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ON THE EMERGENCE OF RANDOM INITIAL CONDITIONS IN FLUID LIMITS

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ABSTRACT. The paper presents a phenomenon occurring in population processes that start near zero and have large carrying capacity. By the classical result of Kurtz (1970), such processes, normalized by the carrying capacity, converge on finite intervals to the solutions of ordinary differential equations, also known as the fluid limit. When the initial population is small relative to carrying capacity, this limit is trivial. Here we show that, viewed at suitably chosen times increasing to infinity, the process converges to the fluid limit, governed by the same dynamics, but with a random initial condition. This random initial condition is related to the martingale limit of an associated linear birth and death process.

1. INTRODUCTION

Many models of population growth can be formulated, following the ideas of McKendrick [13] and Bartlett [3], [4], as Markovian birth and death (BD) processes. The classical Malthusian model can be viewed as a BD process with population birth rate λz and death rate μz depending linearly on the population size z, corresponding to constant *per capita* birth rate λ and death rate μ. This process cannot stabilize near any finite population size, and so non-linear density dependent BD processes $(Z_t,\, t \ge 0)$, with *per capita* birth rates $\lambda - (\lambda-\mu)g_1(z/K)$ and death rates $\mu + (\lambda-\mu)g_2(z/K)$, $z \in \mathbb{Z}_+$, have been introduced to remedy the defect. In such a formulation, $\lambda > \mu \ge 0$ are fixed constants, $g = g_1 + g_2$ is typically an increasing function with $g(0) = 0$ and $g(x_\infty) = 1$ for some $x_\infty \in (0,\infty)$, and K is a parameter, thought of as being large, that is representative of the *carrying capacity.*

The analogue of Verhulst's (1838) model has $g_1(x) = 0$ and $g_2(x) = x$ for all $x \ge 0$, and is known as the stochastic logistic process; it serves as our prototype. Ricker's [15] model has $g_1(x) = \frac{\lambda}{\lambda-\mu}(1-e^{-\alpha x})$ and $g_2(x) = 0$; that of Beverton & Holt [5] has $g_1(x) = \frac{\lambda}{\lambda-\mu}x/(x+m)$ and $g_2(x) = 0$, that of Hassell [7] has $g_1(x) = \frac{\lambda}{\lambda-\mu}\{1-(1+x/m)^{-c}\}$ and $g_2(x) = 0$, and that of Maynard–Smith & Slatkin [14] has $g_1(x) = \frac{\lambda}{\lambda-\mu}\{1-(1+(x/m)^c)^{-1}\}$ and $g_2(x) = 0$.

In these models, when K is large and the initial population size Z_0 is relatively small, the birth rate exceeds the death rate, and the population size begins by growing exponentially, avoiding extinction in the early stages with a significant probability. As the size gets larger, the net birth rate decreases and population growth slows down, settling around the carrying capacity Kx_∞. The population typically fluctuates around the carrying capacity for a very long period of time, until, by chance, it eventually dies out.

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This qualitative behavior can be made precise by considering the normalized *density* process $\overline{Z}_t = Z_t/K$. By the result of Kurtz [12], Theorem 2.11, for any fixed $T > 0$,

$$\sup_{0\le t\le T} \big|\overline{Z}_t - x_t\big| \xrightarrow[K\to\infty]{d} 0, \tag{1.1}$$

where $x = (x_t)_{t\in\mathbb{R}_+}$ is the solution of the o.d.e., or fluid limit,

$$\dot{x}_t \;=\; (\lambda-\mu)x_t\big(1-g(x_t)\big), \quad t\ge 0, \tag{1.2}$$

subject to the initial condition $x_0 := \lim_{K\to\infty}\overline{Z}_0$.

When the initial population size Z_0 is proportional to K, the initial condition x_0 is positive, and the density process $\overline{Z}$ converges to the corresponding positive solution of (1.2). In particular, this implies that extinction prior to any *fixed* time T has vanishing probability. As T increases, the solution of (1.2) approaches its stable equilibrium at x_∞. Since $\overline{Z}$ is a transient Markov chain, it is absorbed at zero eventually. However, large deviation analysis (see, for example, Barbour [1] and Jagers & Klebaner [8]) shows that $\overline{Z}$ does not leave a vicinity of x_∞ for a long period of time, with mean growing exponentially with K.

If the initial population size Z_0 is fixed with respect to K, so that $x_0 = 0$, the limit (1.1) implies that $\overline{Z}$ converges to the zero function on any bounded interval. This implies that those trajectories of Z that stay positive up to time T remain of smaller order than K during that time, so that it takes longer to grow to level comparable with K. Other than that, the convergence (1.1) reveals no information about the behaviour of those trajectories that eventually reach the carrying capacity.

In the present paper, we derive a limit theorem showing that, if the initial population is small when compared to K, so that $x_0 = 0$, the density process nonetheless converges over increasing time intervals to a nontrivial solution of the same o.d.e. (1.2), but now with a *random* initial condition.

The emergence of a random initial condition in the limit can already be seen in the simple model of pure birth processes. This case admits a one page proof, involving nothing more complicated than weighted sums of i.i.d. exponential random variables (Section 3). A completely different approach is required in the more general setup of Theorem 2.1. Here, the proof relies on the approximation of the non-linear BD process by a linear BD process during the initial stages, and by the non-linear deterministic dynamics thereafter.

As pointed out in Barbour *et al.* [2], the idea of such an approximation is not new, going back to the papers of Kendall [9] and Whittle [18] in the mid 1950's. However, its rigorous justification in many of the models where it has heuristically been invoked can be quite involved. Non-linear multidimensional Markov population processes were considered recently in [2], where it was established that, after an initial build up phase, the random population follows the solution of the corresponding deterministic equations, but with a *random* time shift ([2], Theorem 1.1). The proof in [2] relies on an abstract coupling construction (Thorisson [16], Theorem 7.3).

Here, we revisit the one-dimensional setting, in which the argument can be made much simpler; in particular, there is a very neat explicit expression for the random initial condition to be used with the fluid approximation. In addition, the argument can be carried through under somewhat weaker assumptions than are used in [2].

