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ON THE ENTIRE FUNCTION SHARING ONE VALUE CM WITH K-TH DERIVATIVES

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Abstract. In this paper, we investigate some properties of the entire function of the hyper order less than $\frac{1}{2}$ sharing one value CM with its k-th derivative.

1. Introduction and results

Let f and g be two non-constant meromorphic functions, and let a be a finite value in the complex plane. We say that f and g share the value a CM (IM) provided that $f-a$ and $g-a$ have the same zeros counting multiplicities (ignoring multiplicities). Nevanlinna [17] four values theorem says that if two non-constant meromorphic functions f and g share four values CM, then $f \equiv g$ or f is a Möbius transformation of g. The condition "f and g share four values CM" has been weakened to "f and g share two values CM and two values IM" by Gundersen [7, 8], as well as by Mues [15] and Wang [20]. But whether the condition can be weakened to "f and g share three values IM and another value CM" or not, is still an open question. In a special case, it was shown [18] that if an entire function f share two finite values CM with its derivative, then $f \equiv f'$. This result has been generalized to sharing values IM by Gundersen [6] and by Mues-Steinmetz [16] independently.

How is the relation between f with f' if an entire function f share one finite value CM with its derivative f'? In [3], R. Brück raised the following.

Conjecture. Let f be a nonconstant entire function such that the hyper order $\sigma_2(f) < \infty$ and $\sigma_2(f)$ isn't a positive integer. If f and f'

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share the finite value a CM, then

$$\frac{f'-a}{f-a}=c$$

where c is a nonzero constant. Where the notation $\sigma_2(f)$ denotes the hyper-order (see [22]), of $f(z)$, it is defined by

$$\sigma_2(f)=\overline{\lim_{r\to\infty}}\frac{\log\log T(r,f)}{\log r}.$$

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (e.g. see [10, 11]). In addition, we will use the notations $\lambda(f)$ to denote the exponents of convergence of the zero-sequence of the meromorphic function $f(z), \sigma(f)$ to denote the order growth of $f(z)$.

The conjecture for the case that $a=0$ had been proved by Brück in the following theorem.

THEOREM A [3]. *Let f be a non-constant entire function such that the hyper order $\sigma_2(f)<\infty$ and $\sigma_2(f)$ isn't a positive integer. If f and f' share the finite value 0 CM, then $f'=cf$ where c is a nonzero constant.*

From differential equations

$$\frac{f'-1}{f-1}=e^{z^n},\ \frac{f'-1}{f-1}=e^{e^z},$$

we see that when the hyper order $\sigma_2(f)$ of f is a positive integer or infinite, the conjecture of Brück does not hold. For the case that the zero-points of f' are fewness, Brück obtain the following in [3].

THEOREM B. *Let f be a nonconstant entire function. If f and f' share a value 1 CM, and satisfy $N(r,0,f')=S(r,f)$, then*

$$\frac{f'-1}{f-1}=c$$

where c is a nonzero constant.

For entire functions with finite order, Lianzhong Yang proved following two theorems in [21].

THEOREM C. *Let f be a nonconstant entire function with finite order. If f and f' share a finite value a CM, then*

$$\frac{f'-a}{f-a}=c$$

where c is a nonzero constant.

THEOREM D. *Let f be a nonconstant entire function with finite order. If f and $f^{(k)}(k \geq 1)$ share a finite value $a \neq 0$ CM, then*

$$\frac{f^{(k)} - a}{f - a} = c$$

where c is a nonzero constant, k is a positive integer.

In this paper, we investigate the case that an entire function is of infinite order, and get the following theorems.

THEOREM 1. *Let $f(z)$ be a nonconstant entire function with the hyper order $\sigma_2(f)$ isn't a positive integer and $\sigma_2(f) < \infty$. If f and $f^{(k)}(k$ is a positive integer) share the value 0 CM, then*

$$f^{(k)} \equiv cf$$

where c is a nonzero constant.

REMARK. (i) The proof of Theorem 1 is completely different from the proof of Theorem A.

