J. Math. Soc. Japan
Vol. 28, No. 4, 1976

On the equations of compact Riemann surfaces of genus 3 and the generalized Teichmüller spaces

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(Received Oct. 25, 1975)
(Revised Dec. 9, 1975)

The purpose of this paper is first, to classify and determine the equations of compact Riemann surfaces of genus 3 and 4 with non-trivial automorphisms and second, to investigate the analyticity, as functions of moduli, of the coefficients which appear in these equations by applying the results which were introduced in the previous papers [**5**, **6**]. Namely, in [**5**], we introduced a family $\Omega(g', n, \{\nu_i\})$ of Riemann surfaces with non-trivial automorphisms and we gave a necessary and sufficient condition for Ω to be non-empty and we constructed a generalized Teichmüller space of $\Omega(g', n, \{\nu_i\})$. In [**6**], we proved a lemma which gave the number of elements of $\Omega(g', n, \{\nu_i\})$ when we fixed the r points arbitrarily on the Riemann surface of genus g'. In this paper, using these lemmas, we obtain equations of Riemann surfaces of genus 3 and 4 with non-trivial automorphisms. In $\Omega(g', n, \{\nu_i\})$, if $g'=0$ then the forms of equations are quite simple (Theorem (2.2.2)). If $g'=1$ and $g=3$, then we have two types of the equations (Theorems (3.1.4), (3.3.7)):

(i) $y^2=ax^2+bx+c+(x-\delta)\{x(x-1)(x-\beta)\}^{1/2}$

(ii) (1) $y^3=(x-\xi_1)(x-\xi_2)+c\{x(x-1)(x-\beta)\}^{1/2}$

(2) $y^3=ax^2+bx+c+(x-\delta)\{x(x-1)(x-\beta)\}^{1/2}$.

Here, in each case, a, b, c, δ, β, ξ_1, ξ_2 are suitable constants, which are related algebraically. If $g'=2$ and $g=3$, we have two types of equations (Theorem (3.5.2)):

(i) $y^2=(x-\beta_i)(a(x)+b(x)\{(x-\beta_1)\cdots(x-\beta_5)\}^{1/2})^2 \quad (1\leqq i\leqq 5)$

(ii) $y^2=(x-\beta_i)(x-\beta_j)(a(x)+b(x)\{(x-\beta_1)\cdots(x-\beta_5)\}^{1/2})^2 \quad (1\leqq i,\ j\leqq 5;\ i\neq j)$.

Here, in each case, $a(x)$ and $b(x)$ are arbitrary polynomials. The same construction goes in the case of Riemann surfaces of genus 4 with non-trivial automorphisms. However, in the case of Riemann surfaces of genus 5, even if it has non-trivial automorphisms, we cannot any longer expect to obtain, in general, the above form of equations by our method. In fact, in this case, we may have many Riemann surfaces of genus 3 which have no non-trivial automorphisms on the base space. These are contents of § 2 and § 3.

In § 4, we give some theorems which show the analyticity of the coefficients of the equations of Riemann surfaces of our family, as functions on the generalized Teichmüller space. The coefficients in (3.1.4) or (4.5.3), and (3.3.7) or (4.5.4) are single-valued holomorphic functions on the generalized Teichmüller spaces (Theorems (4.5.19), (4.5.21)). We give also explicit representations of these functions by the quotients of Theta constants (Theorem (4.6.7)). Here we considered the Siegel space. However, it would be interesting to study the space which Shimura has introduced in [**9**] instead of the Siegel space.

The author wishes to thank Professor Shoji Koizumi for his generous assistance at several points in this paper and also thank Professor Kotaro Oikawa who taught an intelligible proof of Lemma 4.5.7 to him.

§ 1. Preliminaries.

1.1. Let R be a compact Riemann surface of genus g ($\geqq 2$). Let G be a subgroup of the group of all automorphisms of R, whose order is finite ($\geqq 2$). We shall denote by V the complex vector space of all differentials of the first kind on R. Every element σ of G induces a linear mapping of V onto V; and fixing a basis of V, we obtain a matrix representation Φ of G. If G is a cyclic group of order n, then Φ can be determined by $\Phi(\sigma)$ for a generator σ of G. Further, we can transform $\Phi(\sigma)$ into a diagonal form:

$$\Phi(\sigma)=\begin{bmatrix}\alpha_1 & & 0\\ & \ddots & \\ 0 & & \alpha_g\end{bmatrix}$$

where $\alpha_1, \cdots, \alpha_g$ are n-th roots of unity. Let g' be the number of indices k for which $\alpha_k=1$. It is obvious that g' is equal to the genus of $R'=R/G$. Let K (resp. K') be the algebraic function field of R (resp. R'). Then K is a Galois extension of K' whose Galois group can be identified with G.

1.2. Let n be a prime number, g a positive integer >1, and Φ a complex $g\times g$ matrix such that $\Phi^n=I_g$. Let us consider a couple (R, σ) formed by a compact Riemann surface of genus g and an automorphism σ of order n such that the representation of σ in the vector space V is equivalent to Φ. We say that (R, σ) and (R', σ') are isomorphic if there exists a holomorphic bijection $f: R\to R'$ such that $f\sigma=\sigma' f$. We denote by $\langle R, \sigma\rangle$ the isomorphism class of (R, σ) with given n and Φ, and we denote by $\Omega(n, \Phi)$ the set of all classes $\langle R, \sigma\rangle$.

1.3. In order to obtain all Riemann surfaces of genus g with non-trivial automorphisms, we need only to find all non-empty $\Omega(n, \Phi)$ for every prime number n. In fact, if a prime number n divides the order of G, then G has a subgroup of order n. Therefore, we will give the conditions for $\Omega(n, \Phi)$ to be non-empty.

1.4. For that purpose, we introduce subfamilies $\Omega(n, g', \{\nu_1, \cdots, \nu_r\})$ of $\Omega(n, \Phi)$ as follows. Let n be as before a prime number and $\{\nu_1, \cdots, \nu_r\}$ be a set of positive integers such that $1 \leqq \nu_i < n$ $(1 \leqq i \leqq r)$. We denote by $\Omega(g', n, \{\nu_1, \cdots, \nu_r\})$ the set of all isomorphism classes of (R, σ) satisfying the following conditions:

(1.4.1) σ is an automorphism of order n with r fixed points.

(1.4.2) R/G is of genus g', where G is the cyclic group generated by σ.

(1.4.3) σ can be represented as $t_i \rightarrow \zeta^{\nu_i} t_i + \cdots$ $(\zeta = \exp(2\pi i/n))$, where t_i is a local parameter at q_i, a fixed point of σ and i runs from 1 to r. It is obvious that the coefficient ζ^{ν_i} does not depend on the choice of local parameter.

1.5. There is a relation between the exponents $\{\nu_1, \cdots, \nu_r\}$ and the representation $\Phi(\sigma)$ as follows.

(1.5.1) Let $\mathrm{tr}\,\Phi(\sigma)$ be the trace of the matrix $\Phi(\sigma)$. Then, we have the formula

$$\mathrm{tr}\,\Phi(\sigma) = 1 + \sum_{i=1}^{r} \zeta^{\nu_i}/(1-\zeta^{\nu_i}).$$

PROOF. See [**3**; pp. 267-298].

1.6. Now, the representation $\Phi(\sigma)$ is completely determined by $\{\nu_1, \cdots, \nu_r\}$. Therefore, it is sufficient for our purpose to give conditions for $\Omega(g', n, \{\nu_1, \cdots, \nu_r\})$ to be non-empty. For that we need one more lemma from Galois theory.

(1.6.1) Let K be a Galois extension of a field K'. Suppose that the Galois group is cyclic of order n and K' contains a primitive n-th root of unity ζ. Then for every generator σ of the Galois group, there exists an element y of K such that $\sigma(y) = \zeta y$, $K = K'(y)$ and $y^n \in K'$.

1.7. We shall apply (1.6.1) to the field of algebraic functions which are considered above. Let R, R', K, K', σ, Φ be as above. Let ζ be $\exp(2\pi i/n)$ and t_i be a local parameter at the fixed point q_i of σ. Let the expansion of y at the point q_i be $y = c_i t_i^{k_i} + \cdots$ $(c_i \neq 0)$. Then we have $k_i \nu_i \equiv 1 \pmod{n}$ for $i = 1, \cdots, r$. We can state the conditions for $\Omega(g', n, \{\nu_1, \cdots, \nu_r\})$ to be non-empty in terms of these integers $k_1, \cdots, k_n$ as follows.

(1.7.1) A necessary and sufficient conditions for $\Omega(g', n, \{\nu_i\})$ to be non-empty is $\sum_{i=1}^{r} k_i \equiv 0 \pmod{n}$.

PROOF. See [**5**; pp. 134-135].

Now, let R' be an arbitrarily fixed Riemann surface of genus g' and let $q'_1, \cdots, q'_r$ be an arbitrarily fixed set of r $(\geqq 0)$ points on R'. We wish to count the number of elements $\langle R, \sigma\rangle$ of the family $\Omega(g', n, \{\nu_i\})$ which are ramified only on these points $\{q_i'\}$. This number comes from the following lemma.

(1.7.2) If $r\geqq 1$, then the number of elements $\langle R, \sigma\rangle$ of $\Omega(g', n, \{\nu_i\})$ which satisfy our condition is exactly $n^{2g'}$; and if $r=0$, then the number is exactly $n^{2g'}-1$.

The proof has appeared in [**6**], but we will outline in this paper.

PROOF. Let $\langle R, \sigma\rangle$ of Ω be ramified only on these points $\{q_i'\}$ on R'. Let (R, σ) be a representative of $\langle R, \sigma\rangle$. Then, using the notations as above, there exists an element of y of K such that $K=K'(y)$, $z=y^n\in K'$ and $\sigma(y)=\zeta y$. The element z is a meromorphic function on R' and we have $\mathrm{div}_{R'}(z)=\sum_{i=1}^{r}k_iq_i'+nD'$. Here D' is a divisor on R'. Then, by (1.7.1) we have $\sum_{i=1}^{r}k_i\equiv 0 \pmod n$. Hence we may put $\sum_{i=1}^{r}k_i=nl$ (l is an integer). Let q_0' be an arbitrary point distinct from $\{q_i'\}$ on R'. Then we have $\deg(\sum_{i=1}^{r}k_iq_i'-nlq_0')=0$ and $\deg(-D'-lq_0')=0$. The divisor class group of degree 0 of K' is isomorphic to its Jacobian variety J'. Therefore, the divisor class of $-D'-lq_0'$ determines a point τ of J'. Thus we can associate τ with $K=K'(y)$ of Ω. It is easy to see that is independent of the choice of the function y.

Now, let the point of J' determined by the divisor class of $\sum_{i=1}^{r}k_iq_i'-nq'$ be μ. For the fixed μ, consider $M=\{\tau\in J' \mid n\tau=\mu\}$. We see easily that the above association is surjective. We will check that it is injective. Let $K=K'(y)$, $K_1=K'(y_1)$ be two algebraic function fields to which the same τ is associated. Then we have

$$\mathrm{div}_{R'}(z)=nD'+\sum_{i=1}^{r}k_iq_i', \qquad \deg(-D'-lq_0')=0;$$

$$\mathrm{div}_{R'}(z_1)=nD_1'+\sum_{i=1}^{r}k_iq_i', \qquad \deg(-D_1'-lq_0')=0,$$

and $(-D'-lq_0')$ is linearly equivalent to $(-D_1'-lq_0')$. Hence we have $\mathrm{div}_{R'}(z)-\mathrm{div}_{R'}(z_1)=n(D'-D_1')$. Therefore, $\mathrm{div}_{R'}(z)-\mathrm{div}_{R'}(z_1)$ is linearly equivalent to $n\,\mathrm{div}(u)$ $(u\in K')$. Thus, we can conclude that $K=K_1$.

