ON THE ENDOMORPHISM SEMIGROUP (AND CATEGORY) OF BOUNDED LATTICES

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Recently, a large number of papers have been published on the representations of a semigroup with identity (i.e., a monoid) as the endomorphism semigroup of algebras and relational systems of various kinds. For instance Z. Hedrlín and J. Lambek showed that every monoid can be represented as the endomorphism semigroup of a semigroup.

Conspicuously missing from the list of algebras for which theorems of this type can be proved are lattices. The reason for this is that any constant map is a lattice endomorphism, and therefore endomorphism semigroups of lattices are very special. For partially ordered sets one can eliminate constant maps as endomorphisms by considering only those maps φ that satisfy $x < y$ implies $x\varphi < y\varphi$ (Z. Hedrlín and R. H. McDowell); however, this would not be a very natural condition to impose on lattice endomorphisms.

The approach of this paper[1] is to consider bounded lattices only, that is, lattices with smallest element 0 and largest element 1, and as endomorphisms to admit only those lattice endomorphisms that preserve 0 and 1 (i.e., keep 0 and 1 fixed; this amounts to considering 0 and 1 as nullary operations). Such endomorphisms are usually called {0, 1}-endomorphisms but they are called simply endomorphisms in this paper.

The first result is that every monoid is isomorphic to the monoid of all endomorphisms of a bounded lattice.

One can also consider lattices with complementation $\langle L; \wedge, \vee, ' \rangle$, where $'$ is a complementation, that is, for every $a \in L$, $a \wedge a' = 0$ and $a \vee a' = 1$. For such algebras an endomorphism is a lattice endomorphism φ that preserves $'$, that is, $(a\varphi)' = a'\varphi$ for all $a \in L$. Every lattice with complementation is bounded, and any such endomorphism preserves 0 and 1.

The second result is that every monoid is isomorphic to the endomorphism semigroup of a lattice with complementation.

Both these results are consequences of much stronger theorems proved in this paper.

The proofs use some nontrivial graph theoretic and lattice theoretic results. For the reader's convenience, these are described in detail in §§ 2 and 3. Bounded lattices are dealt with in § 4, and lattices with complementation in § 5. Some further results and

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[1] An alternative approach is investigated in [11].

open problems complete the discussion in § 6.

2. **Graphs.** A *graph* $\langle X; R \rangle$ consists of a set X and a set R of two-element subsets of X. (This is the definition of an unoriented graph; we do not need oriented graphs.) The elements of X are called *vertices*, and the elements of R are called *edges*. A *compatible mapping* φ from the graph $\langle X; R \rangle$ into the graph $\langle Y; S \rangle$ is a map of X into Y satisfying: $\{a, b\} \in R$ implies $\{a\varphi, b\varphi\} \in S$.

We shall denote by G the following category of graphs:

The *objects* of G are graphs $\langle X; R \rangle$ satisfying conditions (1) and (2):

(1) for every $x \in X$ there is a seven element subset $\{x_0, x_1, \cdots, x_6\}$ of X such that $x = x_0$, and $\{x_0, x_1\}$, $\{x_1, x_2\}$, $\cdots$, $\{x_5, x_6\}$, $\{x_6, x_0\} \in R$; in other words, every vertex lies on a *cycle* of length seven.

(2) there is no cycle of length less than seven.

The morphisms of G are all the compatible maps.

Our constructions are based on the following result, which follows easily from the papers [7] and [8] of Z. Hedrlín and A. Pultr:

THEOREM 1. *Every category of algebras is isomorphic to a full subcategory of* G.

In more detail, this states the following. Let K be a class of algebras of a fixed type (see, e.g., [2]), and let K also denote the category whose objects are the algebras in K and the morphisms are all homomorphisms. A *functor* $F: K \to G$ assigns to every algebra $\mathfrak{A} \in K$ a graph $F(\mathfrak{A})$ and to every homomorphism φ of $\mathfrak{A}$ into $\mathfrak{B}$ ($\mathfrak{A}, \mathfrak{B} \in K$) a compatible map $F(\varphi)$ of $F(\mathfrak{A})$ into $F(\mathfrak{B})$ such that if $\varphi\psi = \alpha$, then $F(\varphi)F(\psi) = F(\alpha)$. A functor F is called a *full embedding* if F is one-to-one on objects, one-to-one on morphisms, and, in addition, it is *full*, that is, every morphism $\psi: F(\mathfrak{A}) \to F(\mathfrak{B})$ in G is of the form $F(\varphi)$ for some homomorphism φ of $\mathfrak{A}$ into $\mathfrak{B}$.