(ii) For the problem that f and $f^{(k)}$ share the value 0 CM, $k = 1$ and $k > 1$ are very different. If f and f' share the value 0 CM, then neither f nor f' doesn't have zero. But, if f and $f^{(k)}(k > 1)$ share the value 0 CM, then both of f and $f^{(k)}$ may have many zeros.

THEOREM 2. *Let $f(z)$ be a nonconstant entire function with $\sigma_2(f) = \alpha < \frac{1}{2}$. If f and $f^{(k)}$ share the finite value a CM, then*

$$\frac{f^{(k)} - a}{f - a} \equiv c$$

where c is a nonzero constant.

REMARK. For a finite order entire function, the condition "$a \neq 0$" in Theorem D is deleted by Theorems 1 and 2.

By Theorems 1 and 2, we can obtain the following corollaries.

COROLLARY 1. *Let f be a nonconstant entire function with the hyper order $\sigma_2(f)$ isn't a positive integer and $\sigma_2(f) < \infty$. If f and $f^{(k)}(k$ is a positive integer) share the value 0 CM, and there exists a point z_0 satisfying $f^{(k)}(z_0) = f(z_0) \neq 0$, then $f \equiv f^{(k)}$.*

COROLLARY 2. *Let f be a nonconstant entire function with the hyper order $\sigma_2(f)$ isn't a positive integer and $\sigma_2(f) < \infty$. If f and $f^{(k)}(k$ is* *a positive integer) share the value 0 CM and a finite value $b(\neq 0)$ IM , then $f \equiv f^{(k)}$.*

COROLLARY 3. *Let f be a nonconstant entire function with the hyper order $\sigma_2(f)$ isn't a positive integer and $\sigma_2(f) < \infty$. If f and $f^{(k)}$(k is a positive integer) share the value 0 CM, and there exists a point z_0 and a positive integer m satisfying $f^{(k+m)}(z_0) = f^{(m)}(z_0) \neq 0$, then $f \equiv f^{(k)}$.*

COROLLARY 4. *Let f be a nonconstant entire function with $\sigma_2(f) < \frac{1}{2}$. If f and $f^{(k)}$ share a finite value a CM, and there exists a point z_0 satisfying $f^{(k)}(z_0) = f(z_0) \neq a$, then $f \equiv f^{(k)}$.*

COROLLARY 5. *Let f be a nonconstant entire function with $\sigma_2(f) < \frac{1}{2}$. If f and $f^{(k)}$ share a finite value a CM and a finite value $b(\neq a)$ IM , then $f \equiv f^{(k)}$.*

COROLLARY 6. *Let f be a nonconstant entire function with $\sigma_2(f) < \frac{1}{2}$. If f and $f^{(k)}$ share a finite value a CM, and there exist a point z_0 and a positive integer m satisfying $f^{(k+m)}(z_0) = f^{(m)}(z_0) \neq 0$, then $f \equiv f^{(k)}$.*

2. Lemmas for the proofs of Theorems 1 and 2

The Hadamard theorem of entire functions of infinite order can be found in [12].

LEMMA 1. *Let f be a transcendental entire function of infinite order and $\sigma_2(f) = \alpha < \infty$, then f can be represented in*

$$f(z) = U(z)e^{V(z)}, \tag{2.1}$$

where U and V are entire functions such that

$$\lambda(f) = \lambda(U) = \sigma(U),\ \lambda_2(f) = \lambda_2(U) = \sigma_2(U),$$

$$\sigma_2(f) = \max\{\sigma_2(U), \sigma_2(e^V)\}.$$

where notation $\lambda_2(f)$ denotes the hyper exponent of convergence of zeros of entire function f by

$$\lambda_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log N(r, \frac{1}{f})}{\log r}.$$

LEMMA 2 [4]. *Let $g(z)$ be an entire function of infinite order with $\sigma_2(g) = \sigma$, and let $\nu(r)$ be the central index of g. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log\log\nu(r)}{\log r} = \sigma_2(g) = \sigma. \tag{2.2}$$

Using the similar proof as in the proof of Remark 1 of [5], we can obtain the following Lemma 3.