It is easy to see that the number of elements of M is equal to $n^{2g'}$ if $r\geqq 1$. If $r=0$, we must exclude the case $\tau=0$; and then the number is equal to $n^{2g'}-1$. Here we should keep in mind that we count the number in Ω. If we count the isomorphic function fields in usual sense, then it becomes $(n^{2g'}-1)/(n-1)$.

§ 2. A classification of Riemann surfaces by means of $\Omega(g', n, \{\nu_1, \cdots, \nu_r\})$.

2.1. By the formula of Riemann-Hurwitz, i. e. $2g-2=n(2g'-2)+(n-1)r$, we can classify the Riemann surfaces of genus 3 as follows;

(2.1.1) $g'=0$. We see that only the following three cases may take place:

$$\text{(i)}\quad n=2,\ r=8; \qquad \text{(ii)}\quad n=3,\ r=5; \qquad \text{(iii)}\quad n=7,\ r=3.$$

In each case, $\Omega(g', n, \{\nu_i\})$ is actually non empty.

(2.1.2) $g'=1$. Here only the following three cases may take place:

$$\text{(i)}\quad n=2,\ r=4; \qquad \text{(ii)}\quad n=3,\ r=2; \qquad \text{(iii)}\quad n=5,\ r=1.$$

However, by (1.5.1) the case (iii) cannot happen. On the other hand, by (1.7.1) the cases (i) and (ii) do in fact occur. In the former, we have $\nu_1=\nu_2=\nu_3=\nu_4=1$ and in the latter, we have $\nu_1=1$, $\nu_2=2$ or $\nu_1=2$, $\nu_2=1$.

(2.1.3) $g'=2$. Here the Riemann-Hurwitz formula only allows

$$\text{(i)}\quad n=2,\ r=0.$$

In this case, R is an unramified covering surface of R' and clearly the case (i) can take place.

2.2. In (2.1.1), all the Riemann surfaces are considered as n-sheeted covering surfaces of the Riemann sphere R_0; and we may assume that one of the points of R which are fixed by σ is over the point ∞ of R_0. By (1.6.1), we can write the equation of R in the form

$$\text{(2.2.1)} \qquad y^n=(x-a_1)^{m_1}\cdots(x-a_s)^{m_s}, \qquad n \nmid m_1+\cdots+m_s$$

with distinct complex numbers $a_1, \cdots, a_s$ over which we have all fixed points of R except for the one over the point at infinity and with positive integers $m_1, \cdots, m_s$ less than n. Therefore, we obtain the following theorem.

(2.2.2) THEOREM. *In* (2.2.1), *the equations of Riemann surfaces of genus* 3 *are given by*

$$\text{(i)} \qquad y^2=(x-a_1)\cdots(x-a_7),$$

$$\text{(ii)} \qquad y^3=(x-a_1)^{m_1}\cdots(x-a_4)^{m_4}, \qquad 3\nmid m_1+\cdots+m_4,$$

$$\text{(iii)} \qquad y^7=(x-a_1)^{m_1}(x-a_2)^{m_2}, \qquad 7\nmid m_1+m_2.$$

In (2.1.2), R is considered as an n-sheeted covering surface of R' whose genus is one. On the other hand, R' is considered as a two-sheeted covering surface of the Riemann sphere R_0. Therefore, in (2.1.2) (i), R is considered as a four-sheeted covering surface of R_0; and in (2.1.2) (ii), R is considered as a six-sheeted covering surface of R_0. In each case, we may assume that the equation of R' is given by $v^2=x(x-1)(x-\beta)$. If we put $K'=C(x, v)$, $v^2=x(x-1)(x-\beta)$, then in (i), by (1.6.1) there exists a $y\in K$ such that $K=K'(y)$, $y^2\in K'$ and $\sigma(y)=-y$. Hence $\boldsymbol{C}$ is a complex number field. Hence, we can write the equation of R in the form

$$y^2=a_0(x)+a_1(x)\{x(x-1)(x-\beta)\}^{1/2}, \tag{2.2.3}$$

where $a_0(x)$ and $a_1(x)$ are rational functions in x. Similarly, in (ii), there exists a $y\in K$ such that $K=K'(y)$, $y^3\in K'$ and $\sigma(y)=\zeta y$ ($\zeta=\exp(2\pi i/3)$). Thus, we can write the equation of R in the form

$$y^3=b_0(x)+b_1(x)\{x(x-1)(x-\beta)\}^{1/2}, \tag{2.2.4}$$

where $b_0(x)$ and $b_1(x)$ are rational functions in x. In the next section, we shall determine these functions $a_0(x)$, $a_1(x)$ and $b_0(x)$, $b_1(x)$ in the explicit form.

In (2.1.3), R is considered as a two-sheeted covering surface of R' whose genus is two. On the other hand, R' is a two-sheeted covering surface of R_0. In this case, we have no fixed point on R and we may assume that the equation of R' is given by $v^2=(x-\beta_1)\cdots(x-\beta_5)$ with distinct complex numbers $\beta_1, \cdots, \beta_5$. In the same manner as in (2.1.2), we can write the equation of R in the form

$$y^2=c_0(x)+c_1(x)\{(x-\beta_1)\cdots(x-\beta_5)\}^{1/2}, \tag{2.2.5}$$

where $c_0(x)$ and $c_1(x)$ are rational functions in x. We shall consider these functions in the next section.

2.3. Quite similarly we can classify the Riemann surfaces of genus 4 with non-trivial automorphisms.

(2.3.1) $g'=0$. We see that only the following three cases may take place:

(i) $n=2$, $r=10$; (ii) $n=3$, $r=6$; (iii) $n=5$, $r=4$.

In each case, $\Omega(g', n, \{\nu_i\})$ is actually non-empty.

(2.3.2) $g'=1$. Here only the following three cases may take place:

(i) $n=2$, $r=6$; (ii) $n=3$, $r=3$; (iii) $n=7$, $r=1$.

However, by (1.5.1) we see that the case (iii) cannot happen. On the other hand, by (1.7.1) the cases (i) and (ii) do in fact occur. In the former, we have $\nu_1=\cdots=\nu_6=1$ and in the latter, we have $\nu_1=\nu_2=\nu_3=1$ and $\nu_1=\nu_2=\nu_3=2$.

(2.3.3) $g'=2$. Here only the following two may take place:

(i) $n=2$, $r=2$; (ii) $n=3$, $r=0$.

By (1.7.1), the case (i) does occur and we have $\nu_1=\nu_2=1$. In the case (ii), R

is an unramified covering surface of R' and clearly the case (ii) can take place. We shall give in the next section the equations of these Riemann surfaces.

§ 3. Equations of Riemann surfaces of genus 3 and 4.

3.1. Now, we are in a position to solve the problems in § 3. Let us start with (2.1.2). For this purpose, we consider the divisors of the adjoint functions (2.2.3) and (2.2.4) on R'. In (2.1.2) (i), we have $\mathrm{div}_{R'}(y^2)=k_1q_1'+k_2q_2'+k_3q_3'+k_4q_4'+2D$. Here D is a divisor on R', and for each $i=1, \cdots, 4$, k_i is an integer such that the Laurent expansion of y at q_i in the local parameter t_i is $y=c_it_i^{k_i}+\cdots$ $(c_i\neq 0)$. In (2.1.2) (ii), we have $\mathrm{div}_{R'}(y^3)=k_1q_1'+k_2q_2'+3D$. Here D is again a divisor on R' and k_1, k_2 are integers similar to those in (i).

In each case by the theorem of Abel, we can assume that one of the fixed points, say q_4' in (i), and q_2' in (ii), is equal to q_∞' which is over the point at infinity.

In (i), we may put $k_1=k_2=k_3=1$. We consider points q_α' such that $-(q_1'+q_2'+q_3')+3q_\infty'$ is linearly equivalent to $2(q_\alpha'-q_\infty')$. There are four distinct such points. We denote these points by q_{α_1}', q_{α_2}', q_{α_3}' and q_{α_4}'. Consider the divisors on R'

$$(3.1.1) \qquad q_1'+q_2'+q_3'-5q_\infty'+2q_{\alpha_i}' \qquad (1\leqq i\leqq 4).$$

Any such divisor is the divisor of a function on R'. We denote these functions by z_{α_1}, z_{α_2}, z_{α_3} and z_{α_4}. We see that $\mathrm{div}_{R'}(z_{\alpha_i}/z_{\alpha_j})=2(q_{\alpha_i}'-q_{\alpha_j}')$ for $i\neq j$ $(1\leqq i,\ j\leqq 4)$. Obviously there is no function f on R' that satisfies the relation $\mathrm{div}_{R'}(f)=q_{\alpha_i}'-q_{\alpha_j}'$. On the other hand, it is easy to show that $K'(y_1)=K'(y_2)$ if and only if y_2 is equal to by_1, where b is ane lement of K'. Therefore, we can conclude that $K'(z_{\alpha_i}^{1/2})\neq K'(z_{\alpha_j}^{1/2})$ if $i\neq j$ $(1\leqq i,\ j\leqq 4)$. Consequently, by (1.7.2) we see that the desired function fields are nothing but $K'(z_{\alpha_i}^{1/2})$ $(1\leqq i\leqq 4)$.

Now, the function z_{α_i} $(1\leqq i\leqq 4)$ is represented by $z_{\alpha_i}=r_0(x)+r_1(x)\{x(x-1)(x-\beta)\}^{1/2}$ with rational functions $r_0(x)$ and $r_1(x)$. However, it is easy to show that these are polynomials. In fact, by the form of $\mathrm{div}_{R'}(z_{\alpha_i}^{1/2})$, z_{α_i} is holomorphic at any points except for q_∞'. Then, we see that $r_0(x)-r_1(x)\{x(x-1)(x-\beta)\}^{1/2}$ is also holomorphic at any points except for q_∞'. Therefore, $r_0(x)$ is a polynomial and $r_1(x)\{x(x-1)(x-\beta)\}^{1/2}$ is holomorphic except at q_∞', i. e., $r_1(x)^2x(x-1)(x-\beta)$ is holomorphic except at q_∞'. This shows $r_1(x)$ is a polynomial. Hence, we may denote z_{α_i} $(1\leqq i\leqq 4)$ by $z_{\alpha_i}=a_0(x)+a_1(x)\{x(x-1)(x-\beta)\}^{1/2}$ with polynomials $a_0(x)$ and $a_1(x)$. Then it follows that $a_0(x)$ is of at most degree 2 and $a_1(x)$ is of degree 1. Because $\mathrm{div}_{R'}(z_{\alpha_i})$ is represented by (3.1.1). Thus, we can write

$$(3.1.2) \qquad z_{\alpha_i}=ax^2+bx+c+(x-\delta)\{x(x-1)(x-\beta)\}^{1/2}$$

with complex numbers a, b, c and δ. It remains to be seen what these complex numbers are. To solve this question, consider how R covers the Riemann sphere R_0. Let the projections of q_1, q_2, q_3 and q_∞ on R_0 be t_1, t_2, t_3 and ∞ respectively. Here, q_i is the point of R over q'_i of R' and q_∞ is the point of R over q'_∞ of R'. If any of t_1, t_2, t_3 does not coincide with any of $0, 1, \beta$, then each of q_1, q_2, q_3 must be a ramification point whose index is one. There are no other ramification points on R except for those points which are over the points ∞, β, 0, 1, t_1, t_2 and t_3. Then we have an identity

$$(3.1.3)\qquad (x-\delta)^2x(x-1)(x-\beta)-(ax^2+bx+c)^2 = (x-t_1)(x-t_2)(x-t_3)(x-\alpha_i)^2$$

for any x of R_0. Here α_i is the projection of $q_{\alpha i}$ on R. Hence, we obtain the following theorem.

