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ON THE ENTROPY NORM SPACES AND THE HARDY SPACE $\operatorname{Re} H^1$

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ABSTRACT. R. Dabrowski introduced certain natural multiplier operators which map from the entropy norm spaces of B. Korenblum into the Hardy space $\operatorname{Re} H^1$. We show that the images of the entropy norm spaces in $\operatorname{Re} H^1$ do not include all of that space.

1. INTRODUCTION

We consider the entropy norm spaces of Korenblum [4]. He defined an entropy function $\kappa : [0, 1] \to [0, 1]$ to be a concave, continuous, increasing function with $\kappa(0) = 0$. We denote by K_0 the set of such functions such that $\kappa'(0) = \lim_{x \to 0^+} k(x)/x = \infty$. According to Dabrowski [1] to each $\kappa \in K_0$ there is a unique probability measure $\mu = \mu_\kappa$ such that

$$\kappa(x) = \int_0^x \int_t^1 \frac{d\mu(u)}{u} \, dt.$$

Then the entropy norm of a continuous 1-periodic function $f \in C(T)$ (where $T = R \bmod 1$) is given by

$$||f||_\kappa = \int_0^1 \int_T \Omega_I(f) \, dt \, \frac{d\mu(s)}{s}$$

where $I = [t - s/2, t + s/2]$ and where $\Omega_I(f) = \sup\{|f(u) - f(v)| : u, v \in I\}$. (This norm was introduced by Korenblum [4]; this formula for the norm is due to Dabrowski [4].) We denote by $C_\kappa \subseteq C(T)$ the space of continuous 1-periodic functions of finite entropy norm.

In [2], Dabrowski introduced an operator $T_\kappa : C_\kappa \to \operatorname{Re} H^1$, given by

$$T_\kappa f(t) = \int_T \int_0^1 \frac{\chi_I(t)}{s^2} (f(t) - f(I)) \, d\mu(s) \, dx$$

where $I = [x - s/2, x + s/2]$, $f(I) = \frac{1}{|I|} \int_I f(t) \, dt$ is the average of f over I, and χ_I is the usual characteristic function of I. He showed that T_κ is a

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multiplier with coefficients

$$\beta_n = \beta_n(\kappa) = \frac{1}{2\pi^2 n^2} \int_{(0,1]} (\cos(2\pi n s) - 1 + 2\pi^2 n^2 s^2) \frac{1}{s^3}\, d\mu_\kappa(s)$$

(for $n > 0$ we set $\beta_{-n} = \beta_n$ and $\beta_0 = 0$). In [3], Dabrowski asked the question: given $f \in \operatorname{Re} H^1$, are there $\kappa \in K_0$ and $g \in C_\kappa$ such that $f = T_\kappa g$? (One reason why this question is of interest is because, as Dabrowski remarks, a positive answer would imply the Fefferman duality $(\operatorname{Re} H^1(0))^* = \mathbf{BMO}$.)

2. The main result

We are ready to give a negative answer to this question.

Theorem. *There is a function $f \in \operatorname{Re} H^1$ such that there are no $\kappa \in K_0$ and $g \in C_\kappa$ with $f = T_\kappa g$.*

Proof. We construct f as follows. Let h be the function with Fourier series $\sum_{n=1}^\infty (\sqrt{n} \log(n+1))^{-1} e_n$, where $e_n = e^{2\pi i n t}$. Then $h \in H^2$. So $h^2 \in H^1$ (see, e.g., Zygmund [6, VII (7.22), p. 275]). We let $f = \operatorname{Re}(h^2)$. So of course $f \in \operatorname{Re} H^1$. We have

$$h^2 \sim \sum_{n=1}^\infty \left(\sum_{j=1}^{n-1} b_j b_{n-1} \right) e_n$$

where $b_j = (\sqrt{j} \log(j+1))^{-1}$. It is not hard to show that f has Fourier series $\sum_{n=1}^\infty a_n \cos(2\pi n t)$ where $a_n \geq \text{const.}(\log(n+1))^{-2}$ for $n = 1, 2, 3, \ldots$.

Now we suppose that there is a $\kappa \in K_0$ and a $g \in C_\kappa$ such that $T_\kappa g = f$. We write g as $\sum c_n e_n$. Then since $T_\kappa g = f$ we have $c_n = a_n / \beta_n$, $n \geq 1$. This enables us to write g as $\sum_1^\infty c_n \cos(2\pi n t)$ where $c_n \geq 0$ for all $n > 0$.

We assume that $g \in C_\kappa$ which implies that g is bounded. Consequently (since g has a cosine series with positive coefficients), we must have $\sum c_n < \infty$ or $\sum a_n / \beta_n < \infty$. Therefore

$$\sum_{n=1}^\infty \left(\frac{1}{\log(n+1)} \right)^2 \frac{1}{\beta_n} < \infty. \tag{1}$$

We must also have

$$\sum_{n=1}^\infty \frac{1}{n^2} \beta_n < \infty. \tag{2}$$

[By Lang [5], β_n compares with $n\kappa(1/n) - n^2 \int_0^{1/n} \kappa(t)\, dt = \overline{\kappa}'(1/n)$ where $\overline{\kappa}(x) = \frac{1}{x} \int_0^x \kappa(t)\, dt$. We have $\overline{\kappa}(x) = \int_0^x \overline{\kappa}'(t)\, dt$, so this integral must be convergent; we may estimate this integral by the sum

$$\sum_{n=1}^\infty \left(\frac{1}{n} - \frac{1}{n+1} \right) \overline{\kappa}' \left(\frac{1}{n} \right) \approx \sum_{n=1}^\infty \frac{1}{n^2} \beta_n.$$

(Note that $\overline{\kappa}'(x) = (1/x^2)(x\kappa(1/x) - \int_0^x \kappa(t)\, dt)$ is the product of $1/x^2$ and a function which goes to 0 monotonically as $x \to 0$. So the integral and the sum compare.)]

But (1) and (2) are not compatible. Indeed, suppose the sums (1) and (2) are both finite. Then by the Cauchy-Schwarz inequality

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{\log(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\sqrt{\beta_n}\right) \left(\frac{1}{\log(n+1)} \frac{1}{\sqrt{\beta_n}}\right)$$
$$\leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \beta_n\right)^{1/2} \left(\sum_{n=1}^{\infty} \left(\frac{1}{\log(n+1)}\right)^2 \frac{1}{\beta_n}\right)^{1/2} < \infty,$$

which is nonsense. So there cannot be $\kappa \in K_0$, $g \in C_\kappa$ such that $T_\kappa g = f$, and we are done. □

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