LEMMA 3. *Let $f(z)$ be an entire function with $\sigma(f) = \infty$ and $\sigma_2(f) = \alpha < +\infty$, let a set $E \subset (1,\infty)$ have a finite logarithmic measure. Then there exists $\{z_k = r_k e^{i\theta_k}\}$ such that $|f(z_k)| = M(r_k, f)$, $\theta_k \in [0, 2\pi)$, $\lim_{k\to\infty}\theta_k = \theta_0 \in [0,2\pi)$, $r_k \notin E$, $r_k \to \infty$, if $\alpha > 0$, then for any given $\varepsilon(0 < \varepsilon < \alpha)$, we have as r_k sufficiently large*

$$\exp\{r_k^{\alpha-\varepsilon}\} < \nu(r_k) < \exp\{r_k^{\alpha+\varepsilon}\}; \tag{2.3}$$

if $\alpha = 0$ then for any large $M(>0)$, we have as r_k sufficiently large

$$\nu(r_k) > r_k^M. \tag{2.4}$$

LEMMA 4. (see [14]) *Let*

$$Q(z) = b_n z^n + b_{n-1}z^{n-1} + \cdots + b_0$$

where n is a positive integer and $b_n = \alpha_n e^{i\theta_n}, \alpha_n > 0, \theta_n \in [0, 2\pi)$. For any given $\varepsilon(0 < \varepsilon < \pi/(4n))$, we introduce $2n$ open angles

$$S_j : -\frac{\theta_n}{n}+(2j-1)\frac{\pi}{2n}+\varepsilon < \theta < -\frac{\theta_n}{n}+(2j+1)\frac{\pi}{2n}-\varepsilon \ (j = 0,1,\dots,2n-1).$$

Then there exists a positive number $R = R(\varepsilon)$ such that for $|z| = r > R$,

$$Re\{Q(z)\} > \alpha_n(1-\varepsilon)\sin(n\varepsilon)r^n \tag{2.5}$$

if $z \in S_j$ where j is even; while

$$Re\{Q(z)\} < -\alpha_n(1-\varepsilon)\sin(n\varepsilon)r^n \tag{2.6}$$

if $z \in S_j$ where j is odd.

Now for any given $\theta \in [0, 2\pi)$, if $\theta \neq -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n}$ $(j = 0, 1, \dots, 2n-1)$, then we take ε sufficiently small, there is some $S_j, j \in \{0, 1, \dots, 2n-1\}$ such that $\theta \in S_j$.

LEMMA 5 [1]. *Let $h(z)$ be an entire function with $\sigma(h) = \sigma < \frac{1}{2}$, set*

$$A(r) = \inf_{|z|=r} \log|h(z)|,\ B(r) = \sup_{|z|=r} \log|h(z)|.$$

If $\sigma < \alpha < 1$, *then*

$$\underline{\log dens}\{r : A(r) > (\cos \pi\alpha)B(r)\} \geq 1 - \frac{\sigma}{\alpha}, \tag{2.7}$$

where the lower logarithmic density $\underline{\log dens}H$ *of subset* $H \subset (1, +\infty)$ *is defined by*

$$\underline{\log dens}H = \varliminf_{r\to\infty} \Big(\int_1^r (\chi_H(t)/t)dt\Big)/\log r,$$

and the upper logarithmic density $\overline{\log dens}H$ *of subset* $H \subset (1, +\infty)$ *is defined by*

$$\overline{\log dens}H = \varlimsup_{r\to\infty} \Big(\int_1^r (\chi_H(t)/t)dt\Big)/\log r,$$

where $\chi_H(t)$ *is the characteristic function of the set* H.

LEMMA 6 [2]. *Let* $h(z)$ *be an entire function with the lower order* $\mu = \mu(h) < \frac{1}{2}$, *and* $\mu < \sigma = \sigma(h)$. *If* $\mu \leq \delta < \min(\sigma, \frac{1}{2})$ *and* $\delta < \alpha < \frac{1}{2}$, *then*

$$\underline{\log dens}\{r : A(r) > (\cos \pi\alpha)B(r) > r^{\delta}\} \geq C(\sigma, \delta, \alpha), \tag{2.8}$$

where $C(\sigma, \delta, \alpha)$ *is a positive constant only dependent on* σ, δ, α.