(3.1.4) THEOREM. *In* (2.1.2) (i), *choosing a, b, c and δ which satisfy the identity* (3.1.3) *for each α_i* ($1\leqq i\leqq 4$), *we obtain the equation of R as follows:*

$$y^2=ax^2+bx+c+(x-\delta)\{x(x-1)(x-\beta)\}^{1/2}.$$

Here we should keep in mind that a, b, c and δ can be considered as functions of four parameters t_1, t_2, t_3 and β.

3.2. REMARK. If t_1, t_2, t_3 coincide with $0, 1, \beta$ respectively, then we have $a=b=c=0$ and $\delta=\alpha$. Obviously, we have $0, 1, \beta, \infty$ as the values of four α's. Thus, we obtain as a special case of (3.1.4)

$$(3.2.1)\qquad \begin{aligned} &\text{(i)} && y^2=\{x(x-1)(x-\beta)\}^{1/2} \\ &\text{(ii)} && y^2=x\{x(x-1)(x-\beta)\}^{1/2} \\ &\text{(iii)} && y^2=(x-1)\{x(x-1)(x-\beta)\}^{1/2} \\ &\text{(iv)} && y^2=(x-\beta)\{x(x-1)(x-\beta)\}^{1/2}. \end{aligned}$$

3.3. In 2.1.2 (ii), we may put $k_1=1$ or $k_1=2$. For $k_1=1$, we consider points q'_α such that $-q'_1+q'_\infty$ is linearly equivalent to $3(q'_\alpha-q'_\infty)$. There are nine such points. We denote these points by $q'_{\alpha 1}, \cdots, q'_{\alpha_q}$. Consider the divisor on R'

$$(3.3.1)\qquad q'_1-4q'_\infty+3q'_{\alpha i} \qquad (1\leqq i\leqq 9).$$

Any such divisor is the divisor of a function on R'. We denote these functions by $z_{\alpha 1}, \cdots, z_{\alpha_q}$. By the same manner as in 2.1.2 (i), we can conclude that $K'(z_{\alpha_i}^{1/3})\neq K'(z_{\alpha_j}^{1/3})$ if $i\neq j$ ($1\leqq i, j\leqq 9$). We see that the desired function fields are nothing but $K'(z_{\alpha_i}^{1/3})$ ($1\leqq i\leqq 9$).

Now, the function $z_{\alpha_i}(1\leqq i\leqq 9)$ is represented by $z_{\alpha i}=r_0(x)+r_1(x)\{x(x-1)(x-\beta)\}^{1/2}$ with rational functions $r_0(x)$ and $r_1(x)$. As in (2.1.2) (i), it is easy to see that $r_0(x)$ and $r_1(x)$ are polynomials. Hence, we may denote $z_{\alpha i}$ by $z_{\alpha i}=b_0(x)+b_1(x)\{x(x-1)(x-\beta)\}^{1/2}$ with polynomials $b_0(x)$ and $b_1(x)$. Then it follows $b_0(x)$ is of degree 2 and $b_1(x)$ is of degree 0. Thus we can write

$$z_{\alpha i}=(x-\xi_1)(x-\xi_2)+c\{x(x-1)(x-\beta)\}^{1/2} \tag{3.3.2}$$

with complex numbers ξ_1, ξ_2 and c. We must show what these complex numbers are. As in (i), let the projections of q_1 and q_∞ on R_0 be t_1 and ∞ respectively. If t_1 does not coincide with any of $0, 1, \beta, \infty$, then q_1 must be a ramification points whose index is two. There are no other ramification points except for those points which are over the points $\infty, 0, 1, \beta$ and t_1. Then we have an identity

$$(x-\xi_1)^2(x-\xi_2)^2-c^2x(x-1)(x-\beta)=(x-t_1)(x-\alpha_i)^3 \tag{3.3.3}$$

for any x of R_0. Here α_i is the projection of $q_{\alpha i}$ on R.

For $k_1=2$, we consider points q'_α such that $-2q'_1+2q'_\infty$ is linearly equivalent to $3(q'_\alpha-q'_\infty)$ and consider the divisors on R'

$$2q'_1-5q'_\infty+3q'_{\alpha i} \qquad (1\leqq i\leqq 9). \tag{3.3.4}$$

Any such divisor is the divisor of a function on R'. We denote these functions by $z_{\alpha 1}, \cdots, z_{\alpha q}$. Then we have

$$z_{\alpha i}=ax^2+bx+c+(x-\delta)\{x(x-1)(x-\beta)\}^{1/2} \tag{3.3.5}$$

with complex numbers a, b, c and δ which satisfy an identity

$$(x-\delta)^2x(x-1)(x-\beta)-(ax^2+bx+c)^2=(x-t_1)^2(x-\alpha_i)^3 \tag{3.3.6}$$

for any x of R_0. Hence, we obtain the following theorem.

(3.3.7) THEOREM. *In* (2.1.2) (ii), *for* $k=1$ *choosing* ξ_1, ξ_2 *and* c *which satisfy the identity* (3.3.3) *for each* α_i $(1\leqq i\leqq 9)$, *we obtain the equation of* R *as follows:*

$$y^3=(x-\xi_1)(x-\xi_2)+c\{x(x-1)(x-\beta)\}^{1/2}.$$

Here we should keep in mind that ξ_1, ξ_2 *and* c *can be considered as functions of two parameters* t_1 *and* β. *For* $k=2$, *choosing* a, b, c *and* δ *which satisfy the identity* (3.3.6) *for each* α_i $(1\leqq i\leqq 9)$, *we obtain the equation of* R *as follows:*

$$y^3=ax^2+bx+c+(x-\delta)\{x(x-1)(x-\beta)\}^{1/2}.$$

Here we should keep in mind that a, b, c *and* δ *can be considered as functions of two parameters* t_1 *and* β.

3.4. REMARK. In (3.3.1), if t_1 coincides with any one of $0, 1, \beta$, say $t_1=0$, then we may put $\xi_1=0$ in (3.3.2). It is interesting to investigate the locus of the points $q_{\alpha i}$ $(1 \leqq i \leqq 9)$. We have an equation in α:

$$\alpha^6 - 6\beta\alpha^4 + 4\beta\alpha^3 + 4\beta^2\alpha^3 - 3\beta^2\alpha^2 = 0\,. \tag{3.4.1}$$

This equation has five distinct solutions α_0, α_1, α_2, α_3 and α_4. Here $\alpha_0=0$ and α_1, α_2, α_3, α_4 are distinct from both 1 and β. We have two points of R' over each α_i $(1 \leqq i \leqq 4)$. We denote these points by $q'_{\alpha i_1}$, $q'_{\alpha i2}$. For each α_i, we can determine c and ξ_2. The unknown c has two solutions, say k and $-k$. Thus, we obtain

(3.4.2) (i) $\mathrm{div}_{R'}(z_{\alpha i1}) = q'_1 - 4q'_\infty + 3q'_{\alpha i1}$,

$$y^3_{\alpha i_1} = x(x-\xi_2) + k\{x(x-1)(x-\beta)\}^{1/2}\,.$$

(ii) $\mathrm{div}_{R'}(z_{\alpha i2}) = q'_1 - 4q'_\infty + 3q'_{\alpha i2}$,

$$y^3_{\alpha i2} = x(x-\xi_2) - k\{x(x-1)(x-\beta)\}^{1/2}\,.$$

As in 2.1.2 (i), we can show that the eight function fields $K'(y_{\alpha i_1})$ and $K'(y_{\alpha i2})$ $(1 \leqq i \leqq 4)$ are distinct from each other. For $\alpha_0=0$, we have $c=0$ and $\xi_2=0$. Therefore, the equation is

$$y^3 = x^2\,. \tag{3.4.3}$$

We can replace this equation with

$$y^3 = (x-1)(x-\beta)\{x(x-1)(x-\beta)\}^{1/2}\,. \tag{3.4.5}$$

In fact, we have $\mathrm{div}_{R'}(y^3) = 4q'_{0'} - 4q'_\infty$ and $\mathrm{div}_{R'}(Y^3) = q'_{0'} - 7q'_\infty + 3q'_{1'} + 3q'_{\beta'}$. Thus, we must have $\mathrm{div}_{R'}(y^3/Y^3) = 3(q'_{0'} + q'_\infty - q'_{1'} - q'_{\beta'})$. On the other hand, obviously $q'_{0'} - q'_{1'} - q'_{\beta'} + q'_\infty$ is linearly equivalent to zero. In (3.3.4), if t_1 coincides with any one of 0, 1, β, then we have a similar result.

3.5. Now, we investigate equations in (2.1.3). By the classification in § 2, we have $n=2$ and $r=0$ and if we put $K'=C(x, v)$, $v^2=(x-\beta_1)\cdots(x-\beta_5)$, then there exists an element y of K such that $K=K'(y)$, $y \in K'$ and $\sigma(y)=-y$. We can construct two kinds of functions on R' such that

(3.5.1) (i) $y^2 = (x-\beta_i) \qquad (1 \leqq i \leqq 5)$

(ii) $y^2 = (x-\beta_i)(x-\beta_j) \qquad (1 \leqq i,\ j \leqq 5,\ i \neq j)\,.$

If y_a, y'_a are two functions of (3.5.1) (i), then we see that $K'(y_a)$ is not equal to $K'(y'_a)$ as before. If y_b, y'_b are two of (3.5.1) (ii), then we can write $\mathrm{div}_{R'}(y_b^2/y_b'^2) = 2q'_{\beta_k} + 2q'_{\beta l} - 2q'_{\beta i} - 2q'_{\beta_j}$. Here, k, l, i, j are integers from 1 to 5 and $k \neq l$, $i \neq j$, and $\{k, l\}$ is not equal to $\{i, j\}$. If we put $D=q'_{\beta i}+q'_{\beta_j}$, then we have $l(D)=l(W-D)+1$ by the theorem of Riemann-Roch. On the other hand, we see that $\mathrm{div}_{R'}(dx) = q'_{\beta 1} + \cdots + q'_{\beta 5} - 3q'_\infty$ and $\mathrm{div}_{R'}(v) = q'_{\beta 1} + \cdots + q'_{\beta 5} - 5q'_\infty$. Hence we obtain $W=2q'_\infty$ and $W-D=2q'_\infty-q'_{\beta_i}-q'_{\beta_j}$. If there were a non-zero function f on R' such that $\mathrm{div}_{R'}(f)+W-D>0$, then we would have $\mathrm{div}_{R'}(f)=q'_{\beta_i}+q'_{\beta_j}-2q'_\infty$. This is a contradiction. Hence we have $l(W-D)=0$ and we cannot have $q'_{\beta_k}+q'_{\beta_l}-q'_{\beta_i}-q'_{\beta_j}\sim 0$. This shows that $K'(y_b)$ is not equal to $K'(y'_b)$. If y_a is any one of (3.5.1) (i) and y_b is any one of (ii), then we can write $\mathrm{div}_{R'}(y_a^2/y_b^2)$ $=2q'_\infty+2q'_{\beta_i}-2q'_{\beta_j}-2q'_{\beta_k}$. Here i, j and k are integers from 1 to 5 and $j\neq k$. Again we cannot have $q'_\infty+q'_{\beta_i}-q'_{\beta_j}-q'_{\beta_k}\sim 0$. Thus, we have five y_a's from (i) and ten y_b's from (ii). If we denote these by $y_{a_1},\cdots,y_{a_5}$ and $y_{b_1},\cdots,y_{b_{10}}$, then it follows that $K'(y_{a_1}),\cdots,K'(y_{a_5})$ $K'(y_{b_1}),\cdots,K'(y_{b_{10}})$ are distinct from each other. Hence, we can conclude that besides these there are no other our function fields. Consequently we have the following theorem.