2. The main result

Defining $g_l^+(x) := \sup_{0\le y\le x} |g_l(y)|$, $l = 1, 2$, and recalling that $g = g_1 + g_2$, we work under the following assumptions:

$$\begin{array}{ll} \text{(i)} & g(0) = 0,\ g(x_\infty) = 1 \text{ for some } x_\infty < \infty, \text{ and } g(x) < 1 \text{ for } 0 < x < x_\infty; \\ \text{(ii)} & xg(x) \text{ is uniformly Lipschitz on } [0, x_\infty], \text{ with constant } \theta \ge 1; \\ \text{(iii)} & g^+(x) := g_1^+(x) + g_2^+(x) \text{ is such that } x^{-1}g^+(x) \text{ is integrable from } 0. \end{array} \tag{2.1}$$

In view of (2.1) (ii), the o.d.e. (1.2), with initial condition $x_s = x$, has a unique solution. It is given implicitly by

$$G(x_t) - G(x) \ := \ \int_x^{x_t} \frac{du}{u(1-g(u))} \ = \ (\lambda-\mu)(t-s), \tag{2.2}$$

where the function G is determined up to an additive constant; for any $0 < a < x_\infty$, we can for instance take

$$G(x) \ = \ G_a(x) \ := \ \int_a^x \frac{du}{u(1-g(u))} + \log a \ = \ \log x + H_a(x), \tag{2.3}$$

with

$$H_a(x) \ := \ \int_a^x \frac{g(u)\,du}{u(1-g(u))}. \tag{2.4}$$

With this notation, we can formulate our main result as follows.

Theorem 2.1. *For $\lambda > \mu > 0$ and for $0 \le \alpha < 1$, let $(Z^{(K)}, K \ge 1)$ be a sequence of BD processes with per capita birth rates $\lambda - (\lambda-\mu)g_1(z/K)$ and death rates $\mu + (\lambda-\mu)g_2(z/K)$, started at the initial population size $Z_0 = \lfloor K^\alpha \rfloor$. Let $\overline{Z}^{(K)}(t) := K^{-1}Z^{(K)}(t)$ and $t_1(K) := (\lambda-\mu)^{-1}\log K^{1-\alpha}$. Then, under Assumptions (2.1), the sequence of processes $\overline{Z}^{(K)}(t_1(K)+\cdot)$ converges weakly as $K\to\infty$, in the uniform topology on bounded intervals, to the solution of the o.d.e.* (1.2) *started with the initial condition*

$$w_0 \ := \ \begin{cases} G_0^{-1}\big(\log W\big), & \alpha = 0; \\ G_0^{-1}(0), & \alpha \in (0,1), \end{cases} \tag{2.5}$$

where W is a random variable with $\mathbb{P}[W = 0] = \mu/\lambda$ and $\mathbb{P}[W > w] = (1-\mu/\lambda)e^{-(1-\mu/\lambda)w}$, $w \ge 0$.

Remark 2.2.

(1) The function G_0 is well defined, because of Assumption (2.1) (iii), and is strictly increasing, having $\lim_{x\to 0+} G_0(x) = -\infty$ and $\lim_{x\to x_\infty} G_0(x) = \infty$. The latter limit holds, because $0 \le x_\infty - xg(x) \le \theta(x_\infty - x)$ in $0 \le x \le x_\infty$, from Assumption (2.1) (ii), implying that

$$0 \ \le \ s - sg(s) = \ (s - sg(s)) - (x_\infty - x_\infty g(x_\infty)) \ \le \ (\theta-1)(x_\infty - s), \quad 0 < s < x_\infty, \tag{2.6}$$

and that $sg(s) \geq \frac{1}{2}x_\infty$ in $(1-1/2\theta)x_\infty \leq s \leq x_\infty$. From this it follows that

$$\int_{(1-1/2\theta)x_\infty}^{x} \frac{g(s)}{s(1-g(s))}\,ds \;>\; \frac{1}{x_\infty}\int_{(1-1/2\theta)x_\infty}^{x} \frac{sg(s)}{s-sg(s)}\,ds \;>\; \frac{1}{2}\int_{(1-1/2\theta)x_\infty}^{x} \frac{ds}{(\theta-1)(x_\infty-s)}$$
$$= \frac{1}{2(\theta-1)}\log\Big(\frac{x_\infty/2\theta}{x_\infty-x}\Big) \xrightarrow[x\to x_\infty]{} \infty.$$

Hence $G := G_0$ is a bijection from $(0,x_\infty)$ to $\mathbb{R}$, with bounded continuous inverse $G^{-1}\colon \mathbb{R} \mapsto (0,x_\infty)$.

In particular, $g(x) = x^p$ with $p>0$, satisfies our assumptions, with $G(x) = \frac{1}{p}\log\frac{x^p}{1-x^p}$, giving

$$w_0 \;=\; \begin{cases} \Big(\frac{W^p}{1+W^p}\Big)^{\frac{1}{p}}, & \alpha = 0; \\ \big(\frac{1}{2}\big)^{\frac{1}{p}}, & \alpha \in (0,1). \end{cases}$$

The stochastic logistic process corresponds to taking $p=1$, and yields the initial condition $w_0 = \frac{W}{1+W}$ in (1.2).

(2) It follows from (4.12) below that W has the distribution of the a.s. limit of the martingale $e^{-(\lambda-\mu)t}Y_t$, when Y is the linear BD process with *per capita* birth and death rates λ and μ, starting with $Y_0 = 1 = K^0$. If $Y^{(K)}$ denotes the same process, but with initial condition $Y_0^{(K)} = \lfloor K^\alpha \rfloor$ for some $0<\alpha<1$, then the martingale $e^{-(\lambda-\mu)t}(Y_t^{(K)}/Y_0^{(K)})$ has mean 1 and variance of order $K^{-\alpha}$ as $K\to\infty$, explaining why W is replaced by 1 in (2.5) when $\alpha \in (0,1)$.