REMARK. By definitions of the logarithmic measure and the logarithmic density, we see that if the upper logarithmic density $\overline{\log dens}H > 0$, then the logarithmic measure $lmH = +\infty$.

LEMMA 7 [9]. *Let* f *be a transcendental meromorphic function, and let* $\alpha > 1$ *be a given constant. Then*

(i) *there exists a set* $E \subset [0, 2\pi)$ *with linear measure zero and a constant* $B > 0$ *that depends only on* α *and* $j = 1, \ldots, k$, *such that if* $\varphi_0 \in [0, 2\pi)\backslash E$, *then there is a constant* $R = R(\varphi_0) > 1$ *so that for all* z *satisfying* $\arg z = \varphi_0$ *and* $|z| = r \geq R$, *we have*

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| \leq B\Big(\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\Big)^j \tag{2.9}$$

for all $j = 1, \ldots, k$;

(ii) *there exists a set* $E \subset (1, \infty)$ *with finite logarithmic measure and a constant* $B > 0$ *that depends only on* α *and* $j = 1, \ldots, k$, *such that for all* z *satisfying* $|z| = r \notin [0, 1] \bigcup E$, *we have* (2.9) *holds.*

LEMMA 8 [11]. (Hadamard-Borel-Caratheodory) *Let* $w(z)$ *is a non-constant entire function,* $A(r, w) = \max_{|z|<r}\{Re(w(z))\}$, *then for* $0 \leq$

$r < R$, we have

$$M(r, w) \leq \frac{4r}{R-r} A(R, w) + \frac{R-3r}{R-r} |w(0)|. \tag{2.10}$$

3. Proof of Theorem 1

Since f and $f^{(k)}$ share the value 0 CM, by Lemma 1,we can write

$$\frac{f^{(k)}(z)}{f(z)} = e^{Q(z)} \tag{3.1}$$

where $Q(z)$ is an entire function. First we know f is a transcendental since f and $f^{(k)}$ share the value 0 CM. We divide this into three cases (Q is a constant, or polynomial, or transcendental) to prove.

Case (1): Q is a constant. Then Theorem 1 holds.

Case (2): Q is a polynomial with $\deg Q = n \geq 1$. By Lemma 7, we see that there exists a set $E \subset [0, 2\pi)$ with linear measure zero and a constant $B > 0$ such that if $\theta \in [0, 2\pi)\backslash E$, then there is a constant $R = R(\theta) > 1$ so that for all z satisfying $\arg z = \theta$ and $|z| = r \geq R$, we have

$$|\frac{f^{(k)}(z)}{f(z)}| \leq B(T(2r, f))^{2k}. \tag{3.2}$$

Let

$$Q(z) = \alpha_n e^{i\theta_n} z^n + b_{n-1} z^{n-1} + \cdots + b_0, \ \alpha_n > 0, \ \theta_n \in [0, 2\pi).$$

By Lemma 4, for any given $\varepsilon (0 < \varepsilon < \frac{\pi}{4n})$, there are $2n$ opened angles

$$S_j : \quad -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon < \theta < -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon,$$
$$(j = 0, 1, \ldots, 2n-1).$$

We take the ray $\arg z = \theta_0 \in S_j \backslash E, j (\in \{0, 1, \ldots, 2n-1\})$ is some even, then

$$Re\{Q(z)\} > \alpha_n (1-\varepsilon) \sin(n\varepsilon) r^n \ (|z| = r) \tag{3.3}$$

holds for sufficiently large r. By (3.1)-(3.3), we obtain

$$\exp\{\alpha_n (1-\varepsilon) \sin(n\varepsilon) r^n\} \leq B(T(2r, f))^{2k}. \tag{3.4}$$