Using these concepts Theorem 1 can be restated as follows: There exists a full embedding of K into G for every category K of algebras.

Since every monoid M can be represented as the endomorphism semigroup of some algebra (see, e.g., [2]; the algebra can be constructed on the set M by defining as operations left multiplications by elements of M), Theorem 1 contains as a special case the statement that every monoid can be represented as the semigroup of all compatible maps of a graph $\langle X; R \rangle$ into itself, where $\langle X; R \rangle$ satifies (1) and (2).

3. **Lattices.** We denote by L the category of all bounded

lattices with 0 and 1 preserving homomorphisms as morphisms.

Let $\mathfrak{n}$ be a cardinal. We identify cardinals with initial ordinals. A lattice with $\mathfrak{n}$-complementations is an algebra

$$\langle L; \wedge, \vee, 0, 1, c_0, c_1, \cdots, c_\gamma, \cdots\rangle, \gamma < \mathfrak{n} ,$$

where $\langle L; \wedge, \vee, 0, 1\rangle$ is a bounded lattice and for each $\gamma < \mathfrak{n}$, $a \in L$, $c_\gamma(a)$ is a complement of a. We shall denote by $L_\mathfrak{n}$ the category whose objects are lattices with $\mathfrak{n}$ complementations satisfying conditions (3)–(5):

   (3) $c_i(c_i(a)) = a$ for all $a \in L$; $i < \mathfrak{n}$ (c_i is an involution);
   (4) $c_i(c_j(a)) = c_j(c_i(a))$ for all $a \in L$, $i, j < \mathfrak{n}$ (the c_i commute);
   (5)   for $a \in L$, $a \neq 0, 1$, and $i, j < \mathfrak{n}$, $i \neq j$, $c_i(a) \neq c_j(a)$.

The morphisms of $L_\mathfrak{n}$ are lattice homomorphisms φ preserving all c_i, $i < \mathfrak{n}$ (that is, $c_i(a\varphi) = c_i(a)\varphi$).

For a bounded lattice L set

$$C(L) = \{\{a, b\} \mid \{a, b\} \subseteqq L, a \wedge b = 0, a \vee b = 1\} ,$$

that is, $C(L)$ is the set of complemented pairs of L. The lattice L is said to have *incomparable complements* if

$$\{a, b\}, \{a, c\} \in C(L) \text{ and } b \geqq c \text{ imply that } b = c .$$

The lattice theoretic result we need is the following theorem of C. C. Chen and G. Grätzer [1]:

THEOREM 2. *Let M be a bounded lattice with incomparable complements, and let A be a set disjoint with M. Let $\langle A \cup (M - \{0, 1\}); C\rangle$ be a graph satisfying the following condition:*
   (6) $\{a, b\}$, $\{a, c\} \in C$, $b, c \in M$, and $b \geqq c$ imply that $b = c$, and $\{a, b\} \in C$, $a \in M$ implies that $b \in A$.
   Let L be the bounded lattice freely generated by $M \cup A$, and denote by $\Theta[C]$ the smallest congruence relation on L satisfying
   (7) $\{a, b\} \in C$ implies that $a \wedge b \equiv 0$ $(\Theta[C])$ and $a \vee b \equiv 1(\Theta[C])$.
Let $M(A, C) = L/\Theta[C]$, and identify $a \in M \cup A$ with the congruence class containing it. Then
   (8) M *is a $\{0, 1\}$-sublattice of $M(A, C)$;*
   (9) $A \subseteqq M(A, C)$;
   (10) $A \cap M = \varnothing$ *in $M(A, C)$;*
   (11) $M(A, C)$ *is generated by $A \cup M$;*
   (12) $M(A, C)$ *has incomparable complements;*
   (13) $C(M(A, C)) = C(M) \cup C$.

4.   Full embedding of G into L.   In this section we construct

the functor M which is the required full embedding. First we define M on the objects.

Let $\langle X; R \rangle$ be a graph in G. We apply Theorem 2 in the following way: let $M = \{0, 1\}$, $A = X$, and $C = R$. Then L is the free bounded lattice $L(X)$ generated by X. $M(X, R)$ has the following properties by Theorem 2:

(14) $X \subseteq M(X, R)$;

(15) X generates $M(X, R)$;

(16) $C(M(X, R)) = R \cup \{\{0, 1\}\}$.