(3.5.2) THEOREM. *In* (2.1.3), *we have as the desired form of equations*

$$\text{(i)}\qquad y^2=(x-\beta_i)[a(x)+b(x)\{(x-\beta_1)\cdots(x-\beta_5)\}^{1/2}]^2\,,\qquad (1\leqq i\leqq 5)\,.$$

$$\text{(ii)}\qquad y^2=(x-\beta_i)(x-\beta_j)[a(x)+b(x)\{(x-\beta_1)\cdots(x-\beta_5)\}^{1/2}]^2\,,$$
$$(1\leqq i,\ j\leqq 5,\ i\neq j)\,.$$

Here, in each case, $a(x)$ *and* $b(x)$ *are arbitrary polynomials.*

3.6. By the same method as in genus 3, we obtain the equations of Riemann surfaces of genus 4 with non-trivial automorphisms.

(3.6.1) THEOREM. *In* (2.3.1), *the equations of Riemann surfaces of genus 4 are given by*

$$\text{(i)}\qquad y^2=(x-a_1)\cdots(x-a_9)\,.$$

$$\text{(ii)}\qquad y^3=(x-a_1)^{m_1}\cdots(x-a_5)^{m_5}\,,\qquad 3\nmid m_1+\cdots+m_5\,.$$

$$\text{(iii)}\qquad y^5=(x-a_1)^{m_1}\cdots(x-a_3)^{m_3}\,,\qquad 5\nmid m_1+m_2+m_3\,.$$

(3.6.2) THEOREM. *In* (2.3.2) (i), *choosing* a, b, c, d *and* δ_1, δ_2 *which satisfy the identity*

$$(x-\delta_1)^2(x-\delta_2)^2x(x-1)(x-\beta)-(ax^3+bx^2+cx+d)^2$$
$$=(x-t_1)(x-t_2)\cdots(x-t_5)(x-\alpha_i)^2$$

for each α_i $(1\leqq i\leqq 4)$, *we have the desired equation of* R *as follows:*

$$y^2=ax^2+bx^3+cx+d+(x-\delta_1)(x-\delta_2)\{x(x-1)(x-\beta)\}^{1/2}\,.$$

(3.6.3) THEOREM. *In* (2.3.2) (ii), *we have two types of equations.*

$$\text{(i)}\qquad y^3=(x-\xi_1)(x-\xi_2)+c\{x(x-1)(x-\beta)\}^{1/2}\,.$$

Here ξ_1, ξ_2 *and* c *are constants which satisfy the identity*

$$(x-\xi_1)^2(x-\xi_2)^2-c^2x(x-1)(x-\beta)=(x-t_1)(x-t_2)(x-\alpha_i)^3$$

for each α_i $(1\leqq i\leqq 9)$.

(ii) $$y^3=ax^3+bx^2+cx+d+(x-\delta_1)(x-\delta_2)\{x(x-1)(x-\beta)\}^{1/2}.$$

Here a, b, c, d, δ_1 *and* δ_2 *are constants which satisfy the identity*

$$\begin{aligned}(x-\delta_1)^2(x-\delta_2)^2x(x-1)(x-\beta)-(ax^3+bx^2+cx+d)^2\\ =(x-t_1)^2(x-t_2)^2(x-\alpha_i)^3\end{aligned}$$

for each α_i $(1\leqq i\leqq 9)$.

(3.6.4) THEOREM. *In* (2.3.3) (i) *we have* $k_1=k_2=1$. *Let the points of order* 2 *for the divisor* $-(q_1+q_2)+2q_\infty$ *on the Jacobian variety* $J(R')$ *be* Q_i $(1\leqq i\leqq 2^4)$. *We have two points* $q_{\alpha i1}$, $q_{\alpha i2}$ *on* R' *corresponding to cach* Q_i. *Then each of the divisors* $q_1+q_2-6q_\infty+2(q_{\alpha i1}+q_{\alpha i2})$ *is the divisor of a function on* R'. *If we denote these functions by* z_{α_i} $(1\leqq i\leqq 2^4)$, *then we see that the function fields* $K'(z_{\alpha_i}^{1/2})$ $(1\leqq i\leqq 2^4)$ *are distinct. Choosing* ξ_1, ξ_2, ξ_3 *and* k *which satisfy the identity*

$$\begin{aligned}(x-\xi_1)^2(x-\xi_2)^2(x-\xi_3)^2-k^2(x-\beta_1)(x-\beta_2)\cdots(x-\beta_5)\\ =(x-t_1)(x-t_2)(x-\alpha_{i1})^2(x-\alpha_{i2})^2\end{aligned}$$

for each pair α_{i1}, α_{i2} $(1\leqq i\leqq 2^4)$, *we have the desired equation of* R *as follows:*

$$y^2=(x-\xi_1)(x-\xi_2)(x-\xi_3)+k\{(x-\beta_1)(x-\beta_2)\cdots(x-\beta_5)\}^{1/2}.$$

(3.6.5) THEOREM. *In* (2.3.3) (ii), *we have the equation of* R *as follows:*

(i) $y^3=(x-\beta_i)$, $y^3=(x-\beta_i)^2$, $(1\leqq i\leqq 5)$.

(ii) $y^3=(x-\beta_i)(x-\beta_j)$, $y^3=(x-\beta_i)^2(x-\beta_j)^2$, $(1\leqq i,\ j\leqq 5,\ i\neq j)$.

(iii) $y^3=(x-\beta_i)(x-\beta_j)(x-\beta_k)(x-\beta_l)$,

$y^3=(x-\beta_i)^2(x-\beta_j)^2(x-\beta_k)^2(x-\beta_l)^2$.

Here i, j, k, l *runs from* 1 *to* 5 *and they are distinct from each other.*

(iv) $y^3=(x-\beta_i)^2(x-\beta_j)(x-\beta_l)(x-\beta_k)$,

$y^3=(x-\beta_i)^2(x-\beta_j)^2(x-\beta_l)^2(x-\beta_k)$.

Here i, j, k, l *runs from* 1 *to* 5 *and they are distinct from each other.*

The number of these equations is 80 in all. However, if we consider the isomorphic function fields in usual sense, then we can select 40 equations from these 80 equations.

§4. Analyticity of the coefficients.

4.1. In [8], Riemann stated the problem of describing algebraic functions as a product of Theta functions times an exponential function. Hurwitz also treated this problem in [4]. As an application of our theory, we shall express the functions which are to be adjoint to the algebraic function field K' of a Riemann surface R' (in § 1) explicitly as a product of Theta functions times an exponential function.

4.2. First, we shall give a brief account of the Theta function together with an explanation of notations. Let R be a Riemann surface of genus g and A_k, B_k $(1\leqq k\leqq g)$ be a canonical dissection of R. Let $\boldsymbol{p}_0$ be a common point of all A_k, B_k $(1\leqq k\leqq g)$. The point $\boldsymbol{p}_0$ is also the initial point for integration. Let $dw_1, \cdots, dw_g$ be a basis of the differentials of the first kind of R and let

$$(4.2.1)\qquad w_l(\boldsymbol{p})=\int_{\boldsymbol{p}_0}^{\boldsymbol{p}}dw_l \quad (1\leqq l\leqq g), \qquad \mathfrak{w}(\boldsymbol{p})=\int_{\boldsymbol{p}_0}^{\boldsymbol{p}}d\mathfrak{w},$$

where $\boldsymbol{p}$ is the variable point on R and the integration paths are to be selected on the canonically dissected Riemann surface R^*. We assume that with this basis the period matrix has the form $[E, Z]$, where E is the unit matrix of size g, and $Z=X+iY$ satisfies the Riemann's relation: $X={}^tX$, $Y={}^tY$ and $Y>0$. The Theta function formed with Z is defined by

$$(4.2.2)\qquad \theta(\mathfrak{s})=\theta(\mathfrak{s}, Z)=\sum_{\mathfrak{n}}\exp(\pi i{}^t\mathfrak{n}Z\mathfrak{n}+2\pi i{}^t\mathfrak{n}\mathfrak{s})$$

and $\theta(\mathfrak{s})$ satisfies the functional relation

$$(4.2.3)\qquad \theta(\mathfrak{s}+\mathfrak{g}+Z\mathfrak{h})=\theta(\mathfrak{s})\exp(-\pi i{}^t\mathfrak{h}Z\mathfrak{h}-2\pi i{}^t\mathfrak{h}\mathfrak{s}),$$

where ${}^t\mathfrak{g}=(g_1, \cdots, g_g)$, ${}^t\mathfrak{h}=(h_1, \cdots, h_g)$ are arbitrary integer vectors. We put $f(\boldsymbol{p})=\theta(\mathfrak{w}(\boldsymbol{p})-\mathfrak{s})$, where ${}^t\mathfrak{w}(\boldsymbol{p})=(w_1(\boldsymbol{p}), \cdots, w_g(\boldsymbol{p}))$ and ${}^t\mathfrak{s}=(s_1, \cdots, s_g)$. With a circuit of $\boldsymbol{p}$ along a closed curve on R, $\mathfrak{w}(\boldsymbol{p})-\mathfrak{s}$ changed by a summand of the form $\mathfrak{g}+Z\mathfrak{h}$, and $f(\boldsymbol{p})$ is multiplied by the non-zero factor $\exp(-i\pi{}^t\mathfrak{h}Z\mathfrak{h}-2\pi i{}^t\mathfrak{h}(\mathfrak{w}(p)-\mathfrak{s}))$. If, for a fixed $\mathfrak{s}$, $f(\boldsymbol{p})$ does not vanish identically in $\boldsymbol{p}$, then it has exactly g zeros on R. Let $\mathfrak{c}$ be the vector consisting of the g quantities

$$(4.2.4)\qquad c_k=\sum_{l=1}^{g}\int_{A_l}w_k(\boldsymbol{p})dw_l-(1/2)z_{kk} \qquad (1\leqq k\leqq g).$$

Now, if we choose vector $\mathfrak{s}$ such that the function $f(\boldsymbol{p})=\theta(\mathfrak{w}(\boldsymbol{p})-\mathfrak{s}+\mathfrak{c})$ does not vanish identically in $\boldsymbol{p}$, then its g zeros $\boldsymbol{q}_1, \cdots, \boldsymbol{q}_g$ satisfy the congruence

$$(4.2.5)\qquad \sum_{l=1}^{g}\mathfrak{w}(\boldsymbol{q}_l)\equiv\mathfrak{s}.$$

It is known that $\theta(\mathfrak{w}(\boldsymbol{p})-\mathfrak{s}+\mathfrak{c})$ does not vanish identically in $\boldsymbol{p}$, if and only if the inverse problem for the vector $\mathfrak{s}$ is uniquely determined, i. e., the equation (4.2.5) has a unique solution $\boldsymbol{q}_1, \cdots, \boldsymbol{q}_g$. In this case, the divisor $D=\boldsymbol{q}_1+\cdots+\boldsymbol{q}_g$ is general. We shall call an integral divisor $D=\boldsymbol{p}_1+\cdots+\boldsymbol{p}_m$ general if there are no non-constant meromorphic functions f on R with the property div$(f)+D>0$.