(3) Theorem 2.1 implies that the trajectories that survive early extinction reach the magnitude of the carrying capacity at times of order $\frac{1}{\lambda-\mu}\log K^{1-\alpha}$. For $\alpha=0$, since $G^{-1}(-\infty)=0$, it follows from (2.5) that the trajectories that vanish are those corresponding to the set $\{W=0\}$. This set is exactly the set of extinction of the linear branching process Y. For $\alpha>0$, the probability of early extinction vanishes as $K\to\infty$ for both $Z^{(K)}$ and $Y^{(K)}$.

(4) The Lipschitz assumption on the function $xg(x)$ can be replaced by assuming that it is increasing, and has finite derivative at x_∞: see Remark 4.1.

3. A preview: pure Birth Process

This subsection is a short detour from our main setup, which provides an additional insight into the structure of the limit. Consider a non-linear pure birth process Z that jumps from an integer z to $z+1$ at rate $\lambda(z) = az(1-g(z/K))$, $z = 1,2,\dots,[Kx_\infty]$, where $a>0$ is a constant and g is a function satisfying the assumption of Theorem 2.1. Let $Z_0 = 1$ and define $\lambda(z)=0$ for $z > [Kx_\infty]$, so that Z is absorbed, once it exceeds the level $[Kx_\infty]$. The holding time in state z equals $\tau_z/\lambda(z)$, where $\tau_z \sim \mathrm{Exp}(1)$ and τ_z's are i.i.d. Consider also a linear pure birth process Y with $Y_0 = 1$ and birth rates az, $z\in\mathbb{Z}_+$. It is well known that $e^{-at}Y_t$ is an L^2 bounded martingale which has an almost sure limit W, and that W has Exp(1) distribution.

Proposition 3.1. *Let G be defined as in Theorem 2.1, and $Z_0 = 1$ then*

$$\frac{1}{K}Z_{\frac{1}{a}\log K} \xrightarrow[K\to\infty]{d} G^{-1}(\log W). \tag{3.1}$$

Proof. Let Y and Z be defined as above, using the same sequence of random variables (τ_i). Due to monotonicity of a pure birth process for $t \geq 0$

$$\{Z_t > n\} = \left\{\sum_{i=1}^{n} \frac{1}{\lambda_i}\tau_i < t\right\} = \{T_n < t\}, \quad n \leq [x_\infty K]$$
$$\{Y_t > n\} = \left\{\sum_{i=1}^{n} \frac{1}{ai}\tau_i < t\right\} = \{\widetilde{T}_n < t\}, \quad n \in \mathbb{N},$$

where T_n and $\widetilde{T}_n$ are the times of the n-th jump of Z and Y respectively:

$$T_n = \sum_{i=1}^{n} \frac{1}{\lambda_i}\tau_i \quad \text{and} \quad \widetilde{T}_n = \sum_{i=1}^{n} \frac{1}{ai}\tau_i. \tag{3.2}$$

Note that the coefficients in the first sum T_n in (3.2) depend on K, whereas in the second sum $\widetilde{T}_n$ they do not. Therefore we establish convergence of the second sum first, and then show that their difference converges to a constant. Since $Y_{\widetilde{T}_n} = n$, $\lim_{n\to\infty} \widetilde{T}_n = \infty$ and $\lim_{t\to\infty} e^{-at}Y_t = W$

$$\widetilde{T}_n - \frac{1}{a}\log n = -\frac{1}{a}\log\left(e^{-a\widetilde{T}_n}Y_{\widetilde{T}_n}\right) \xrightarrow[n\to\infty]{a.s.} -\frac{1}{a}\log W. \tag{3.3}$$

Let us show that for any $x \in (0, x_\infty)$

$$T_{[xK]} - \widetilde{T}_{[xK]} \xrightarrow[K\to\infty]{L^2} \frac{1}{a}\int_0^x \frac{1}{s}\frac{g(s)}{1-g(s)}ds. \tag{3.4}$$

Indeed, denoting by $h(s) = \frac{g(s)}{s(1-g(s))}$, we have

$$a\mathbb{E}\left(T_{[xK]} - \widetilde{T}_{[xK]}\right) = \sum_{i=1}^{[xK]} \frac{1}{i}\frac{g(i/K)}{1-g(i/K)}\mathbb{E}\tau_i = \sum_{i=1}^{[xK]} \frac{1}{i/K}\frac{g(i/K)}{1-g(i/K)}\frac{1}{K}$$
$$= \sum_{i=1}^{[xK]} \frac{1}{K}h(i/K) \xrightarrow[K\to\infty]{} \int_0^x h(s)ds,$$

where we used Assumption (2.1) (iii). Similarly,

$$a^2 \operatorname{Var}\left(T_{[xK]} - \widetilde{T}_{[xK]}\right) = \sum_{i=1}^{[xK]} \left(\frac{1}{i}\frac{g(i/K)}{1-g(i/K)}\right)^2 = \frac{1}{K}\sum_{i=1}^{[xK]} \frac{1}{K}h^2(i/K) \xrightarrow[K\to\infty]{} 0. \tag{3.5}$$

This can be seen as follows. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that $0 \leq g(x) \leq \varepsilon$ for all $x \in [0, \delta]$. This is possible because g is continuous and $g(0) = 0$. It is clear that $\frac{1}{K}\sum_{i=[\delta K]}^{[xK]} \frac{1}{K}h^2(i/K) \xrightarrow[K\to\infty]{} 0$, because the function $h^2(s) = (g(s)/s)^2$ is bounded and integrable on $[\delta, x]$. The residual sum satisfies

$$\frac{1}{K}\sum_{i=1}^{[\delta K]} \frac{1}{K}h^2(i/K) \leq \frac{1}{K}\Big(\max_{1\leq i\leq[\delta K]} h(i/K)\Big)\sum_{i=1}^{[\delta K]} \frac{1}{K}h(i/K).$$

By Assumption (2.1) (iii), the sum in the right hand side converges to $\int_0^\delta h(s)ds < \infty$ and

$$\frac{1}{K}\Big(\max_{1\le i\le[\delta K]} h(i/K)\Big) \le C\frac{1}{K}\max_{1\le i\le[\delta K]}\frac{g(i/K)}{i/K} \le C\max_{1\le i\le[\delta K]} g(i/K) \le C\varepsilon,$$

with a constant C independent of K. Thus the convergence in (3.5) holds by arbitrariness of ε and the limit (3.4) follows.