We define M on $\langle X; R \rangle$ to be $M(X, R)$. Let $\pi(X, R)$ denote the natural homomorphism of $L(X)$ onto $M(X, R)$ (with $\Theta[R]$ as induced congruence relation).

To define M on morphisms, let $\varphi : X \to Y$ be a compatible map from $\langle X, R \rangle$ into $\langle Y; S \rangle$ in G. Then φ has a natural extension to a homomorphism $L(\varphi)$ of $L(X)$ into $L(Y)$ (see the diagram). Let Θ

$$
\begin{array}{ccc}
L(X) & \xrightarrow{\ L(\varphi)\ } & L(Y) \\
{\scriptstyle \pi(X,R)} \big\downarrow & & \big\downarrow {\scriptstyle \pi(Y,S)} \\
M(X, R) & \dashrightarrow[M(\varphi)] & M(Y, S)
\end{array}
$$

be the congruence relation induced by $L(\varphi)\pi(Y, S)$ (that is, for $a, b \in L(X)$, $a \equiv b(\Theta)$ is equivalent to $aL(\varphi)\pi(Y, S) = bL(\varphi)\pi(Y, S)$). If $\{a, b\} \in R$, then (since φ is compatible) $\{a\varphi, b\varphi\} \in S$, hence $aL(\varphi)\pi(Y, S)$ is a complement of $bL(\varphi)\pi(Y, S)$. Thus Θ is a congruence relation under which $a \wedge b \equiv 0(\Theta)$ and $a \vee b \equiv 1(\Theta)$ for any $\{a, b\} \in R$. By the definition of $\Theta[R]$ we conclude that $\Theta[R] \subseteq \Theta$. Hence by the Second Isomorphism Theorem (see e.g. [2, Th. 11.4]) there is a unique homomorphism $M(\varphi)$ such that $L(\varphi)\pi(Y, S) = \pi(X, R)M(\varphi)$. We define $M(\varphi)$ to be the value of M at φ.

It is easy to check that M is a functor from G into L, and M is one-to-one on objects. Since $X \subseteq M(X, R)$, and $M(\varphi)$ restricted to X is φ, we get that M is one-to-one on morphisms.

To prove that M is full, let ψ be a homomorphism of $M(X, R)$ into $M(Y, S)$. Take $x \in X$; we wish to show that $x\psi \in Y$. Recall that by (1) and (16) there exist $x_0 = x, x_1, \cdots, x_6 \in X$ such that $\{x_0, x_1\}, \{x_1, x_2\}, \cdots, \{x_5, x_6\} \in C(M(X, R))$. Therefore,

$$\{x_0\psi, x_1\psi\}, \cdots, \{x_6\psi, x_0\psi\} \in C(M(Y, S)) .$$

By (16) this is possible only if $x_0\psi, \cdots, x_6\psi \in Y \cup \{0, 1\}$; so $x_0\psi \in Y \cup \{0, 1\}$. We claim that $x_0\psi \in Y$. Indeed, otherwise $x_0\psi \in \{0, 1\}$, say $x_0\psi = 0$. Then $x_1\psi = 1$ (as the only complement of 0 is 1), $x_2\psi = 0$, $x_3\psi = 1$, $x_4\psi = 0$, $x_5\psi = 1$, $x_6\psi = 0$, $x_0\psi = 1$, a contradiction. Thus ψ restricted

to X maps X into Y; let φ denote this map. Then φ is a compatible map between $\langle X; R \rangle$ and $\langle Y; S \rangle$ and $M(\varphi) = \psi$ follows from (15).

Combining this with Theorem 1 we have

THEOREM 3. *Every category of algebras can be fully embedded into the category L of bounded lattices. In particular, every monoid is isomorphic to the endomorphism semigroup of some bounded lattice.*

5. **Full embedding of G into L_n.** We want to construct the functor N_n which fully embeds G into L_n. To outline the idea of the proof let us take the case $n = 1$ (this eliminates the necessity to consider condition (4)). Let $\langle X; R \rangle$ be a graph in G, then $M_0 = M(X, R)$ has the same endomorphism semigroup as $\langle X; R \rangle$. We define for every $a \in M_0$, $a \neq 0, 1$ a new element $c(a)$, and apply Theorem 2 with $M = M_0$, $A = \{c(a) \mid a \in M_0, a \neq 0, 1\}$, and

$$C = \{\{a, c(a)\} \mid a \in M_0, a \neq 0, 1\} .$$

Then we define c on $M_1 = M(A, C)$ as follows: $c(0) = 1$, $c(1) = 0$, for $a \in M_0$, $a \neq 0, 1$, $c(a)$ is the new element, $c(c(a)) = a$, and $c(a)$ is not defined for any other element. Then we define $A_1 = \{c(a) \mid a \in M_1,$ $c(a)$ is not defined$\}$ to get M_2, and so on. $N_1(X, R)$ will be $\bigcup(M_i \mid i < \omega)$. The endomorphisms will work out all right because the elements of X are still distinguished by being on seven cycles of complementary elements.