4.3. Now, for our investigation the following lemma [10] is useful.

(4.3.1) Let $f(\boldsymbol{p})$ be a meromorphic function on R with the zeros $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m$ and the poles $\boldsymbol{b}_1, \cdots, \boldsymbol{b}_m$. We choose a general integral divisor $\boldsymbol{x}_1+\cdots+\boldsymbol{x}_{g-1}$ of degree $g-1$ such that $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{g-1}$ are different from the points $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m$ and $\boldsymbol{b}_1, \cdots, \boldsymbol{b}_m$. Then, after suitable choice of paths of integration, we have

$$\text{(i)} \qquad \sum_{k=1}^{m} \mathfrak{w}(\boldsymbol{a}_k)=\sum_{k=1}^{m} \mathfrak{w}(\boldsymbol{b}_k)$$

and

$$\text{(ii)} \qquad f(\boldsymbol{p})=\eta \prod_{k=1}^{m} \frac{\theta(\mathfrak{w}(\boldsymbol{p})+\sum_{l=1}^{g-1} \mathfrak{w}(\boldsymbol{x}_l)-\mathfrak{w}(\boldsymbol{a}_k)-\mathfrak{c})}{\theta(\mathfrak{w}(\boldsymbol{p})+\sum_{l=1}^{g-1} \mathfrak{w}(\boldsymbol{x}_l)-\mathfrak{w}(\boldsymbol{b}_k)-\mathfrak{c})}.$$

Here, η is a quantity independent of $\boldsymbol{p}$, and $\mathfrak{c}$ is defined in (4.2.4).

We can obtain from (4.3.1) directly a fact if $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are both neither zero nor poles of $f(\boldsymbol{p})$, then we have a representation

(4.3.2)

$$\frac{f(\boldsymbol{p}_1)}{f(\boldsymbol{p}_2)}=\prod_{k=1}^{m} \frac{\theta(\mathfrak{w}(\boldsymbol{p}_1)+\sum_{l=1}^{g-1} \mathfrak{w}(\boldsymbol{x}_l)-\mathfrak{w}(\boldsymbol{a}_k)-\mathfrak{c})}{\theta(\mathfrak{w}(\boldsymbol{p}_1)+\sum_{l=1}^{g-1} \mathfrak{w}(\boldsymbol{x}_l)-\mathfrak{w}(\boldsymbol{b}_k)-\mathfrak{c})} \cdot \prod_{k=1}^{m} \frac{\theta(\mathfrak{w}(\boldsymbol{p}_2)+\sum_{l=1}^{g-1} \mathfrak{w}(\boldsymbol{x}_l)-\mathfrak{w}(\boldsymbol{b}_k)-\mathfrak{c})}{\theta(\mathfrak{w}(\boldsymbol{p}_2)+\sum_{l=1}^{g-1} \mathfrak{w}(\boldsymbol{x}_l)-\mathfrak{w}(\boldsymbol{a}_k)-\mathfrak{c})}.$$

This connotes that there is a space of moduli in the rearwards.

We must remark that $\mathfrak{c}$ is also a quantity such that $2\mathfrak{c}\equiv \sum_{l=1}^{2g-2} \mathfrak{w}(\boldsymbol{y}_l)$ with zeros $\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{2g-2}$ of an arbitrary differential of the first kind.

Now, we consider in the case (2.1.1). We can write the equation of R in the form (2.2.1). The divisor of the function y on R is

$$\text{(4.3.3)} \qquad \operatorname{div}_R(y)=m_1\boldsymbol{q}_1+\cdots+m_s\boldsymbol{q}_s-(m_1+\cdots+m_s)\boldsymbol{q}_\infty,$$

where $\boldsymbol{q}_1, \cdots, \boldsymbol{q}_s, \boldsymbol{q}_\infty$ are points on R which are over the points $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_s, \infty$ on the Riemann sphere R_0. Therefore, by (4.3.1) we have

$$\text{(4.3.4)} \qquad y(\boldsymbol{p})=\eta \prod_{i=1}^{s} \prod_{q_i}^{m_i} \frac{\theta(\mathfrak{w}(\boldsymbol{p})+\sum_{l=1}^{g-1} \mathfrak{w}(\boldsymbol{x}_l)-\mathfrak{w}(\boldsymbol{q}_i)-\mathfrak{c})}{\theta(\mathfrak{w}(\boldsymbol{p})+\sum_{l=1}^{g-1} \mathfrak{w}(\boldsymbol{x}_l)-\mathfrak{w}(\boldsymbol{q}_\infty)-\mathfrak{c})}.$$

We consider in the case (2.1.2) (i). The other case can be done similarly. The divisor of the function y^2 on R' is written by

$$\text{(4.3.5)} \qquad \operatorname{div}_{R'}(y^2)=\boldsymbol{q}'_1+\boldsymbol{q}'_2+\boldsymbol{q}'_3-5\boldsymbol{q}'_\infty+2\boldsymbol{q}'_\alpha.$$

Therefore, by (4.3.1) we have

$$y^2(\boldsymbol{p})=\eta\prod_{i=1}^{5}\frac{\theta(\mathfrak{w}(\boldsymbol{p})-\mathfrak{w}(\boldsymbol{q}_i')-\mathfrak{c})}{\theta(\mathfrak{w}(\boldsymbol{p})-\mathfrak{w}(\boldsymbol{q}_\infty')-\mathfrak{c})}. \tag{4.3.6}$$

Here, $\boldsymbol{q}_4'=\boldsymbol{q}_5'=\boldsymbol{q}_\alpha'$. We consider in the case (2.1.3). The divisor of y^2 is

$$\mathrm{div}_{R'}(y^2)=2\boldsymbol{q}_{\beta i}'-2\boldsymbol{q}_\infty'. \tag{4.3.7}$$

The other case can be done similarly. We have by (4.3.1)

$$y^2(\boldsymbol{p})=\eta\prod_{i=1}^{2}\frac{\theta(\mathfrak{w}(\boldsymbol{p})+\mathfrak{w}(\boldsymbol{x}_1)-\mathfrak{w}(\boldsymbol{q}_{\beta i}')-\mathfrak{c})}{\theta(\mathfrak{w}(\boldsymbol{p})+\mathfrak{w}(\boldsymbol{x}_1)-\mathfrak{w}(\boldsymbol{q}_\infty')-\mathfrak{c})}. \tag{4.3.8}$$

Here, we should keep in mind that $\mathfrak{w}(\boldsymbol{q}_{\beta i}')+\mathfrak{w}(\boldsymbol{q}_{\beta i}')=\mathfrak{w}(\boldsymbol{q}_\infty')+\mathfrak{w}(\boldsymbol{q}_\infty')$.

Now, we proceed further in the case (2.1.2) (i). We can denote (4.3.6) by

(4.3.9)

$$y^2(\boldsymbol{p})=\eta\prod_{i=1}^{3}\frac{\theta\left(\int_{x_0}^{x}v^{-1}dx-\int_{x_0}^{t_i}v^{-1}dx+\mathfrak{c}\right)}{\theta\left(\int_{x_0}^{x}v^{-1}dx-\int_{x_0}^{\infty}v^{-1}dx+\mathfrak{c}\right)}\cdot\prod_{\alpha}^{2}\frac{\theta\left(\int_{x_0}^{x}v^{-1}dx-\int_{x_0}^{\alpha}v^{-1}dx+\mathfrak{c}\right)}{\theta\left(\int_{x_0}^{x}v^{-1}dx-\int_{x_0}^{\infty}v^{-1}dx+\mathfrak{c}\right)}.$$

Here we must notice that α is determined by

$$\int_\infty^{t_1}v^{-1}dx+\int_\infty^{t_2}v^{-1}dx+\int_\infty^{t_3}v^{-1}dx+2\int_\infty^{\alpha}v^{-1}dx\equiv 0, \tag{4.3.10}$$

and c is determined by β analytically. Hence, we can put the right side of (4.3.9) in the form $\eta\ F(x\,;t_1,t_2,t_3,\beta)$, where F is analytic in x, t_1, t_2, t_3 and β. Furthermore, considering our assumption we have

$$\begin{aligned}
&c=\eta F(0\,;t_1,t_2,t_3,\beta),\\
&a+b+c=\eta F(1\,;t_1,t_2,t_3,\beta),\\
&a\beta^2+b\beta+c=\eta F(\beta\,;t_1,t_2,t_3,\beta),\\
&at_1^2+bt_1+c+(t_1-\delta)\{t_1(t_1-1)(t_1-\beta)\}^{1/2}=\eta F(t_1\,;t_1,t_2,t_3,\beta),\\
&at_2^2+bt_2+c+(t_2-\delta)\{t_2(t_2-1)(t_2-\beta)\}^{1/2}=\eta F(t_2\,;t_1,t_2,t_3,\beta),\\
&at_3^2+bt_3+c+(t_3-\delta)\{t_3(t_3-1)(t_3-\beta)\}^{1/2}=\eta F(t_3\,;t_1,t_2,t_3,\beta).
\end{aligned} \tag{4.3.11}$$

Since the family of Riemann surfaces $\Omega(1, 2, \{\nu_1, \nu_2, \nu_3, \nu_4\})$ has four parameters, there is a relation in a, b, c and δ as we can see in (4.3.11).

Now, the function y^2 is also a meromorphic function on R. The divisor of y^2 on R is written by

$$\mathrm{div}_R(y^2)=2\boldsymbol{q}_1+2\boldsymbol{q}_2+2\boldsymbol{q}_3-10\boldsymbol{q}_\infty+2\boldsymbol{q}_\alpha^1+2\boldsymbol{q}_\alpha^2. \tag{4.3.12}$$

Therefore, by (4.3.1) we have

$$(4.3.13)\qquad y^2(\boldsymbol{p})=\eta\prod_i\prod_\alpha\frac{\Theta(\mathfrak{w}(\boldsymbol{p})+\sum_{l=1}^{2}\mathfrak{w}(\boldsymbol{x}_l)-\mathfrak{w}(\boldsymbol{q}_i)-\mathfrak{c})}{\Theta(\mathfrak{w}(\boldsymbol{p})+\sum_{l=1}^{2}\mathfrak{w}(\boldsymbol{x}_l)-\mathfrak{w}(\boldsymbol{q}_\infty)-\mathfrak{c})}.$$

Then, we can obtain a relation between θ of R' and Θ of R by comparing (4.3.6) with (4.3.12).

4.4. Now, we return to the problem stated at the beginning of this section. The divisor of the function $z=y^n$ on R' is written by

$$(4.4.1)\qquad \operatorname{div}_{R'}(z)=nD+\sum_{i=1}^{r}k_i\boldsymbol{q}_i.$$

We can assume that k_i $(1\leqq i\leqq r)$ are all positive and $\operatorname{supp}D$ contains no $\boldsymbol{q}_i$ $(1\leqq i\leqq r)$.