Now (3.3) and (3.4) imply

$$T_{[xK]} - \frac{1}{a}\log K \xrightarrow[K\to\infty]{\mathbb{P}} \frac{1}{a}\left(\int_0^x \frac{1}{s}\frac{g(s)}{1-g(s)}ds + \log x - \log W\right) = \frac{1}{a}\Big(G(x) - \log W\Big).$$

Since W has a continuous distribution,

$$\mathbb{P}\left(\frac{1}{K}Z_{\frac{1}{a}\log K} > x\right) = \mathbb{P}\left(Z_{\frac{1}{a}\log K} > [xK]\right) = \mathbb{P}\left(T_{[xK]} < \frac{1}{a}\log K\right) \xrightarrow[K\to\infty]{} \mathbb{P}\Big(G(x) - \log W < 0\Big) = \mathbb{P}\Big(G^{-1}(\log W) > x\Big), \quad \forall x \in (0, x_\infty)$$

which proves (3.1). □

4. Proof of Theorem 2.1

The main idea of the proof is to construct the process $Z^{(K)}$, together with an auxiliary linear BD process $Y^{(K)}$, on the same probability space, in such a way that $Z^{(K)}$ is well approximated by $Y^{(K)}$ on the interval $[0, t_0(K)]$, where $t_0(K) := \frac{1}{\lambda-\mu}\log K^c$, and $c > 0$ is a constant such that $\alpha + c$ is less than, but close enough to 1: more precisely, such that

$$0 < \{1 - (\alpha + c)\}(1 + \theta) < 1/2, \tag{4.1}$$

for θ as in Assumption (2.1) (ii). Thereafter, we extrapolate this approximation on $[t_0(K), t_1(K)]$, using the flow generated by the o.d.e. (1.2). Our proof shows that this approximation is enough to establish Theorem 2.1. The main effort is in proving that

$$\frac{1}{K}Z_{t_1} \xrightarrow[K\to\infty]{d} w_0, \tag{4.2}$$

where w_0 is as in (2.5). Once this is done, the rest is immediate from Kurtz [12], Theorem 2.11.

To this end, for each K, we construct a process $(Y^{(K)}, Z^{(K)}, U^{(K)}, V^{(K)})$ with the following properties:

(a) $Y_0^{(K)} = Z_0^{(K)} = U_0^{(K)} = V_0^{(K)} = \lfloor K^\alpha \rfloor$;

(b) $Y^{(K)}$ is the linear BD process with *per capita* birth rate λ and death rate μ;

(c) $Z^{(K)}$ is the non-linear BD process with *per capita* birth rate $\lambda - (\lambda-\mu)g_1(z/K)$ and death rate $\mu + (\lambda-\mu)g_2(z/K)$, $z \in \mathbb{Z}_+$;

(d) $U^{(K)}$ is the linear BD process with *per capita* birth rate $\{\lambda + \lambda_K\}$ and death rate $\{\mu - \mu_K\}$, where

$$\lambda_K := (\lambda-\mu)g_1^+\big(K^{\alpha+c+\eta-1}\big), \quad \mu_K := (\lambda-\mu)g_2^+\big(K^{\alpha+c+\eta-1}\big),$$

and where η is a constant satisfying $0 < \eta < 1 - \alpha - c$;

(e) $V^{(K)}$ is the linear BD process with *per capita* birth rate $\{\lambda-\lambda_K\}$ and death rate $\{\mu+\mu_K\}$;

(f) $V_t^{(K)} \le Y_t^{(K)} \le U_t^{(K)}$ for all $t \ge 0$;

(g) $V_t^{(K)} \le Z_t^{(K)} \le U_t^{(K)}$ for $t \le \tau^{(K)}$, where $\tau^{(K)}$ is the first time at which $Z^{(K)}$ hits the level $K^{\alpha+c+\eta}$:

$$\tau^{(K)} \;:=\; \inf\left\{t \ge 0 : Z_t^{(K)} \ge K^{\alpha+c+\eta}\right\}. \tag{4.3}$$

The coupling is described in Section 4.3.

Suppressing the dependence on K where possible, define $\overline{Y}_t := \frac{1}{K}Y_t$ and another auxiliary process

$$\widetilde{Z}_t \;:=\; \begin{cases} \overline{Y}_t, & t \le t_0; \\ \phi_{t_0,t}(\overline{Y}_{t_0}), & t > t_0, \end{cases} \tag{4.4}$$

where $\phi_{s,t}(x)$ is the flow generated by the o.d.e. (1.2); that is, using (2.2),

$$G(\phi_{s,t}(x)) - G(x) \;=\; (\lambda-\mu)(t-s), \tag{4.5}$$

if $x > 0$, and $\phi_{s,t}(0) = 0$ for all $t > s$.