Now we proceed with a formal construction of the functor N_n.

Recall that n is identified with the initial ordinal of cardinality n, hence n is a set of ordinals. Let E_n denote the set of all finite subsets of n. For $a, b \in E_n$ let $a + b$ denote the symmetric difference. We define a graph $\langle E_n; D_n \rangle$, where $\{a, b\} \in D_n$ if and only if $|a + b| = 1$. Finally, for $a \in n$ we define the function $\varepsilon_a : E_n \to E_n$ by

$$\varepsilon_a(x) = \begin{cases} x \cup \{a\} & \text{if } a \notin x ; \\ x - \{a\} & \text{if } a \in x . \end{cases}$$

Let us make a few observations about the ε_a:

(17) $\varepsilon_a(\varepsilon_a(x)) = x$ for all $x \in E_n$;

(18) $\varepsilon_a(\varepsilon_b(x)) = \varepsilon_b(\varepsilon_a(x))$ for all $x \in E_n$, $a \neq b$;

(19) $\varepsilon_a(x) \neq \varepsilon_b(x)$ for all $x \in E_n$, $a \neq b$;

(20) if $x = \{a_1, \cdots, a_n\}$, then $\varepsilon_{a_1}(\varepsilon_{a_2}(\cdots(\varepsilon_{a_n}(\phi)\cdots)) = x$;

(21) all cycles in $\langle E_n; D_n \rangle$ are of even length.

All these are obvious; to prove (21) observe that $|\varepsilon_a(x)| = |x| \pm 1$.

For $\langle X; R \rangle$ in G set $M_0 = M(X, R)$ as defined in § 4. We define a graph $\langle (M_0 - \{0, 1\}) \times E_n; C_1 \rangle$ by

$$C_1 = \{\{\langle a, e_1 \rangle, \langle a, e_2 \rangle\} \mid a \in M_0 - \{0, 1\}, \, e_1, e_2 \in E_{\mathfrak{n}}, \, \{e_1, e_2\} \in D_{\mathfrak{n}}\} \, .$$

Identifying $a \in M_0 - \{0, 1\}$ with $\langle a, \varnothing \rangle$ we can apply Theorem 2 to M_0, $(M_0 - \{0, 1\}) \times (E_{\mathfrak{n}} - \{\varnothing\})$, and C_1; to see that we only have to check (6) for C_1: if $b, c \in M$, $\{a, b\}, \{a, c\} \in C_1$, and $b \geqq c$, then $b = \langle b, \varnothing \rangle$, $c = \langle c, \varnothing \rangle$, $a = \langle b, e_1 \rangle = \langle c, e_1 \rangle$, hence $b = c$. Thus by Theorem 2 we get the lattice $M_1 = L(X)/\Theta[C_1]$.

For all $a \in \mathfrak{n}$ we define c_a on a part of M_1:

(22) $c_a(\langle b, e \rangle) = \langle b, \varepsilon_a(e) \rangle$,

(23) $c_a(0) = 1, \, c_a(1) = 0$.

Observe that $c_a(c_a(x)) = x$ by (17), $c_a(c_b(x)) = c_b(c_a(x))$ by (18). Also, $c_a(x)$ is a complement of x because $\mid e + \varepsilon_a(e) \mid = 1$, hence

$$\{\langle b, e \rangle, \langle b, \varepsilon_a(e) \rangle\} \in C_1 \, ;$$

finally, by (13), $C(M_1) = C(M) \cup C_1$. Note that c_a is defined exactly on $M_0 \cup C_1$. Let $\bar{M}_1$ be M_1 with all the c_a, $a \in \mathfrak{n}$ as partial unary operations.