Let the zeros of the function $z=y^n$ be $\boldsymbol{p}_1,\cdots,\boldsymbol{p}_s$. Then the points $\boldsymbol{q}_1,\cdots,\boldsymbol{q}_r$ are contained in the set $\{\boldsymbol{p}_1,\cdots,\boldsymbol{p}_s\}$, i.e., $\boldsymbol{q}_i$ is contained k_i times for all i $(1\leqq i\leqq r)$ in $\{\boldsymbol{p}_1,\cdots,\boldsymbol{p}_s\}$. We have $\sum_{i=1}^{r}k_i=nl$, where l is a positive integer. We choose a sufficiently large integer l_1 such that $n\mid l_1$, $n^2\nmid l_1$ and $nl\equiv l_1g'$ (mod n). Moreover, we may assume that the number of zeros of z is equal to l_1g'. Then, we divide the set $\{\boldsymbol{p}_1,\cdots,\boldsymbol{p}_s\}$ into the l_1 blocks: $\{\boldsymbol{p}_1^{(1)},\cdots,\boldsymbol{p}_{g'}^{(1)}\},\cdots,\{\boldsymbol{p}_1^{(l_1)},\cdots,\boldsymbol{p}_{g'}^{(l_1)}\}$, and we put

$$(4.4.2)\qquad \mathfrak{e}^{(1)}=\mathfrak{w}(\boldsymbol{p}_1^{(1)})+\cdots+\mathfrak{w}(\boldsymbol{p}_{g'}^{(1)}),\ \cdots,\ \mathfrak{e}^{(l_1)}=\mathfrak{w}(\boldsymbol{p}_1^{(l_1)})+\cdots+\mathfrak{w}(\boldsymbol{p}_{g'}^{(l_1)}).$$

Here we can assume that all the divisors $D_k=\boldsymbol{p}_1^{(k)}+\cdots+\boldsymbol{p}_{g'}^{(k)}$ $(1\leqq k\leqq l_1)$ are general. In fact, we may put

$$(4.4.3)\qquad \boldsymbol{p}_1^{(1)}=\cdots=\boldsymbol{p}_1^{(k_1)}=\boldsymbol{q}_1,$$
$$\boldsymbol{p}_1^{(k_1+1)}=\cdots=\boldsymbol{p}_1^{(k_1+k_2)}=\boldsymbol{q}_2,\ \cdots,\ \boldsymbol{p}^{(k_1+\cdots+k_{r-1}+1)}=\cdots=\boldsymbol{p}^{(k_1+\cdots+k_r)}=\boldsymbol{q}_r.$$

Hence, by the method of the choice of l_1 and the zeros we obtain our assertion. If $g'=1$, then every prime divisor is general and so we need no consideration as above.

Now put

$$(4.4.4)\qquad \mathfrak{k}=(1/l_1)\sum_{k=1}^{l_1}\mathfrak{e}^{(k)}$$

and consider the representation

$$(4.4.5)\qquad \exp(-2\pi i{}^t\mathfrak{w}(\boldsymbol{p})\cdot\mathfrak{h}l_1)\prod_{k=1}^{l_1}\frac{\theta(\mathfrak{w}(\boldsymbol{p})-\mathfrak{e}^{(k)}+\mathfrak{c})}{\theta(\mathfrak{w}(\boldsymbol{p})-\mathfrak{k}+\tilde{\mathfrak{g}}+Z\tilde{\mathfrak{h}}+\mathfrak{c})}.$$

Here the components of the vectors ${}^t\tilde{\mathfrak{g}}=(\tilde{g}_1,\cdots,\tilde{g}_{g'})$, ${}^t\tilde{\mathfrak{h}}=(\tilde{h}_1,\cdots,\tilde{h}_{g'})$ run through all the numbers $0,1/n,\cdots,(n-1)/n$ independently. Further we can assume that each $\theta(\mathfrak{w}(\boldsymbol{p})-\mathfrak{k}+\mathfrak{g}+Z\mathfrak{h}+\mathfrak{c})$ does not vanish identically in p. In fact, we can take $\{\boldsymbol{p}_2^{(1)}, \cdots, \boldsymbol{p}_{g'}^{(1)}\}, \cdots, \{\boldsymbol{p}_2^{(l_1)}, \cdots, \boldsymbol{p}_{g'}^{(l_1)}\}$ sufficiently arbitrary. If $g'=1$, then there is no trouble.

Now, it is easy to show that the representation (4.4.5) is a single valued meromorphic function on R'. We denote this function by $F(\boldsymbol{p})$. The set of all zeros of $F(\boldsymbol{p})$ coincides with the set $\{\boldsymbol{p}_1, \cdots, \boldsymbol{p}_s\}$. The number of poles of $F(\boldsymbol{p})$ is, of course, $s=l_1 g'$ and so is a multiple of n. Therefore, we can write

$$\text{div}_{R'}(F)=nD_0+\sum_{i=1}^{r} k_i \boldsymbol{q}'_i, \tag{4.4.6}$$

with a divisor D_0 on R'. Hence, if we adjoint $F^{1/n}$ to K', then we can obtain a desired function field. If we take another column vector $\mathfrak{g}'$, $\tilde{\mathfrak{h}}'$ and denote by $F'(\boldsymbol{p})$ the resulting function, then we have as above

$$\text{div}_{R'}(F')=nD'_0+\sum_{i=1}^{r} k_i \boldsymbol{q}'_i, \tag{4.4.7}$$

with a divisor D'_0 on R'. Again, if we adjoint $F'^{1/n}$ to K', then we can obtain another desired function field. We claim that $K'(F^{1/n})\neq K'(F'^{1/n})$ if $(\mathfrak{g}, \tilde{\mathfrak{h}})=(\mathfrak{g}', \tilde{\mathfrak{h}}')$. In fact, let the zeros of $\theta(\mathfrak{w}(\boldsymbol{p})-\mathfrak{f}+\mathfrak{g}+Z\tilde{\mathfrak{h}}+\mathfrak{c})$ be $\boldsymbol{s}_1, \cdots, \boldsymbol{s}_{g'}$ and the zeros of $\theta(\mathfrak{w}(\boldsymbol{p})-\mathfrak{f}+\mathfrak{g}'+Z\tilde{\mathfrak{h}}'+\mathfrak{c})$ be $\boldsymbol{s}'_1, \cdots, \boldsymbol{s}'_{g'}$. By the assumption on l_1, we have with an integer m

$$\begin{aligned} l_1(\boldsymbol{s}_1+\cdots+\boldsymbol{s}_{g'})&=nm(\boldsymbol{s}_1+\cdots+\boldsymbol{s}_{g'}), \\ l_1(\boldsymbol{s}'_1+\cdots+\boldsymbol{s}'_{g'})&=nm(\boldsymbol{s}'_1+\cdots+\boldsymbol{s}'_{g'}). \end{aligned} \tag{4.4.8}$$

Here we must note that n cannot divide m. By (4.4.6~7), $D_0\sim D'_0$ if and only if $m(\boldsymbol{s}_1+\cdots+\boldsymbol{s}_{g'})\sim m(\boldsymbol{s}'_1+\cdots+\boldsymbol{s}'_{g'})$. On the other hand, we have

$$\sum_{k=1}^{g'} \mathfrak{w}(\boldsymbol{s}_k)\equiv\mathfrak{f}-\mathfrak{g}-Z\tilde{\mathfrak{h}}, \qquad \sum_{k=1}^{g'} \mathfrak{w}(\boldsymbol{s}'_k)\equiv\mathfrak{f}-\mathfrak{g}'-Z\tilde{\mathfrak{h}}'. \tag{4.4.9}$$

Therefore, $D_0\sim D'_0$ if and only if $m(\mathfrak{g}+Z\tilde{\mathfrak{h}})=m(\mathfrak{g}'+Z\tilde{\mathfrak{h}}')$. However, the latter is not possible if $(\mathfrak{g}, \tilde{\mathfrak{h}})\neq(\mathfrak{g}', \tilde{\mathfrak{h}}')$. Hence, if $(\mathfrak{g}, \tilde{\mathfrak{h}})\neq(\mathfrak{g}', \tilde{\mathfrak{h}}')$, then $K'(F^{1/n})\neq K'(F'^{1/n})$. Thus, by (1.7.2) we obtain the following lemma.

(4.4.10) If $r>0$, then the functions given by (4.4.5) are nothing but the n-th powers of the desired functions. The number of these functions is exactly $n^{2g'}$.

If $r=0$, then we consider the representation

$$\exp(-2\pi i^t\mathfrak{w}(\boldsymbol{p})\cdot\tilde{\mathfrak{h}}n)\prod^{n}\frac{\theta(\mathfrak{w}(\boldsymbol{p})-\mathfrak{s}+\mathfrak{c})}{\theta(\mathfrak{w}(\boldsymbol{p})-\mathfrak{s}+\mathfrak{g}+Z\tilde{\mathfrak{h}}+\mathfrak{c})}. \tag{4.4.11}$$

Here the notations are the same as in the representation (4.4.5), and $\mathfrak{s}$ is a vector for which the inverse problem is uniquely determined and further

$\mathfrak{s}-\tilde{\mathfrak{g}}-Z\tilde{\mathfrak{h}}$ has the same property. Then, (4.4.11) is again a single valued meromorphic function on R'. We denote this function by $G(\boldsymbol{p})$. By the same method as in the case of $r>0$, we obtain exactly $n^{2g'}-1$ distinct function fields. Indeed, we must exclude the case $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})=(0, 0)$. Hence, by (1.7.2) we obtain the following lemma.

(4.4.12) If $r=0$, then the functions given by (4.4.11) are nothing but the n-th powers of the desired functions. The number of these functions is exactly $n^{2g'}-1$.

For a fixed $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$, the cases of $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$, $(2\tilde{\mathfrak{g}}, 2\tilde{\mathfrak{h}})$, $\cdots$, $((n-1)\tilde{\mathfrak{g}}, (n-1)\tilde{\mathfrak{h}})$ give us the isomorphic function field in the usual sence. However, in our case they must be regarded as distinct.

4.5. Now, we shall investigate analyticity of the coefficients of equations. For that purpose, we give a brief account of the generalized Teichmüller space Λ. We fix a couple (R_0, σ_0) such that $\langle R_0, \sigma_0\rangle$ belongs to $\Omega(g', n, \{\nu_1, \cdots, \nu_r\})$ and denote by $\Lambda(R_0, \sigma_0)$ the set of all the elements $\langle R, \sigma\rangle$ of $\Omega(g', n, \{\nu_i\})$ such that (R, σ) is topologically equivalent to (R_0, σ_0), i. e., there exists a topological mapping $f: R_0\to R$ such that $f\sigma_0=\sigma f$. We consider further a triple (R, σ, α) formed by a couple (R, σ) such that $\langle R, \sigma\rangle$ of $\Gamma(R_0, \sigma_0)$ and a homotopy class α of topological mappings of (R_0, σ_0) onto (R, σ). We say that (R, σ, α) is isomorphic to (R', σ', α') if there exists an isomorphism of (R, σ) onto (R', σ') which belongs to the homotopy class $\alpha'^{-1}\alpha$. We denote by $\langle R, \sigma, \alpha\rangle$ the isomorphism class of (R, σ, α) and the set of all classes $\langle R, \sigma, \alpha\rangle$ is denoted by $\Lambda(g', n, \{\nu_i\}; (R_0, \sigma_0))$ or briefly $\Lambda(R_0, \sigma_0)$. We can also define $\Lambda(R_0, \sigma_0)$ by using a canonical homotopy basis of (R, σ). These two definitions are equivalent.

The space $\Lambda(R_0, \sigma_0)$ is a metric space with the distance function defined as follows: For arbitrary $\langle R, \sigma, \alpha\rangle$ and $\langle R', \sigma', \alpha'\rangle$, consider all the quasiconformal mappings f of R onto R' which belongs to the homotopy class $\alpha'\alpha^{-1}$ and are such that $f\sigma=\sigma' f$. Define the distance between $\langle R, \sigma, \alpha\rangle$ and $\langle R', \sigma', \alpha'\rangle$ by means of $\inf\log K_f$, where K_f is the maximum dilatation of f, and the infimum is taken with respect to all the f mentioned above. It is known that the infimum is the minimum, and is attained by only one f, which will be called the extremal quasi-conformal mapping. The space $\Lambda(R_0, \sigma_0)$ carries further an analytic structure and we have the following lemma [5].

