Theorem 1 is measured by the identification space M^* whose elements are the leaves of N and the components of $M-N$. If M^* is metrizable, it can be shown to have inductive dimension 1. In this case the argument above can be replaced by an application of the Vietoris mapping theorem.

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ON THE EMBEDDABILITY OF THE REAL PROJECTIVE SPACES[1]

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In a paper of the same title, Massey [4] proved that if $2^{k-1}+2^{k-2}-1 \leqq n < 2^k$ then P_n cannot be differentiably embedded in R^{2^k}. By using the technique of Massey in a different way we can prove the following theorem which clearly includes Massey's.

THEOREM. *If $2^{k-1}<n<2^k$ then P_n cannot be embedded differentiably in Euclidean space of dimension 2^k.*

Besides the result of Massey, the main result in this direction is if $2^{k-1}<n<2^k$ then P_n cannot be embedded differentiably in R^{2^k-1}. Our result yields, in particular, that for P_{2^k+1}, the embedding in $R^{2^{k+1}+1}$ given by Hopf and James [1] is the best possible.

The following information from [3; 4] will be needed. Let M be a n-manifold differentiably embedded in R^{n+k+1}; and let $p: E \rightarrow M$ denote the bundle of unit normal vectors. Then there exist subalgebras $A^*(E, Z) \subset H^*(E, Z)$ and $A^*(E, Z_2) \subset H^*(E, Z_2)$ which satisfy the following conditions:

1. $A^0(E, G) = H^0(E, G)$,
2. $H^q(E, G) = A^q(E, G) + p^*(H^q(B, G))$ $(0<q<n+k)$,
3. $A^q(E, G) = 0$, $q \geqq n+k$,

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where $G=Z$ or Z_2. Moreover the algebra A^* is closed under all natural cohomology operations.

By the well-known theorem of Seifert and Whitney, the characteristic class of a normal bundle vanishes hence the Gysin sequence breaks up into parts of length three, $0\to H^{i+k}(M)\to^{p^*} H^{i+k}(E) \to^{\psi} H^i(M)\to 0$. Because of (2) ψ must be an isomorphism on A^*. Each element in $H^*(E)$ can be written in the form $p^*(b_1)+a\cdot p^*(b_2)$ where a is the unique element in A^k such that $\psi(a)=1$. For simplicity we will suppress the map p^* and write a general element in $H^*(E)$ as $b_1+a\cdot b_2$.

The proof of the theorem will consist in showing that P_m $(m=2^{k-1}+1)$ cannot be differentiably embedded in R^{2^k}. The result will then follow from the fact that P_m can be differentiably embedded in P_n for $n>2^{k-1}$.

A simple computation shows that the Stiefel-Whitney classes for the tangent bundle for $P_{2^{k-1}+1}$ are as follows: $W_0=1$, $W_2=\alpha^2$, $W_{m-1}=\alpha^{m-1}$ where α is the nonzero element of $H^1(P_m, Z_2)$ and all other $W_i=0$. Using the fact that $W\cdot\overline{W}=1$ we see that $\overline{W}_{2i}=\alpha^{2i}$, $0\leqq i\leqq(m-3)/2$ and $\overline{W}_j=0$ for all other j.

Suppose P_m is differentiably embedded in R^{2^k}. Let E be the bundle of unit normal vectors over P_m for this embedding. Then E is a $(m-3)$-sphere bundle and since $\overline{W}_{m-2}=0$, the characteristic class vanishes. Let $a\in A^{m-3}$ be the element such that $\psi(a)=1$. Suppose that the nonzero element of A^{m-2} is of the form $\alpha^{m-2}+a\cdot\alpha$; then, since $Sq^1(\alpha^{m-2}+a\cdot\alpha)=\alpha^{m-1}+a\cdot\alpha^2$ because a is an integer class, we have that the nonzero element of A^{m-1} must be $\alpha^{m-1}+a\cdot\alpha^2$. But $(\alpha^{m-2}+a\cdot\alpha)(\alpha^{m-1}+a\cdot\alpha^2)=a\cdot\alpha^m+a^2\cdot\alpha^3+a\cdot\alpha^m$. This element must be in A^{2m+3} which is zero by (3).

But it equals $a^2\cdot\alpha^3=(Sq^{m-3}a)\cdot\alpha^3=a\cdot\overline{W}_{m-3}\cdot\alpha^3$ by a result due to Liao [2], and this equals $a\cdot\alpha^m\neq 0$. Hence the nonzero element of A^{m-2} is of the form $a\cdot\alpha$. But then $Sq^1 a\cdot\alpha=a\cdot\alpha^2$ is the nonzero element of A^{m-1} and $(a\cdot\alpha^2)(a\cdot\alpha^1)\in A^{2m-3}=0$ but it, as before, equals $a\cdot\alpha^m$ which is not zero.

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