It thus follows from (4.4), with our choices of t_0 and t_1, that, on the set $\{\overline{Y}_{t_0} > 0\}$,

$$G(\widetilde{Z}_{t_1}) \;=\; G(\overline{Y}_{t_0}) + (1-\alpha-c)\log K;$$

the equation is also trivially true when $\{\overline{Y}_{t_0} = 0\}$, since then $\widetilde{Z}_{t_1} = \overline{Y}_{t_0} = 0$, and $G(0) = -\infty$. Thus, introducing $W_K := K^{1-\alpha-c}\chi_\alpha(K)\overline{Y}_{t_0} = e^{-(\lambda-\mu)t_0}(Y_{t_0}/Y_0)$, with $\chi_\alpha(K) := K^\alpha/\lfloor K^\alpha\rfloor$, we obtain

$$G(\widetilde{Z}_{t_1}) \;=\; \log W_K - \log\chi_\alpha(K) + H\big(K^{\alpha+c-1}W_K/\chi_\alpha(K)\big). \tag{4.6}$$

It follows from Remark 2.2 (2) that

$$W_K \;=\; e^{-(\lambda-\mu)t_0(K)}(Y_{t_0(K)}^{(K)}/Y_0^{(K)}) \;\xrightarrow[K\to\infty]{d}\; \begin{cases} W, & \text{if } \alpha = 0; \\ 1 & \text{if } \alpha \in (0,1). \end{cases} \tag{4.7}$$

Hence, since H is continuous and $H(0) = 0$, since $\alpha + c < 1$ and since $\lim_{K\to\infty}\chi_\alpha(K) = 1$, the last term in (4.6) converges in distribution to zero as $K \to \infty$. Thus, again using (4.7) in (4.6), and because the function G^{-1} is continuous, it follows that

$$\widetilde{Z}_{t_1(K)}^{(K)} \xrightarrow[K\to\infty]{d} G^{-1}\big(\mathbf{1}_{\{\alpha=0\}}\log W\big).$$

It remains to show that $\widetilde{Z}^{(K)}$ is an appropriate approximation for $\overline{Z}^{(K)}$ at time $t_1(K)$; we use the coupling to show that

$$\overline{Z}_{t_1(K)}^{(K)} - \widetilde{Z}_{t_1(K)}^{(K)} \;\xrightarrow[K\to\infty]{d}\; 0.$$

Since

$$\big|\overline{Z}_{t_1} - \widetilde{Z}_{t_1}\big| \;=\; \big|\overline{Z}_{t_1} - \phi_{t_0,t_1}(\overline{Y}_{t_0})\big| \;\le\; \big|\overline{Z}_{t_1} - \phi_{t_0,t_1}(\overline{Z}_{t_0})\big| + \big|\phi_{t_0,t_1}(\overline{Z}_{t_0}) - \phi_{t_0,t_1}(\overline{Y}_{t_0})\big|,$$

it is enough to show that

$$\overline{Z}_{t_1(K)}^{(K)} - \phi_{t_0,t_1}(\overline{Z}_{t_0}^{(K)}) \;\xrightarrow[K\to\infty]{d}\; 0, \tag{4.8}$$

and, using the coupling of $Y^{(K)}, Z^{(K)}, U^{(K)}$ and $V^{(K)}$ constructed in Section 4.3, that

$$\phi_{t_0,t_1}(\overline{Z}^{(K)}_{t_0}) - \phi_{t_0,t_1}(\overline{Y}^{(K)}_{t_0}) \xrightarrow[K\to\infty]{d} 0. \tag{4.9}$$

These two relations are proved in Sections 4.1 and 4.2. The proof of (4.2) is then complete.

Before proving (4.8) and (4.9), we collect some useful facts. First, for $0 < x < 1$, we have

$$\tfrac12 g^+(x)\log(1/x) \;\le\; \int_x^{\sqrt{x}} \frac{g^+(s)}{s}\,ds \;\le\; \int_0^{\sqrt{x}} \frac{g^+(s)}{s}\,ds,$$

so that, from Assumption (2.1) (iii), $\lim_{x\to0} g^+(x)\log(1/x) = 0$. This, in particular, implies that $\lim_{K\to\infty} g_l^+(K^{-\gamma})\log K = 0$ for any $\gamma > 0$ and $l \in \{1,2\}$, and hence that

$$\lim_{K\to\infty} (\lambda_K + \mu_K)\log K \;=\; 0; \tag{4.10}$$

it also follows that g is continuous at 0. Then, in view of Assumptions (2.1) (i) and (ii), we have

$$xg(x) \;\ge\; \max\{-\theta x, x_\infty + \theta(x - x_\infty)\} \;=:\; f_g(x), \quad 0 \le x \le x_\infty. \tag{4.11}$$

Note that f_g is convex, and that $\tilde g(x) := x^{-1}f_g(x)$ is increasing, since $\theta \ge 1$.

Next, let $Z^{(\gamma,\beta)} := (Z^{(\gamma,\beta)}_t,\, t \ge 0)$ denote the linear birth and death process with *per capita* birth rate γ and death rate β, and with $Z^{(\gamma,\beta)}_0 = 1$; suppose that $\gamma > \beta$. Then, writing $\eta_t := (\beta/\gamma)e^{-(\gamma-\beta)t}$, we have

$$\begin{aligned} \frac{\beta}{\gamma} - \mathbb{P}[Z^{(\gamma,\beta)}_t = 0] \;&=\; \frac{\eta_t}{1-\eta_t}\left(1 - \frac{\beta}{\gamma}\right); \\ \mathbb{P}[Z^{(\gamma,\beta)}_t > r] \;&=\; \frac{1-\beta/\gamma}{1-\eta_t}\left\{1 - \left(\frac{\gamma}{\beta} - 1\right)\frac{\eta_t}{1-\eta_t}\right\}^r, \quad r \ge 1; \end{aligned} \tag{4.12}$$

furthermore, $\mathbb{E}Z^{(\gamma,\beta)}_t = e^{(\gamma-\beta)t}$ and $\mathrm{Var}\, Z^{(\gamma,\beta)}_t \le \frac{\gamma+\beta}{\gamma-\beta}e^{2(\gamma-\beta)t}$ (see Grimmett & Stirzaker (1982, p.159)). The process $(Z^{(\gamma,\beta;M)}_t,\, t \ge 0)$ with the same birth and death rates, but starting with $Z^{(\gamma,\beta;M)}_0 = M$, is distributed as the sum of M independent copies of $Z^{(\gamma,\beta)}$; hence, by Chebyshev's inequality,

$$\mathbb{P}[|M^{-1}e^{-(\gamma-\beta)t}Z^{(\gamma,\beta;M)}_t - 1| \ge \varepsilon] \;\le\; \frac{\gamma+\beta}{M\varepsilon^2(\gamma-\beta)}. \tag{4.13}$$