(4.5.1) The generalized Teichmüller space $\Lambda(g', n, \{\nu_1, \cdots, \nu_r\}; (R_0, \sigma_0))$ is a simply connected $3g'-3+r$ dimensional complex analytic manifold. The mapping ι of $\Lambda(R_0, \sigma_0)$ into the ordinally Teichmüller space T_g, defined by corresponding $\langle R, \sigma, \alpha\rangle$ to $\langle R, \alpha\rangle$ is isomorphic.

If g' is 0, then each member of $\Omega(g', n, \{\nu_i\})$ is of the form $\langle R, \sigma\rangle$ where R is given by the equation

(4.5.2) $$y^n=(x-a_0)^{m_0}(x-a_1)^{m_1}\cdots(x-a_{s+1})^{m_{s+1}}, \qquad n\nmid m_0+\cdots+m_{s+1}$$

with distinct complex numbers $a_0, a_1, \cdots, a_{s+1}$. Here, $r=s+3$. In the equivalence class, we can find a representative (R, σ) such that the equation of R is

(4.5.3) $$y^n=x^{m_0}(x-z_1)^{m_1}\cdots(x-z_s)^{m_s}(x-1)^{m_{s+1}}.$$

We call this form a normal one and denote the Riemann surface defined by this equation by $R(z)$. It is easy to see that in the representation (4.5.3) for $\langle R, \sigma\rangle$, $z_1, \cdots, z_s$ are determined as a set but not as a vector $(z_1, \cdots, z_s)$, i.e., if $m_i=m_j$ then we cannot distinguish z_i from z_j. As we see later, we can distinguish them at least in the generalized Teichmüller space Λ, i.e., $z_1, \cdots, z_s$ are functions in Λ. Here we must note that the dimension of Λ is s by (4.5.1).

If $g'>0$, for example in the case (2.1.2) (i), we obtain as the equation of R

(4.5.4) $$y^2=ax^2+bx+c+(x-\delta)\{x(x-1)(x-\beta)\}^{1/2}.$$

We call this form a normal one for $\langle R, \sigma\rangle$. By the same method as in (4.5.3) we see that a, b, c, δ and β are determined uniquely for $\langle R, \sigma\rangle$. Therefore, it is a matter of course that they are functions in the generalized Teichmüller space $\Lambda(R_0, \sigma_0)$. Here we must note that the dimension of $\Lambda(R_0, \sigma_0)$ is 4 and we have the relation (4.3.11).

Next, we give a brief account of the generalized upper half-plane H_g. Let Z be a complex matrix of size g and satisfy the Riemann's relation: $Z={}^tZ$ and $\operatorname{Im} Z>0$. H_g is defined by $\{Z\}$ and considered as a parameter space of the family of polarized abelian varieties with the period matrix $[E\ \ Z]$. It is known that we have a holomorphic mapping of the Teichmüller space T_g into H_g [2]. By (4.5.1) we have a holomorphic mapping of the generalized Teichmüller space into H_g. Let $\mathfrak{G}$ be the Siegel modular group $Sp(2g, Z)$. Then, two polarized abelian varieties A_1 and A_2 with $Z_1, Z_2\in H_g$ are isomorphic if and only if Z_1 and Z_2 are equivalent under $\mathfrak{G}$. We have the family of abelian varieties of the quotient space $H_g/\mathfrak{G}$. By the Torelli's theorem we see that there is an injection of the space of moduli of Riemann surfaces into $H_g/\mathfrak{G}$. Therefore, there is an injection of the quotient space $\Gamma(R_0, \sigma_0)/\sim$ into $H_g/\mathfrak{G}$. Here the relation $\sim$ means ordinary conformal equivalence. We assume that to a point Z of H_g there exists a point λ of Λ which corresponds to Z. We denote the coordinates of Z by $(\mathfrak{z}_1, \cdots, \mathfrak{z}_N)$, where $N=g(g+1)/2$. In a neighborhood of a point of Λ, $\mathfrak{z}_1, \cdots, \mathfrak{z}_N$ are holomorphic functions of λ.

Now, we denote one of the normal form of (R_0, σ_0) by

(4.5.5) $$y^n=x^{m_0}(x-z_1^{(0)})^{m_1}\cdots(x-z_s^{(0)})^{m_s}(x-1)^{m_{s+1}}, \qquad n\nmid m_0+\cdots+m_{s+1}.$$

Let λ_0 be $\langle R_0, \sigma_0, \alpha_0\rangle$ and let $\lambda=\langle R, \sigma, \alpha\rangle$ be an arbitrary element of $\Lambda(R_0, \sigma_0)$. In the homotopy class $\alpha\alpha_0^{-1}$ there exists one and only one extremal quasi-conformal mapping $f: \lambda_0 \rightarrow \lambda$ which has the property $f\sigma_0=\sigma f$. Then f can be considered as a mapping of the Riemann sphere to the Riemann sphere. Put $f(0)=0$, $f(1)=1$, $f(\infty)=\infty$ and $f(z_i^{(0)})=z_i$ $(1 \leqq i \leqq s)$. We have as the equation of λ

$$(4.5.6) \qquad y^n=x^{m_0}(x-z_1)^{m_1} \cdots (x-z_s)^{m_s}(x-1)^{m_{s+1}}, \qquad n \nmid m_0+ \cdots +m_{s+1}$$

with these $0, z_1, \cdots, z_s$ and 1. Obviously they are distinct from each other, and the parameters z_i $(1 \leqq i \leqq s)$ can be considered as functions on the generalized Teichmüller space $\Lambda(R_0, \sigma_0)$. Then we have the following lemma.

(4.5.7) LEMMA. *z_i $(1 \leqq i \leqq s)$ are continuous on $\Lambda(R_0, \sigma_0)$.*

PROOF. For arbitrary $\lambda=\langle R, \sigma, \alpha\rangle$ and $\lambda'=\langle R', \sigma', \alpha'\rangle$, let g be the extremal quasi-conformal mapping of λ to λ'. As in the proof of (4.5.5), g can be considered as a mapping of the Riemann sphere onto itself, and satisfies $g(0)=0$, $g(1)=1$, $g(\infty)=\infty$ and $g(z_i)=z_i'$ $(1 \leqq i \leqq s)$. On the other hand, g, as a mapping of the Riemann sphere onto itself, satisfies the inequality

$$(4.5.8) \qquad [g(z), z] \leqq C(K_g-1)/(K_g+1).$$

Here [,] indicates the spherical distance, C is a numerical constant, and K_g is the maximal dilatation of g ([1], p. 398, Formula (39)). Accordingly we have $[g(z_i), z_i] \leqq C(e^d-1)/(e^d+1)$, where $d=\log K_g$ is the distance between λ and λ'. We conclude that $z_i' \rightarrow z_i$ as $\lambda' \rightarrow \lambda$ for $i=1, \cdots, s$.

Now, we construct a basis of the first homology group of (R, σ) as follows. Let x_0 be an arbitrary point on the x-sphere different from $0, z_1, \cdots, z_s$, 1 and ∞. We connect x_0 to these points by curves which have no intersection with each other except for x_0. Denote these curves by $\alpha_0, \alpha_1, \cdots, \alpha_{s+1}, \alpha_{s+2}$ respectively. Fix a point $p_0=(x_0, y_0)$ on R and denote by $\tilde{\alpha}_i$ the lift of α_i with the initial point at p_0. We put

$$(4.5.9) \quad \text{(i)} \quad C_i=\tilde{\alpha}_i+\sigma^{m_i}\tilde{\alpha}_i+ \cdots +\sigma^{m_i(\nu_i-1)}\tilde{\alpha}_i \qquad (0 \leqq i \leqq s+2)$$

$$\text{(ii)} \quad Z_j=C_{j-1}-C_j \qquad (1 \leqq j \leqq s+1).$$

Then we see easily that C_j is a curve on R connecting p_0 and $(x_0, \zeta y_0)$, and that Z_j is a closed curve. Moreover we can prove $Z_1, \cdots, Z_{s+1}$ are a basis over $K=\boldsymbol{Q}(\zeta)$ and $Z_1, \sigma Z_1, \cdots, \sigma^{n-2}Z_1; \cdots; Z_{s+1}, \sigma Z_{s+1}, \cdots, \sigma^{n-2}Z_{s+1}$ are a basis over the ring of integers Z.

Take a point $z=(z_1, \cdots, z_s)$ of $\boldsymbol{C}^s$. Here we assume that $z_1, \cdots, z_s$, 0 and 1 are distinct from each other. We denote by $\dot{\boldsymbol{C}}^s$ the set which consists of such points. Let $z=(z_1, \cdots, z_s)$ be an arbitrary point of $\dot{\boldsymbol{C}}^s$. We can associate to z a point $\langle R, \sigma\rangle$ of $\Gamma(R_0, \sigma_0)$ by the equation (4.5.6). There exists a quasi-conformal mapping of (R_0, σ_0) defined by (4.5.5) to (R, σ) defined by (4.5.6) [5]. Therefore there exists at least one λ of $\Lambda(R_0, \sigma_0)$ such that the equation of λ

is (4.5.6).

Take a point z of $\dot{C}^s$. Let $z'=(z'_1, \cdots, z'_s)$ be a point of $\dot{C}^s$ in a neighborhood of z. We denote by $R(z')$ the Riemann surface defined by (4.5.3) with this z'. We can take a basis of differentials of $R(z')$, $\{\omega_1, \cdots, \omega_g\}$, each of which is a differential of the followg form:

$$\omega=x^{k_0}(x-z'_1)^{k_1}\cdots(x-z'_s)^{k_s}y^{-l}dx\,, \tag{4.5.10}$$

where $0<l\leqq n-1$, $0\leqq k_0, \cdots, k_s<n$. On the other hand, we know that there exists a basis of differentials

$$p_1(\zeta, \lambda)d\zeta, \cdots, p_g(\zeta, \lambda)d\zeta \tag{4.5.11}$$

and a canonical homology basis

$$A_1(\zeta, \lambda), \cdots, A_g(\zeta, \lambda)\,; \qquad B_1(\zeta, \lambda), \cdots, B_g(\zeta, \lambda)\,, \tag{4.5.12}$$

such that

$$\int_\zeta^{A_i(\zeta,\lambda)} p_j(\zeta, \lambda)d\zeta=\delta_{ij}\,, \qquad \int_\zeta^{B_i(\zeta,\lambda)} p_j(\zeta, \lambda)d\zeta=z_{ij}(\lambda)\,. \tag{4.5.13}$$

Here $p_j(\zeta, \lambda)$ and $A_j(\zeta, \lambda)$, $B_j(\zeta, \lambda)$ $(1\leqq j\leqq g)$ are holomorphic in λ for every fixed ζ in a bounded Jordan domain $D(\lambda)$ [2].

For the canonical homology basis $\{\gamma_1(z), \cdots, \gamma_{2g}(z)\}$ such that

$$\int_{\gamma_i(z)}\omega_j(z)=\delta_{ij} \qquad (1\leqq i,\ j\leqq g)\,, \tag{4.5.14}$$

$$\int_{\gamma_i(z)}\omega_j(z)=t_{ij}(z) \qquad (1\leqq j\leqq g,\ g+1\leqq i\leqq 2g)\,,$$

we have with a constant matrix S in a neighborhood of λ_0

$$(\gamma_1(z), \cdots, \gamma_{2g}(z))=(A_1(\zeta, \lambda), \cdots, B_g(\zeta, \lambda))S\,. \tag{4.5.15}$$

Then, we have

$$(\omega_1(z'), \cdots, \omega_g(z'))=(p_1(\zeta, \lambda)d\zeta, \cdots, p_g(\zeta, \lambda)d\zeta)M(\lambda) \tag{4.5.16}$$

with a holomorphic matrix $M(\lambda)$ in the neighborhood of λ_0. In fact, put $S=\begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$. Then we have $M(\lambda)=(S_1+ZS_3)^{-1}$ with $Z=(z_{ij}(\lambda))$.