From (3.4), we have

$$\sigma_2(f) \geq n. \tag{3.5}$$

On the other hand, from the Wiman-Valiron theory (see [11, 13, 19]), there is a set $E_1 \subset (1, \infty)$ having logarithmic measure $lmE_1 < \infty$, we choose z satisfying $|z| = r \notin [0,1] \bigcup E_1$ and $|f(z)| = M(r, f)$, then we have as r sufficiently large

$$\frac{f^{(k)}(z)}{f(z)} = (\frac{\nu(r)}{z})^k (1 + o(1)), \tag{3.6}$$

where $\nu(r)$ is the central index of f. For any given $\varepsilon(> 0)$, as r sufficiently large, we have

$$|e^{Q(z)}| \leq e^{r^{n+\varepsilon}}. \tag{3.7}$$

Since ε is arbitrary, by (3.1), (3.6) and (3.7), we have

$$\sigma_2(f) \leq n. \tag{3.8}$$

Hence by (3.5) and (3.8) we get

$$\sigma_2(f) = n$$

which contradict the condition that $\sigma_2(f)$ isn't a positive integer.

Case (3): $Q(z)$ is transcendental. By Lemma 7, we know that there exists a set $E_2 \subset (1, \infty)$ with finite logarithmic measure, and a constant $B > 0$, such that for all z satisfying $|z| = r \notin [0,1] \bigcup E_2$, we have

$$|\frac{f^{(k)}(z)}{f(z)}| \leq B(T(2r, f))^{2k}. \tag{3.9}$$

We can choose z_r satisfying $|z_r| = r \in (1, \infty) \backslash E_2$ and

$$A(r, Q) = Re\{Q(z_r)\} = \max_{|z| \leq r}\{Re(Q(z))\},$$

by Lemma 8, we have

$$M(\frac{r}{2}, Q) \leq 4Re\{Q(z_r)\} + O(1). \tag{3.10}$$

By (3.1), (3.9) and (3.10), we obtain

$$e^{\frac{1}{4}M(\frac{r}{2}, Q)} \leq e^{Re\{Q(z_r)\}} = |e^{Q(z_r)}| \leq B[T(2r, f)]^{2k}. \tag{3.11}$$

From Q is transcendental, by (3.11), we get $\sigma_2(f) = \infty$. This contradict the condition $\sigma_2(f) \neq \infty$. Theorem 1 is thus proved.

4. Proof of Theorem 2

Suppose f and $f^{(k)}$ share the finite value a CM. If $a = 0$ by Theorem 1, we see Theorem 2 holds. The case that f is of finite order and $a \neq 0$ had been proved by Liang Zhong Yang [21]. Now we suppose $a \neq 0$ and $\sigma(f) = \infty$. By Lemma 1 we can write

$$\frac{f^{(k)} - a}{f - a} = e^{Q(z)} \tag{4.1}$$

where $Q(z)$ is an entire function. Set $F = \frac{f}{a} - 1$, then F is an entire function,

$$\sigma(F) = \sigma(f) = \infty,\ \sigma_2(F) = \sigma_2(f) = \alpha < \frac{1}{2}, \tag{4.2}$$

and F satisfies the linear differential equation

$$F^{(k)} - e^{Q(z)}F = 1. \tag{4.3}$$

Because of $\sigma_2(f) = \alpha < \frac{1}{2}$, we know that for $Q(z)$, there are three cases: (1) $Q(z)$ is a constant; (2) $Q(z)$ is a polynomial with degree $\deg Q \geq 1$; (3) $Q(z)$ is a transcendental entire function with order

$$\sigma(Q) = \beta \leq \alpha < \frac{1}{2},\ \sigma_2(e^Q) = \sigma(Q) = \beta.$$

Now we split this into three cases to prove.

Case (1). $Q(z)$ is a constant. Then Theorem 2 holds.

Case (2). $Q(z)$ is a polynomial with $\deg Q = n \geq 1$. We will get a contradiction with $\sigma_2(F) = \sigma_2(f) = \alpha < \frac{1}{2}$.