Now recall the well known semimartingale decomposition of Z:

$$Z_t \;=\; Z_0 + (\lambda - \mu)\int_0^t \Big(Z_s - Z_s g(Z_s/K)\Big)ds + M_t, \quad t \ge 0, \tag{4.14}$$

(see, for example, Klebaner [11] p.360), where $M = (M_t)_{t\ge0}$ is a martingale with $M_0 = 0$ a.s. and

$$\langle M\rangle_t \;=\; \int_0^t \Big((\lambda+\mu)Z_s + (\lambda-\mu)Z_s(g_2(Z_s/K) - g_1(Z_s/K))\Big)\,ds.$$

Dividing both sides of (4.14) by K, we see that the density process $\overline{Z}_t = K^{-1}Z_t$ satisfies the equation

$$\overline{Z}_t \;=\; \overline{Z}_0 + (\lambda - \mu)\int_0^t \Big(\overline{Z}_s - \overline{Z}_s g(\overline{Z}_s)\Big)\,ds + \frac{1}{\sqrt{K}}\widehat{M}_t, \quad t \ge 0, \tag{4.15}$$

where the martingale $\widehat{M}$ has zero mean and predictable quadratic variation

$$\begin{aligned}\langle\widehat{M}\rangle_t &= \int_0^t \Big((\lambda+\mu)\overline{Z}_s + (\lambda-\mu)\overline{Z}_s(g_2(\overline{Z}_s) - g_1(\overline{Z}_s)\Big)\,ds \\ &= \int_0^t \Big((\lambda+\mu)\overline{Z}_s + (\lambda-\mu)\overline{Z}_s g(\overline{Z}_s)\Big)\,ds. \end{aligned} \tag{4.16}$$

Taking expectations in (4.15), and recalling (4.11), we see that

$$\begin{aligned}\mathbb{E}\overline{Z}_t &= \mathbb{E}\overline{Z}_0 + (\lambda-\mu)\int_0^t \Big(\mathbb{E}\overline{Z}_s - \mathbb{E}\{\overline{Z}_s g(\overline{Z}_s)\}\Big)\,ds \\ &\le \mathbb{E}\overline{Z}_0 + (\lambda-\mu)\int_0^t \Big(\mathbb{E}\overline{Z}_s - \mathbb{E}f_g(\overline{Z}_s)\Big)\,ds \\ &\le \mathbb{E}\overline{Z}_0 + (\lambda-\mu)\int_0^t \Big(\mathbb{E}\overline{Z}_s - f_g(\mathbb{E}\overline{Z}_s)\Big)\,ds, \end{aligned} \tag{4.17}$$

this last because f_g is convex. Hence $\mathbb{E}\overline{Z}_t$ satisfies the integral inequality

$$\mathbb{E}\overline{Z}_t \le \mathbb{E}\overline{Z}_0 + (\lambda-\mu)\int_0^t \Big(\mathbb{E}\overline{Z}_s - \mathbb{E}\overline{Z}_s\tilde{g}(\mathbb{E}\overline{Z}_s)\Big)ds, \tag{4.18}$$

so that $\mathbb{E}\overline{Z}_t \le \tilde{\phi}_{0,t}(\mathbb{E}\overline{Z}_0)$, where $\tilde{\phi}_{0,t}(x)$ is the flow generated by replacing g by $\tilde{g}$ in the o.d.e. (1.2). Thus, in particular, since $\tilde{g}(x_\infty) = 1$, $0 \le \mathbb{E}\overline{Z}_t \le x_\infty$ for all $t \ge 0$. This in turn implies, using (4.15), that

$$(\lambda-\mu)\int_{t_0}^t \mathbb{E}\{\overline{Z}_s g(\overline{Z}_s)\}\,ds = \mathbb{E}\overline{Z}_{t_0} + (\lambda-\mu)\int_{t_0}^t \mathbb{E}\overline{Z}_s\,ds - \mathbb{E}\overline{Z}_t \le x_\infty\{1+(\lambda-\mu)(t-t_0)\}. \tag{4.19}$$

4.1. **Proof of** (4.8)**.** Write $\delta_t := \overline{Z}_t - \widehat{Z}_t$, where $\widehat{Z}_t := \phi_{t_0,t}(\overline{Z}_{t_0})$ satisfies the equation

$$\widehat{Z}_t = \overline{Z}_{t_0} + (\lambda-\mu)\int_{t_0}^t \Big(\widehat{Z}_s - \widehat{Z}_s g(\widehat{Z}_s)\Big)ds, \quad t \ge t_0,$$

so that, using (4.15),

$$\delta_t = (\lambda-\mu)\int_{t_0}^t \Big(\delta_s + \widehat{Z}_s g(\widehat{Z}_s) - \overline{Z}_s g(\overline{Z}_s)\Big)\,ds + \frac{1}{\sqrt{K}}\big(\widehat{M}_t - \widehat{M}_{t_0}\big).$$

Applying Itô's formula to δ_t^2 as a function of δ_t, (see, e.g., eq. (8.58) [11]) we obtain

$$\delta_t^2 = 2(\lambda-\mu)\int_{t_0}^t \Big(\delta_s^2 - \big(\overline{Z}_s g(\overline{Z}_s) - \widehat{Z}_s g(\widehat{Z}_s)\big)(\overline{Z}_s - \widehat{Z}_s)\Big)\,ds + \frac{1}{K}\sum_{t_0 \le s \le t}\Big(\Delta\widehat{M}_s\Big)^2. \tag{4.20}$$