Now, we construct by the period vectors (4.5.14) the coordinates $G_1(z')$, $\cdots$, $G_N(z')$ of a point of the Siegel domain H_g. Here $G_1, \cdots, G_N$ are holomorphic functions in z' in a neighborhood of z which corresponds to λ, and N is equal to $g(g+1)/2$. Thus we have the following relation:

$$\mathfrak{z}_1(\lambda')=G_1(z'), \cdots, \mathfrak{z}_N(\lambda')=G_N(z') \tag{4.5.17}$$

in a neighborhood of λ. Put $G(z')=(F_1(z'), \cdots, F_N(z'))$. Then we have

$$
(4.5.18)\qquad
\begin{array}{ccccc}
 & \overset{z}{\swarrow} & A(R_0, \sigma_0) & \xrightarrow{\ w\ } & H_g \\
\dot{C}^s & & \downarrow \pi_1 & & \downarrow \pi_2 \\
 & \underset{\pi}{\searrow} & \Gamma(R_0, \sigma_0)/\sim & \xrightarrow{\ \mu\ } & H/\mathfrak{G}
\end{array}
$$

where w, π, π_2, G, $\mu(=\pi_2 G\pi^{-1})$ are holomorphic and z, $\pi_1(=\pi z)$ are continuous and each diagram is commutative. In fact the mapping μ is injective. The notations used in (4.5.18) will need no explanations. By considering the fact that μ is a holomorphic injection and the theorem of Riemann on the removable singularity we can prove that π_1 is holomorphic. π is a holomorphic covering of $\dot{C}^s$ onto $\Gamma(R_0, \sigma_0)/\sim$, and z is a continuous mapping of $A(R_0, \sigma_0)$ into $\dot{C}^s$. Therefore, by the same method as above we obtain the following theorem.

(5.5.19) THEOREM. *The parameters* $z_1, \cdots, z_s$ *in* (4.5.6) *are single valued holomorphic functions on the generalized Teichmüller space* $A(R_0, \sigma_0)$ *which is constructed from* $\Omega(g', n, \{\nu_i\})$. *Here* $g'=0$.

If $g'>0$, for example in the case (2.1.2) (i), we obtain as the equation of $\langle R, \sigma\rangle$, (4.5.4). The quantities a, b, c and δ are determined by four parameters t_1, t_2, t_3 and β. By the same method as in the case of $g'=0$, we see that the parameters t_1, t_2, t_3 and β are continuous functions in the generalized Teichmüller space $A(R_0, \sigma_0)$. Hence we can conclude that a, b, c and δ are continuous in $A(R_0, \sigma_0)$. Now, we can take a basis of differentials of the first kind, $\{\omega_1, \omega_2, \omega_3\}$ as follows:

$$
(4.5.20)\qquad
\begin{aligned}
\omega_1 &= \{x(x-1)(x-\beta)\}^{-1/2}dx \\
\omega_2 &= (x-\alpha)\{x(x-1)(x-\beta)\}^{-1/2}y^{-1}dx \\
\omega_3 &= (\{x(x-1)(x-\beta)\}^{1/2}-\{\alpha(\alpha-1)(\alpha-\beta)\}^{1/2})\{x(x-1)(x-\beta)\}^{-1/2}y^{-1}dx\,.
\end{aligned}
$$

Here α is one of four α's in 3.1. Hence by the same method as above we can get an analogous relation to (4.5.17) and we can conclude that t_1, t_2, t_3 and β are holomorphic functions on the generalized Teichmüller space. Thus we have

(4.5.21) THEOREM. *The coefficients in* (4.5.4) *are single valued holomorphic functions of the generalized Teichmüller space.*

4.6. Finally, we shall express the parameters of (4.5.3) by theta constants. In (4.5.3), we apply (4.3.1) to a meromorphic function on R

$$
(4.6.1)\qquad f(\boldsymbol{p})=1-x(\boldsymbol{p})\,, \qquad \boldsymbol{p}=(x, y)\,.
$$

Let the points on R which are over the points $0, z_1, \cdots, z_s, 1, \infty$ on the x-sphere be $\boldsymbol{q}_0, \boldsymbol{q}_{z_1}, \cdots, \boldsymbol{q}_{z_s}, \boldsymbol{q}_1, \boldsymbol{q}_\infty$ respectively. Then the zeros of $f(\boldsymbol{p})$ are n points

$\boldsymbol{q}_1, \cdots, \boldsymbol{q}_1$ and the poles of $f(\boldsymbol{p})$ are n points $\boldsymbol{q}_\infty, \cdots, \boldsymbol{q}_\infty$. Therefore, by (4.2.1) we have

$$f(\boldsymbol{q}_0)=\eta\prod^{n}\frac{\theta(\mathfrak{w}(\boldsymbol{q}_0)+\sum_{l=1}^{g-1}\mathfrak{w}(\boldsymbol{x}_l)-\mathfrak{w}(\boldsymbol{q}_1)-\mathfrak{c})}{\theta(\mathfrak{w}(\boldsymbol{q}_0)+\sum_{l=1}^{g-1}\mathfrak{w}(\boldsymbol{x}_l)-\mathfrak{w}(\boldsymbol{q}_\infty)-\mathfrak{c})}, \tag{4.6.2}$$

and

$$f(\boldsymbol{q}_{z_i})=\eta\prod^{n}\frac{\theta(\mathfrak{w}(\boldsymbol{q}_{z_i})+\sum_{l=1}^{g-1}\mathfrak{w}(\boldsymbol{x}_l)-\mathfrak{w}(\boldsymbol{q}_1)-\mathfrak{c})}{\theta(\mathfrak{w}(\boldsymbol{q}_{z_i})+\sum_{l=1}^{g-1}\mathfrak{w}(\boldsymbol{x}_l)-\mathfrak{w}(\boldsymbol{q}_\infty)-\mathfrak{c})}. \tag{4.6.3}$$

Thus, we have the following formula.

$$1-z_i=\prod^{n}\frac{\theta(\mathfrak{w}(\boldsymbol{q}_{z_i})+\sum_{l=1}^{g-1}\mathfrak{w}(\boldsymbol{x}_l)-\mathfrak{w}(\boldsymbol{q}_1)-\mathfrak{c})}{\theta(\mathfrak{w}(\boldsymbol{q}_{z_i})+\sum_{l=1}^{g-1}\mathfrak{w}(\boldsymbol{x}_l)-\mathfrak{w}(\boldsymbol{q}_\infty)-\mathfrak{c})} \cdot\prod^{n}\frac{\theta(\mathfrak{w}(\boldsymbol{q}_0)+\sum_{l=1}^{g-1}\mathfrak{w}(\boldsymbol{x}_l)-\mathfrak{w}(\boldsymbol{q}_\infty)-\mathfrak{c})}{\theta(\mathfrak{w}(\boldsymbol{q}_0)+\sum_{l=1}^{g-1}\mathfrak{w}(\boldsymbol{x}_l)-\mathfrak{w}(\boldsymbol{q}_1)-\mathfrak{c})}. \tag{4.6.4}$$

Now, we take (4.5.11) as the basis of differentials and take (4.5.12) as the canonical homology basis in (4.6.4). First, we see that $\mathfrak{c}$ is holomorphic in λ by (4.2.4). Second, we have for j $(1\leqq j\leqq g)$

$$w_j(\boldsymbol{q}_{z_i})-w_j(\boldsymbol{q}_1)=\int_{q_1}^{q_{z_i}}p_j(\zeta, \lambda)d\zeta, \tag{4.6.5}$$

$$w_j(\boldsymbol{q}_0)-w_j(\boldsymbol{q}_\infty)=\int_{q_\infty}^{q_0}p_j(\zeta, \lambda)d\zeta.$$

These are half periods along the closed curves which can be represented by linear combinations of $A_1(\zeta, \lambda), \cdots, B_g(\zeta, \lambda)$ with constants coefficient respectively in a neighborhood of λ_0. Therefore they are holomorphic in λ. Third, put

$${}^t\mathfrak{p}(\zeta, \lambda)=(p_1(\zeta, \lambda), \cdots, p_g(\zeta, \lambda)).$$

Then we can select a general divisor $D=\boldsymbol{x}_1+\cdots+\boldsymbol{x}_{g-1}$ such that the integral

$$\mathfrak{w}(\boldsymbol{x}_l)=\int_{p_0}^{x_l}\mathfrak{p}(\zeta, \lambda)d\zeta \qquad (1\leqq l\leqq g-1) \tag{4.6.6}$$

is holomorphic in a neighborhood of λ_0. Because we know that if

$$\tilde{D}=\boldsymbol{x}_1+\cdots+\boldsymbol{x}_{g-1}+\boldsymbol{x}_g$$

is general, then $D=\boldsymbol{x}_1+\cdots+\boldsymbol{x}_{g-1}$ is general. We assume that $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_g$ are distinct g points on the Riemann surface. A necessary and sufficient condition for $\tilde{D}$ to be general is that the determinant

$$\det(p_i(\zeta_j, \lambda)) \qquad (1 \leqq i, j \leqq g)$$

does not vanish. Here, $\zeta_1, \cdots, \zeta_g$ are the coordinates of $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_g$ in the domain $D(\lambda)$ respectively. Then, $\det(p_i(\zeta_j, \lambda))$ is a holomorphic function in $\zeta_1, \cdots, \zeta_g, \lambda_1, \cdots, \lambda_s$. Here $\lambda=(\lambda_1, \cdots, \lambda_s)$. Therefore, if this function does not vanish at $x_0=(\zeta_1^{(0)}, \cdots, \lambda_s^{(0)})$, then it does not vanish at all points $x=(\zeta_1, \cdots, \lambda_s)$ in a neighborhood of x_0. Hence we get the assertion.

Summarizing, we obtain the following theorem.

(4.6.7) THEOREM. *The single valued holomorphic functions which we have obtained in* (4.5.21) *can be expressed in the form* (4.6.4).

We may say that (4.6.4) is an extension of the representation of the Lambda function by Theta constants.

4.7. REMARK. We shall give a remark on the formulas (4.5.17) and (4.6.4). We know that the function

$$1-z$$

from the parameter z of the family of Riemann surfaces defined by

$$y^2 = x(x-1)(x-z), \qquad z \in C - \{0, 1\} \tag{4.7.1}$$

is the Lambda function on the upper half plane and is represented by a quotient of Theta constants. (4.6.4) is surely its extension.

In the family of Riemann surfaces defined by (4.7.1), the differential of the first kind is given by $y^{-1}dx$ and the integral

$$\int_g^h y^{-1}dx, \tag{4.7.2}$$

where g and h are two quantities of 0, 1, ∞, are solutions of the differential equation of the second order:

$$z(z-1)\frac{d^2w}{dz^2} + (2z-1)\frac{dw}{dz} + \frac{w}{4} = 0. \tag{4.7.3}$$

Let $w_1(z)$, $w_2(z)$ be two suitable independent solutions of (4.7.3), and denote the ratios of $w_1(z)$ and $w_2(z)$ by

$$\tau = w_1(z)/w_2(z). \tag{4.7.4}$$

Then $z(\tau)$ is a holomorphic function on the upper half plane $\operatorname{Im}\tau > 0$ [7].

(4.5.17) is surely an extension of (4.7.4) and (4.5.21) is an extension of $z(\tau)$. In this case we have a partial differential equation of Appellian type instead of (4.7.3). It would be interesting to investigate (4.5.4) from the view-point of the differential equation as above.

We shall discuss these problems in another place.

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