From the Wiman-Valiron theory (see [11, 13, 19]), there is a set $E_3 \subset (1, \infty)$ having logarithmic measure $lmE_3 < \infty$, we choose z satisfying $|z| = r \notin [0, 1] \bigcup E_3$ and $|F(z)| = M(r, F)$, then we have

$$\frac{F^{(k)}(z)}{F(z)} = (\frac{\nu(r)}{z})^k(1 + o(1)), \tag{4.4}$$

where $\nu(r)$ is the central index of F. Substituting (4.4) into (4.3), we obtain

$$(\frac{\nu(r)}{z})^k(1 + o(1)) = e^{Q(z)} + \frac{1}{F(z).} \tag{4.5}$$

Since $\sigma(F) = \sigma(f) = \infty, |F(z)| = M(r, F)$ and $\deg Q = n \geq 1$, for sufficiently large $|z| = r$ and any given $\varepsilon_1(> 0)$, by (4.5), we have

$$(\frac{\nu(r)}{r})^k \leq e^{r^{n+\varepsilon_1}}. \tag{4.6}$$

Since ε_1 is arbitrary, by (4.6) and Lemma 2, we have $\sigma_2(F) \le n$.

By Lemma 3, there is a point range $\{z_m = r_m e^{i\theta_m}\}$ such that $|F(z_m)| = M(r_m, F), \theta_m \in [0, 2\pi), \lim_{m\to\infty} \theta_m = \theta_0 \in [0, 2\pi), r_m \notin E_3 \bigcup [0,1], r_m \to \infty$, for any given ε satisfying that if $\alpha = 0$, then

$$0 < 3\varepsilon < \min\{\varepsilon_1, \frac{\pi}{4n}\};$$

if $\alpha > 0$, then

$$0 < 3\varepsilon < \min\{\alpha, \varepsilon_1, \frac{1}{2} - \alpha, \frac{\pi}{4n}\},$$

we see that if $\alpha > 0$, then we have

$$\exp\{r_m^{\alpha-\varepsilon}\} < \nu(r_m) < \exp\{r_m^{\alpha+\varepsilon}\}; \tag{4.7}$$

if $\alpha = 0$, then for any large $M(> 1)$, we have as r_m sufficiently large

$$\nu(r_m) > r_m^M. \tag{4.8}$$

Let

$$Q(z) = \alpha_n e^{i\theta_n} z^n + b_{n-1} z^{n-1} + \cdots + b_0,\ \alpha_n > 0,\ \theta_n \in [0, 2\pi).$$

By Lemma 4, there are $2n$ open angles for above ε,

$$S_j:\ -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon < \theta < -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon, \quad (j = 0, 1, \dots, 2n-1). \tag{4.9}$$

For the above θ_0, there are three cases: (i) $\theta_0 \in S_j$ where j is odd; (ii) $\theta_0 \in S_j$ where j is even; (iii) $\theta_0 = -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n}$ for some j. We again divide this into three cases.

Case (i): $\theta_0 \in S_j$ where j is odd. Since S_j is an open set and $\lim_{m\to\infty} \theta_m = \theta_0$, there is a $M_0 > 0$ such that $\theta_m \in S_j$ when $m > M_0$, by Lemma 4, we see that

$$Re\{Q(r_m e^{i\theta_m})\} < -d r_m^n \tag{4.10}$$

where $d = \alpha_n(1-\varepsilon)\sin(n\varepsilon) > 0$. For $\{z_m = r_m e^{i\theta_m}\}$, by (4.5) and $|F(z_m)| = M(r_m, F)$, we have

$$(\frac{\nu(r_m)}{z_m})^k (1 + o(1)) = e^{Q(r_m e^{i\theta_m})} + o(1). \tag{4.11}$$

If $\alpha > 0$, then by $3\varepsilon < \alpha$, (4.7), (4.10) and (4.11) , we have

$$\exp\{k r_m^{\alpha-\varepsilon}\} < (\nu(r_m))^k (1 + o(1)) < r_m^k \exp\{-d r_m^n\} + o(r_m^k). \tag{4.12}$$

Hence (4.12) is a contradiction. If $\alpha = 0$, then by (4.8), (4.10) and (4.11), we have

$$(4.13) \qquad r_m^{k(M-1)} < (\frac{\nu(r_m)}{r_m})^k(1+o(1)) < \exp\{-dr_m^n\} + o(1).$$

(4.13) is also a contradiction.

