi^1 + \frac{\partial S}{\partial I'}(I', \phi^1) = \phi^2 + \frac{\partial S}{\partial I'}(I', \phi^2). \tag{A.4}$$

Pick, γ, a path contained in $W(r_k^2, \xi_k - \delta - 1; \{I_k\})$, joining ϕ^1 to ϕ^2, and consisting of N pieces, γ_j, along which only one component of ϕ varies. As before γ_j may be picked so that the length of γ_j is $|\phi_j^1 - \phi_j^2|$, and applying the fundamental theorem of calculus just as above we find that the map is $1-1$ if $\sup \sum_i |(\partial^2 S/\partial I_i' \partial \phi_j)| \leqq (1/2)$ for all j on $W(r_k^2, \xi_k - \delta - 1; \{I_k\})$. Thus, the second equation in (A.1) can be inverted as

$$\phi = \phi' + \Delta(I', \phi'), \tag{A.5}$$

with $\Delta(I', \phi')$ analytic on the range of the original map. If $\sup |(\partial S/\partial I_j')| < (1/2)$ for all j (with the supremum running over $W(r_k^2, \xi_k - \delta - 1; \{I_k\})$), $\Delta(I', \phi')$ must be analytic at least on $W(r_k^2, \xi_k - \delta - 2; \{I_k\})$.

Appendix B. The Proof of Lemma 7.1

Using the random walk expansion of [2] we write

$$(\mathbb{1} - M)_{ij}^{-1} = \sum_{\Omega: i \to j} \Lambda_{jj}^{-1} \left(\prod_{\ell \in L} \Lambda_{\ell\ell}^{-n(\ell, \Omega)} \right) \left(\prod_{s \in \Omega} \tilde{M}_s \right). \tag{B.1}$$

On the right-hand side of (A.1), $\Lambda_{jj} = 1 - M_{jj}$ and Ω is a random walk on the lattice $L = \{1, \ldots, N\}$, i.e. a set of pairs $\{(i_1, i_2), \ldots, (i_k, i_{k+1})\}$, $i_j \in \{1, \ldots, N\}$. Each of the pairs is referred to as a step, s, with $|\Omega|$ the number of steps in the walk, and Ω: $i \to j$ means $i_1 = i$, $i_{k+1} = j$. Finally $\tilde{M}_s = \tilde{M}_{i_j, i_{j+1}} = (1 - \delta_{i_j i_{j+1}}) M_{i_j i_{j+1}}$, $|s| = |i_{j+1} - i_j|$, and $n(j, \Omega)$ is the number of times j appears as the first element in some step in Ω.

The second bound on the right hand side of (7.14) follows by estimating the sum on the right hand side of (B.1) exactly as was done in proving Lemma 2.6 in [7], so we don't repeat that here.

The other bound in (7.14) follows by noting that if $N = |\Omega|$, and $L = \sum_{s \in \Omega} |s|$, there is exactly one walk with $M = 0$, and its contribution to the sum is $\Lambda_{jj} \delta_{ij}$. Every walk which contributes to (B.1) must have $L \geqq |i - j|$. Furthermore, since any walk with a step of zero length gives no contribution to the sum ($\tilde{M}_{jj} = 0$), there are at most $2^L 2^N$ walks with fixed L. We bound factors of $|\Lambda_{jj}|^{-1}$ by $(1-c)^{-1}$, and factors of $\prod_{s \in \Omega} |M_s|$ by c^N if $L(\Omega) \leqq \bar{L}$, or by $c_1^N e^{-\kappa L(\Omega)}$ if $L(\Omega) > \bar{L}$, where $\bar{L}$ is defined in (7.14). Thus,

$$\begin{aligned} |(\mathbb{1} - M)^{-1}{}_{ij} - \delta_{ij}| \leqq \delta_{ij} |(1 - \Lambda_{jj})| &+ \sum_{M=1}^{\infty} \sum_{L=|i-j|}^{\bar{L}} (1-c)^{-(N+1)} c^N 2^L 2^N \\ &+ \sum_{N=1}^{\infty} \sum_{L=\bar{L}+1}^{\infty} (1-c)^{-(N+1)} c_1^N e^{-\kappa L} 2^L 2^N, \end{aligned} \tag{B.2}$$

and the first bound on the right hand side of (7.14) follows by summing the geometric series.

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