Now assume that $M_0, M_1, \cdots, M_i, \bar{M}_i$ have already been defined and M_i has no comparable complements. Let A_i be the subset of M_i consisting of all noncomplemented elements of M_i. Then we form the set $(M_i - A_i) \cup (A_i \times E_{\mathfrak{n}})$; by identifying $a \in A$ with $\langle a, \varnothing \rangle$, M_i is a subset of this set. Define $C_{i+1} = \{\{\langle a, e_1 \rangle, \langle a, e_2 \rangle\} \mid a \in A_i, \{e_1, e_2\} \in D_{\mathfrak{n}}\}$. Again we apply Theorem 2 to M_i, $(A_i - \{\varnothing\}) \times E_{\mathfrak{n}}$, and C_{i+1}, getting the lattice M_{i+1}. We define $c_a(a \in \mathfrak{n})$ as a partial unary operation on M_{i+1} as follows: for the complemented elements of M_i we have already defined c_a; if $x \in M_i$, but x is not complemented, then $x \in A_i$ and we set $c_a(x) = \langle x, \{a\} \rangle$; in general $c_a(\langle x, e \rangle) = \langle x, \varepsilon_a(e) \rangle$. This defines c_a for all complemented elements of M_{i+1}. Let $\bar{M}_{i+1}$ denote the M_{i+1} with all the c_a.

Finally, we set

$$N_{\mathfrak{n}}(\langle X; R \rangle) = \bigcup (M_i \mid 1 \leqq i < \omega) \, .$$

Observe that $\bar{M}_{\omega} = \bigcup (\bar{M}_i \mid i < \omega)$ is an algebra defined on the bounded lattice $M_{\omega} = \bigcup (M_i \mid i < \omega)$. Since the domain of c_a in M_{i+1} included M_i, c_a is fully defined on M_{ω}. Thus $\bar{M}_{\omega}$ is a bounded lattice with $\mathfrak{n}$ complementations. It is clear from (17)-(19) that in fact (3)-(5) are satisfied, that is, $\bar{M}_{\omega}$ is in $L_{\mathfrak{n}}$.

The next task is to construct $N_{\mathfrak{n}}$ on morphisms. To facilitate this we need a special case of a result of [4], which we shall give as a corollary to Theorem 2:

COROLLARY TO THEOREM 2. *Let M, A, C, and also M_1, A_1, C_1 satisfy the condition of Theorem 2. Let φ be a map of $M \cup A$ into*

$M_1 \cup A_1$ such that $M\varphi \subseteqq M_1$, φ is a homomorphism of M into M_1, and $\{a, b\} \in C$ implies that $\{a\varphi, b\varphi\} \in C_1$. Then there exists a unique extension of φ to a homomorphism $\bar{\varphi}$ of $M(A, C)$ into $M(A_1, C_1)$.

Now let $\varphi: X \to Y$ be a compatible map of $\langle X; R \rangle$ into $\langle Y; S \rangle$, and let $\bar{M}_0, \bar{M}_1, \cdots$ be the sequence of lattices with complementations constructed from $\langle X; R \rangle$ and $\bar{P}_0, \bar{P}_1, \cdots$ the sequence for $\langle Y; S \rangle$. We shall construct mappings $\varphi_i: M_i \to P_i$. In §4 we constructed a homomorphism $\varphi_0: M_0 \to P_0$ extending φ. We extend φ_0 to

$$M_0 \cup ((M_0 - \{0, 1\}) \times (E_\mathfrak{n} - \{\varnothing\}))$$

as follows:

$$\langle b, e \rangle \bar{\varphi}_0 = \begin{cases} \langle b\varphi_0, e \rangle & \text{if } b\varphi_0 \neq 0, 1; \\ 0 & \text{if } b\varphi_0 = 0 \text{ and } |e| \text{ is even, or } b\varphi_0 = 1 \text{ and } |e| \text{ is odd}; \\ 1 & \text{if } b\varphi_0 = 1 \text{ and } |e| \text{ is even, or } b\varphi_0 = 0 \text{ and } |e| \text{ is odd}. \end{cases}$$

Clearly, $\bar{\varphi}_0$ maps $M_0 \cup ((M_0 - \{0, 1\}) \times (E_\mathfrak{n} - \{\varnothing\}))$ into

$$P_0 \cup ((P_0 - \{0, 1\}) \times (E_\mathfrak{n} - \{\varnothing\}))$$

and it satisfies the requirements of the Corollary to Theorem 2, hence it can be extended to a homomorphism φ_1 of M_1 into P_1. It is also clear, that for all $x \in M_1$, $c_a(x)\varphi_1 = c_a(x\varphi_1)$, provided $c_a(x)$ is defined. Hence φ_1 is a homomorphism of $\bar{M}_1$ into $\bar{P}_1$. Thus applying the Corollary to Theorem 2 we get all the φ_i, and $\bar{\varphi} = \bigcup (\varphi_i \mid i < \omega)$ is defined to be $N_\mathfrak{n}(\varphi)$. This completes the definition of $N_\mathfrak{n}$.