Case (ii): $\theta_0 \in S_j$ where j is even. Since S_j is an open set and $\lim_{m\to\infty}\theta_m = \theta_0$, there is $M_0 > 0$ such that $\theta_m \in S_j$ when $m > M_0$. By Lemma 4, we have

$$(4.14) \qquad Re\{Q(r_m e^{i\theta_m})\} > dr_m^n,$$

where $d = a_n(1-\varepsilon)\sin(n\varepsilon) > 0$. For $\{z_m = r_m e^{i\theta_m}\}$, by (4.7), (4.11) and (4.14), we have

$$(4.15) \qquad \exp\{kr_m^{\alpha+\varepsilon}\} > (\nu(r_m))^k(1+o(1)) > r_m^k \exp\{dr_m^n\} - o(r_m^k).$$

(4.15) contradicts the condition $\alpha + \varepsilon < \frac{1}{2}$.

Case (iii): $\theta_0 = -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n}$ for some $j \in \{0, 1, \ldots, 2n-1\}$. Since $Re\{a_n e^{i\theta_n}(re^{i\theta_0})^n\} = 0$, there are two subcases: iii(a) there is some $s(n-1 \geq s \geq 1)$ such that $Re\{b_\delta(re^{i\theta_0})^\delta\} = 0$ $(\delta = n-1, \ldots, s+1)$ and $Re\{b_s(re^{i\theta_0})^s\} \neq 0$; iii(b) $Re\{b_{n-1}(re^{i\theta_0})^{n-1}\} = \cdots = Re\{b_1(re^{i\theta_0})\} = 0$.

In subcase iii(a), if $Re\{b_s(re^{i\theta_0})^s\} < 0$, then when m is sufficiently large,

$$(4.16) \qquad Re\{b_s(r_m e^{i\theta_m})^s + \cdots + b_0\} < -d_1 r_m^s \ (d_1 > 0).$$

We use the notations $d_{n,m}$, $d_{n-1,m}$, $\ldots$, $d_{s+1,m}$ to denote the distances that the points

$$a_n e^{i\theta_n}(r_m e^{i\theta_m})^n, \ b_{n-1}(r_m e^{i\theta_m})^{n-1}, \ \ldots, \ b_{s+1}(r_m e^{i\theta_m})^{s+1}$$

go to the imaginary axis respectively. Since

$$Re\{a_n e^{i\theta_n}(re^{i\theta_0})^n\} = Re\{b_{n-1}(re^{i\theta_0})^{n-1}\} = \cdots = Re\{b_{s+1}(re^{i\theta_0})^{s+1}\} = 0$$

and

$$\lim_{m\to\infty}\theta_m = \theta_0,$$

we see that a ray $\arg z = \theta_0$ is an asymptotic line of $\{r_m e^{i\theta_m}\}$, i.e., the imaginary axis is an asymptotic line of

$$\{a_n e^{i\theta_n}(r_m e^{i\theta_m})^n\}, \ \{b_{n-1}(r_m e^{i\theta_m})^{n-1}\}, \ \ldots, \ \{b_{s+1}(r_m e^{i\theta_m})^{s+1}\}$$

respectively. So, for $j = n, n-1, \ldots, s+1$, when $m \to \infty$, we have

$$d_{j,m} \to 0.$$

Therefore, when m is sufficiently large,

$$(4.17)\qquad -1 < Re\{\alpha_n e^{i\theta_n}(r_m e^{i\theta_m})^n\} = d_{n,m} < 1,$$

$$(4.18)\qquad -1 < Re\{b_j(r_m e^{i\theta_m})^j\} = d_{j,m} < 1,\ (j = n-1, \ldots, s+1).$$

By (4.16), (4.17) and (4.18), we get that when m is sufficiently large,

$$(4.19)\qquad Re\{Q(r_m e^{i\theta_m})\} < -\frac{d_1}{2} r_m^s.$$

If $Re\{b_s(re^{i\theta_0})^s\} > 0$, by the arguing similarly as above, we see that, when m is sufficiently large,

$$(4.20)\qquad Re\{Q(r_m e^{i\theta_m})\} > \frac{d_1}{2} r_m^s\ (d_1 > 0).$$

By (4.19), (4.20) and the arguing similarly as in the proof of Cases (i) and (ii), we can get a contradiction.