Taking expectations of both sides, and using Assumption (2.1) (ii), we obtain the inequality

$$\mathbb{E}\delta_t^2 \le 2(\lambda-\mu)(1+\theta)\int_{t_0}^t \mathbb{E}\delta_s^2\,ds + \frac{1}{K}\int_{t_0}^t \mathbb{E}\Big((\lambda+\mu)\overline{Z}_s + (\lambda-\mu)\overline{Z}_s g(\overline{Z}_s)\Big)ds, \quad t \ge t_0; \tag{4.21}$$

for the last integral, we have used the formulae $\sum_{s\le t}(\Delta\widehat{M}_s)^2 = \big[\widehat{M}\big]_t$ and $\mathbb{E}\big[\widehat{M}\big]_t = \mathbb{E}\big\langle\widehat{M}\big\rangle_t$, together with (4.16). Using $\mathbb{E}\overline{Z}_s \le x_\infty$ and (4.19) in (4.21), we obtain

$$\mathbb{E}\delta_t^2 \;\le\; 2(\lambda-\mu)(1+\theta)\int_{t_0}^t \mathbb{E}\delta_s^2\,ds + \frac{1}{K}x_\infty\{1+2\lambda(t-t_0)\}.$$

The Grönwall inequality now yields

$$\mathbb{E}\delta_{t_1}^2 \;\le\; \frac{1}{K}x_\infty\{1+2\lambda(t_1-t_0)\}e^{2(\lambda-\mu)(1+\theta)(t_1-t_0)}.$$

Since $(\lambda-\mu)(t_1-t_0) = \log K^{1-\alpha-c}$ and by the choice (4.1) of c, it follows that

$$\mathbb{E}\delta_{t_1}^2 \;\le\; x_\infty\left\{1+\frac{2\lambda}{\lambda-\mu}\log K\right\}K^{2(1+\theta)(1-\alpha-c)-1} \xrightarrow[K\to\infty]{} 0,$$

and (4.8) is proved.

4.2. **Proof of** (4.9)**.** In this section, we use the coupling of (Y,Z,U,V) established in Section 4.3.

First, we show that $\lim_{K\to\infty}\mathbb{P}[\tau^{(K)}\le t_0(K)] = 0$, where $\tau^{(K)}$ is as in (4.3). Because, from property (g), $Z_t\le U_t$ for all $0\le t\le t^{(K)}$, we have

$$\begin{aligned}\mathbb{P}[\tau^{(K)}\le t_0(K)] \;&\le\; \mathbb{P}\Big(\sup_{0\le t\le t_0} U_t \ge K^{\alpha+c+\eta}\Big)\\ &\le\; \mathbb{P}\Big(\sup_{0\le t\le t_0} e^{-\gamma_K t}U_t \ge e^{-\gamma_K t_0}K^{\alpha+c+\eta}\Big),\end{aligned}$$

where $\gamma_K := \lambda-\mu+\lambda_K+\mu_K$ is the exponential growth rate of the birth and death process U. Applying Doob's inequality to the martingale $e^{-\gamma_K t}U_t$ thus shows that

$$\mathbb{P}[\tau^{(K)}\le t_0(K)] \;\le\; K^\alpha K^{-(\alpha+c+\eta)}e^{\gamma_K t_0} \;\sim\; K^{-\eta} \;\to 0,$$

as $K\to\infty$, because $(\lambda_K+\mu_K)\log K\to 0$ from (4.10). In view of (4.5) and of properties (f) and (g) of the coupling, it is thus enough for (4.9) to show that

$$\phi_{t_0,t_1}(\overline{U}_{t_0}^{(K)}) - \phi_{t_0,t_1}(\overline{V}_{t_0}^{(K)}) \xrightarrow[K\to\infty]{d} 0, \tag{4.22}$$

where $\overline{U}_t^{(K)} := K^{-1}U_t^{(K)}$ and $\overline{V}_t^{(K)} := K^{-1}V_t^{(K)}$.

If $\alpha=0$, by (2.2) and (2.3), and on the set $\{U_{t_0}>0\}$, we have

$$G\big(\phi_{t_0,t_1}(\overline{U}_{t_0})\big) \;=\; \log\overline{U}_{t_0} + H(\overline{U}_{t_0}) + (\lambda-\mu)(t_1-t_0) \;=\; \log(K^{-c}U_{t_0}) + H(\overline{U}_{t_0}).$$

Define

$$\Psi(x) \;:=\; \begin{cases} G^{-1}(\log x), & x>0;\\ 0, & x=0.\end{cases}$$

This is a continuous function from $[0,\infty)$ to $[0,x_\infty)$, and

$$\phi_{t_0,t_1}(\overline{U}_{t_0}) \;=\; \Psi\left(K^{-c}U_{t_0}e^{H(\overline{U}_{t_0})}\right); \quad \phi_{t_0,t_1}(\overline{V}_{t_0}) \;=\; \Psi\left(K^{-c}V_{t_0}e^{H(\overline{U}_{t_0})}\right), \tag{4.23}$$

irrespective of whether U_{t_0} and V_{t_0} are zero or positive. Now, from (4.12) and (4.10),

$$\lim_{K\to\infty}\mathbb{P}[U_{t_0(K)}^{(K)}=0] \;=\; \frac{\mu}{\lambda}; \quad \lim_{K\to\infty}\mathbb{P}[K^{-c}U_{t_0(K)}^{(K)}>x] \;=\; (1-\mu/\lambda)\exp\{-x(1-\mu/\lambda)\}, \quad x>0. \tag{4.24}$$

The same argument then shows that the limits (4.24) also hold if U_{t_0} is replaced by V_{t_0}. Since $K^{-c}U_{t_0} \geq K^{-c}V_{t_0}$ a.s., and they both have the same limits in distribution, it follows that $K^{-c}(U_{t_0} - V_{t_0}) \xrightarrow{d} 0$ as $K \to \infty$. Note also, from (4.24), that $\overline{U}_{t_0} \xrightarrow{d} 0$ as $K \to \infty$, and thus $e^{H(\overline{U}_{t_0})} \xrightarrow{d} 1$ also; and that the same relations are true if $\overline{U}$ is replaced by $\overline{V}$. From (4.23) and from the continuity of Ψ, the convergence (4.22) now follows. For $0 < \alpha < 1$, K^{-c} is replaced by $K^{-c-\alpha}$ in (4.23), and the convergence in distribution of both $K^{-c-\alpha}U_{t_0}$ and $K^{-c-\alpha}V_{t_0}$ to the constant 1 follows from (4.13), in view of (4.10).