To show that $N_\mathfrak{n}$ is a functor one can use a direct computation or employ an argument similar to the one we used for M. Namely, one can represent $N_\mathfrak{n}(\langle X; R \rangle)$ as a quotient structure $F(X, R)/\Theta[R]$, where $F(X, R)$ is the free $\mathfrak{m}$-complemented lattice generated by $M(X, R)$ and $\Theta[R]$ is the smallest congruence relation forced by (3) and (4). (Note that (5) will be satisfied by $F(X, R)/\Theta[R]$.) Then we get $N_\mathfrak{n}(\varphi)$ by a diagram similar to the one in §4 which implies that $N_\mathfrak{n}$ is a functor.

$N_\mathfrak{n}$ is one-to-one on objects and maps since X generates $N_\mathfrak{n}(\langle X; R \rangle)$. It remains to show that $N_\mathfrak{n}$ is full. Let φ be a homomorphism of $N_\mathfrak{n}(\langle X; R \rangle)$ into $N_\mathfrak{n}(\langle Y; S \rangle)$ just as in §4 it is enough to show that $X\varphi \subseteqq Y$. Every $x \in X$ lies on a cycle of complementary pairs of length seven. Hence $x\varphi$ lies on a quotient of a cycle of length seven, which is either a cycle of length seven, and in this case $x\varphi \in Y$ by (21), or the quotient contains a cycle of length 3 or 5, contradicting (2) and (21).

Combining the full embedding $N_\mathfrak{n}$ with the one given in Theorem 1 we conclude

THEOREM 4. *Let $\mathfrak{n}$ be a cardinal. Any category of algebras is isomorphic to a full subcategory of $L_{\mathfrak{n}}$. Consequently, every monoid is isomorphic to the endormorphism semigroup of a lattice with $\mathfrak{n}$ complementations.*

6. **Concluding remarks.** Utilizing the fact that $|M(X, R)| = |X|$ provided that $|X|$ is infinite, and that $|N_{\mathfrak{n}}(X, R)| = |X|$ provided that $|X| \geq \mathfrak{n} + \aleph_0$ we can combine Theorems 3 and 4 with the results of [9] to get the number of nonisomorphic objects in L and $L_{\mathfrak{n}}$ with a given endomorphism semigroup.

THEOREM 5. *Let S be a monoid with $\mathfrak{p}$ elements and let $\mathfrak{m}$ be an infinite cardinal with $\mathfrak{m} \geq \mathfrak{p}$. Then there are exactly $2^{\mathfrak{m}}$ pairwise nonisomorphic bounded lattices (and bounded lattices with $\mathfrak{n}$ complementation satisfying (3)–(5), provided that $\mathfrak{n} \leq \mathfrak{m}$) of cardinality $\mathfrak{m}$ whose endomorphism semigroup is isomorphic to S.*

Note that Theorem 5 includes the statement that S can be represented as an endormorphism semigroup of arbitrarily large bounded lattices.

The method of this paper is not applicable, however, to get results on the endomorphism semigroups of lattices of finite length. Indeed, $M(X, R)$ and $N_{\mathfrak{n}}(X, R)$ are of infinite length unless $\langle X; R \rangle$ is very close to the complete graph. A new method to deal with this problem is proposed in [10] which yields, for instance, that every group is isomorphic to the endomorphism semigroup of a lattice of finite length (which also happens to be complemented).

Note that the functors M and $N_{\mathfrak{n}}$ take onto maps to onto maps, but almost none of the one-to-one maps are carried into one-to-one maps.

PROBLEM 1. Characterize the endomorphism semigroups of lattices of finite length.

PROBLEM 2. For what categories K of algebras does there exist a full embedding of K into L and $L_{\mathfrak{n}}$ which takes one-to-one maps into one-to-one maps. Could such full embeddings be constructed that take onto maps to onto maps?

PROBLEM 3. Characterize the endomorphism semigroups of complemented lattices.

PROBLEM 4. Find classes of monoids for which the conclusion of Theorem 5 holds under the assumption $\mathfrak{p} < \mathfrak{m} \leq 2^{\mathfrak{p}}$?

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