In subcase iii(b), we see that there is a $M_1(> 0)$ such that when m is sufficiently large,

$$-M_1 < Re\{Q(r_m e^{i\theta_m})\} < M_1,$$

$$(4.21)\qquad \frac{1}{e^{M_1}} \le |e^{Q(r_m e^{i\theta_m})}| \le e^{M_1}.$$

By (4.7) (or (4.8)), (4.11) and (4.21), we have

$$(4.22)$$

$$\frac{1}{r_m^k}\exp\{kr_m^{\alpha-\varepsilon}\} - o(1) \le (\frac{\nu(r_m)}{r_m})^k(1+o(1)) - o(1) \le |e^{Q(r_m e^{i\theta_m})}| \le e^{M_1},$$

or

$$(4.23)\quad r_m^{k(M-1)} - o(1) \le (\frac{\nu(r_m)}{r_m})^k(1+o(1)) - o(1) \le |e^{Q(r_m e^{i\theta_m})}| \le e^{M_1}.$$

But both (4.22) and (4.23) are contradictory.

Case (3): $Q(z)$ is a transcendental entire function with $\sigma(Q) = \beta \le \alpha < \frac{1}{2}$. By the equation (4.3), we have

$$(4.24)\qquad e^{Q(z)} = \frac{F^{(k)}}{F} - \frac{1}{F}.$$

As in the proof of Case (2), we choose z satisfying $|z| = r \notin [0,1] \bigcup E_4$, ($E_4 \subset (1, \infty)$ having finite logarithmic measure and $|F(z)| = M(r, F)$, by the Wiman-Valiron Theorem, we get

$$(4.25)\qquad e^{Q(z)} = (\frac{\nu(r)}{z})^k(1+o(1)) + o(1),$$

where $\nu(r)$ is the central index of F. Since F is of infinite order, we see that $\nu(r) \geq |z|^M$ for any large $M > 0$. So that we can take a principal branch of $Log((\frac{\nu(r)}{z})^k(1+o(1))+o(1))$, and get

$$Q(z) = \log((\frac{\nu(r)}{z})^k(1+o(1))+o(1)). \tag{4.26}$$

Hence we have

$$|Q(z)| \leq |\log|(\frac{\nu(r)}{z})^k(1+o(1))+o(1)||+2\pi \leq k\log\nu(r)+O(1). \tag{4.27}$$

By Lemma 2 and $\sigma_2(F) = \alpha$, we have

$$\frac{\log\log\nu(r)}{\log r} \leq \alpha + 1$$

for sufficiently large r, by (4.27), we get

$$|Q(z)| \leq kr^{\alpha+1} + O(1). \tag{4.28}$$

But by Lemma 5(or 6), we know that there exists a set $H \subset (1, \infty)$ that have a logarithmic measure $lmH = \infty$, such that for all z satisfying $|z| = r \in H$, we have

$$|Q(z)| \geq M(r,Q)^c, \tag{4.29}$$

where $c(0 < c < 1)$ is a positive constant. Now for all z satisfying $|z| = r \in H\backslash E_4$ and $|F(z)| = M(r,F)$, by (4.28) and (4.29), we get

$$\frac{M(r,Q)^c}{r^{\alpha+1}} \leq k. \tag{4.30}$$

Since $Q(z)$ is transcendental, we see that

$$\frac{M(r,Q)^c}{r^{\alpha+1}} \to \infty,$$

which contradict (4.30). Theorem 2 is thus proved.

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