Remark 4.1. Assumption (2.1) (ii) is used to justify (4.21) on the basis of (4.20), to guarantee the uniqueness of the solutions of (1.2), and to show that G_0 maps to the whole of $\mathbb{R}$. However, if $xg(x)$ is non-decreasing in x, it follows from (4.20) that (4.21) holds with θ replaced by zero. Furthermore, if $\overline{Z}$ is replaced by any solution of the o.d.e. (1.2) other than $\widehat{Z}$, but also starting at $\overline{Z}_{t_0}$, the difference $\delta_t := \overline{Z}_t - \widehat{Z}_t$ satisfies (4.20), with M the zero function, from which it follows, using Gronwall's inequality, that $\delta_t = 0$ for all $t \geq t_0$, implying uniqueness of the solutions to the o.d.e.

4.3. **Coupling birth and death processes.** The proof of (4.2) will be completed, once we construct the processes $Y^{(K)}, Z^{(K)}, U^{(K)}$ and $V^{(K)}$, all on the same probability space, with the properties (a)–(g). The basic element of our construction is a coupling of two birth and death processes, one of which has greater birth rate and smaller death rate than the other. Such a coupling has been suggested, e.g., in [6]. For the sake of completeness we give a construction in much the same spirit. As usual, we suppress the index K as far as we can.

The coupling is based on a collection of four sequences of independent standard Poisson processes $(\Pi^l_n,\ n \geq 1)$, $l \in \{1,2,3,4\}$, together with two double sequences $(J^l_n(i),\ i \geq 0, n \geq 1)$, $l \in \{3,4\}$, of independent uniform $U[0,1]$ random variables, all of which are mutually independent. We then define processes $(J^l_{nt},\ t \geq 0)$ by

$$J^3_{nt} \ := \ J^3_n(\Pi^3_n(2\lambda_K t)); \quad J^4_{nt} \ := \ J^4_n(\Pi^4_n(2\mu_K t)),$$

where λ_K and μ_K are as defined in property (d). The definitions of the BD processes U, Y and V are now simple to write down:

$$\begin{aligned}
U_t \ &:= \ U_0 + \sum_{n\geq 1}\int_0^t \mathbf{1}_{\{n\leq U_{s-}\}}\{d\Pi^1_n((\lambda-\lambda_K)s) + d\Pi^3_n(2\lambda_K s)\} - \sum_{n\geq 1}\int_0^t \mathbf{1}_{\{n\leq U_{s-}\}} d\Pi^2_n((\mu-\mu_K)s);\\
Y_t \ &:= \ Y_0 + \sum_{n\geq 1}\int_0^t \mathbf{1}_{\{n\leq Y_{s-}\}}\{d\Pi^1_n((\lambda-\lambda_K)s) + \mathbf{1}_{\{J^3_{ns}\leq 1/2\}} d\Pi^3_n(2\lambda_K s)\}\\
&\qquad - \sum_{n\geq 1}\int_0^t \mathbf{1}_{\{n\leq Y_{s-}\}}\{d\Pi^2_n((\mu-\mu_K)s) + \mathbf{1}_{\{J^4_{ns}\leq 1/2\}} d\Pi^4_n(2\mu_K s)\}; \qquad (4.25)\\
V_t \ &:= \ V_0 + \sum_{n\geq 1}\int_0^t \mathbf{1}_{\{n\leq V_{s-}\}}\, d\Pi^1_n((\lambda-\lambda_K)s) - \sum_{n\geq 1}\int_0^t \mathbf{1}_{\{n\leq V_{s-}\}}\{d\Pi^2_n((\mu-\mu_K)s) + d\Pi^4_n(2\mu_K s)\}.
\end{aligned}$$

That these representations yield the distributions of the corresponding BD processes follows because they define Markov processes having the right jump rates. These processes only have jumps of ± 1, so that, for two of them to cross each other, there have to be times at which they have the same values. However, if $U_t = Y_t$, the next transition either leaves their values the same, or increases U by 1, leaving Y unchanged, or reduces Y by 1, leaving U unchanged: so, if $U_0 \ge Y_0$, then $U_t \ge Y_t$ for all $t \ge 0$. The same considerations yield $Y_t \ge V_t$ for all $t \ge 0$, if $Y_0 \ge V_0$, and property (f) follows, assuming property (a).

In order to define the process Z, let

$$p^3(t) \;:=\; (\lambda_K - (\lambda-\mu)g_1(Z_{t-}/K))/(2\lambda_K); \quad p^4(t) \;:=\; (\lambda_K + (\lambda-\mu)g_2(Z_{t-}/K))/(2\mu_K),$$

noting that, if $0 \le t \le \tau^{(K)} := \inf\{s > 0\colon Z_s \ge K^{\alpha+c+\eta}\}$, as defined in property (g), then $0 \le p^l(t) \le 1$, $l \in \{3,4\}$. Then the process

$$\begin{aligned} Z_t \;&:=\; Z_0 + \sum_{n\ge 1}\int_0^t \mathbf{1}_{\{n\le Z_{s-}\}}\{d\Pi^1_n((\lambda-\lambda_K)s) + \mathbf{1}_{\{J^3_{ns}\le p^3(s)\}}d\Pi^3_n(2\lambda_K s)\} \\ &\qquad - \sum_{n\ge 1}\int_0^t \mathbf{1}_{\{n\le Z_{s-}\}}\{d\Pi^2_n((\mu-\mu_K)s) + \mathbf{1}_{\{J^4_{ns}\le p^4(s)\}}d\Pi^4_n(2\mu_K s)\} \end{aligned} \tag{4.26}$$

is Markovian and has the correct transition rates for $0 \le t \le \tau^{(K)}$; after that time, Z can be continued in any way that reproduces the correct distribution. The argument used to show that $U_t \ge Y_t \ge V_t$ for all $t \ge 0$ also shows that $U_t \ge Z_t \ge V_t$ for all $0 \le t \le \tau^{(K)}$, if $U_0 \ge Z_0 \ge V_0$, and property (g) follows, assuming property (a).

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