Computer-Aided Design 63 (2015) 118–128

Contents lists available at ScienceDirect

Computer-Aided Design

journal homepage: www.elsevier.com/locate/cad

On the equilibrium of funicular polyhedral frames and convex polyhedral force diagrams[☆]

Masoud Akbarzadeh, Tom Van Mele, Philippe Block [*]

Block Research Group, Institute of Technology in Architecture, ETH Zurich, Switzerland

HIGHLIGHTS

- Three-dimensional extension of graphic statics using polyhedral form and force diagrams.
- Defining the topological and geometrical relationships of 3D reciprocal diagrams.
- Design of compression and tension-only spatial structures with externally applied loads.
- Designing complex funicular spatial forms by aggregating convex force polyhedral cells.
- CAD implementation to manipulate the geometry of the force and explore its effects on the forms.

ARTICLE INFO

Article history:
Received 14 June 2014
Accepted 23 January 2015

Keywords:
Three-dimensional graphic statics
Reciprocal diagrams
Spatial systems of forces
Convex force polyhedra
Rankine's principle of equilibrium of polyhedral frames

ABSTRACT

This paper presents a three-dimensional extension of graphic statics using polyhedral form and force diagrams for the design of compression-only and tension-only spatial structures with externally applied loads. It explains the concept of 3D structural reciprocity based on Rankine's original proposition for the equilibrium of spatial frames. It provides a definition for polyhedral reciprocal form and force diagrams that allows including external forces and discusses their geometrical and topological characteristics. This paper furthermore provides a geometrical procedure for constructing a pair of reciprocal polyhedral diagrams from a given polyhedron representing either the form or force diagram of a structural system. Using this method, this paper furthermore suggests a design strategy for finding complex funicular spatial forms in pure compression (or tension), based on the construction of force diagrams through the aggregation of convex polyhedral cells. Finally, it discusses the effect of changes in the geometry of the force diagram on the geometry of the form diagram and the distribution of forces in it.

1. Introduction

Graphic statics is a method for structural form finding that originates in the pre-digital era [1–4], but continues to be used and developed even today [5–7].

In graphic statics, the geometry and equilibrium of forces of a structural system are represented by two reciprocal diagrams: the *form* and the *force* diagram. Since the geometrical relationship between these diagrams provides explicit control over both form and forces of a structure simultaneously, graphic statics is considered as an intuitive technique for structural design, relevant to architects, engineers, researchers, and students.

Despite its clear strengths and advantages, existing method of graphic statics has some limitations. The most important one is that a designer can only design three-dimensional structures by reducing them to a combination of two-dimensional sub-systems, for example, by using projections [8,9].

This paper therefore presents a three-dimensional version of graphic statics using reciprocal polyhedral form and force diagrams for the design and analysis of spatial frames with externally applied loads.

1.1. Previous work

Reciprocal diagrams are the basis of the conventional graphical methods of structural form finding. This section provides a brief summary of the use of reciprocal diagrams and other geometric techniques in structural design.

2D reciprocal diagrams

The concept of investigating static equilibrium through geometric constructions involving polygons of forces first appeared

[☆] This paper has been recommended for acceptance by Helmut Pottmann.

[*] Corresponding author.
E-mail addresses: amasoud@ethz.ch (M. Akbarzadeh), vanmelet@ethz.ch (T. Van Mele), pblock@ethz.ch (P. Block).

http://dx.doi.org/10.1016/j.cad.2015.01.006
0010-4485/

in the drawings by Varignon [1]. Rankine used this concept in *Theory of Equilibrium of Polygonal Frames* and applied it to the design of framed structures [10]. However, the first person who formalized the *reciprocal relationship* between form and force diagrams was Maxwell [3]. He used the concept of duality developed by Möbius [11], and showed that 2D reciprocal diagrams must not only be topologically dual of each other, but each line in one diagram should be perpendicular to the corresponding line in the other diagram. These 2D diagrams became the basis for 2D graphic statics [2,4,12].

Unloaded, pre-stressed, reciprocal surface structures

Reciprocal diagrams on the local coordinates of a surface can also be used to geometrically calculate the states of stress in surfaces which are only loaded in their boundaries. If two unloaded pre-stressed surfaces are reciprocal, the equilibrium of a node on one surface is ensured by a closed polygon of forces on the other [13].

Surface structures and parallel loading

The problem of finding funicular structural forms in three dimensions using geometric approaches has received a lot of attention in different fields of research. Thrust Network Analysis developed by Block and Ochsendorf [14] is a graphic statics-based method for finding compression-only funicular network of forces for given loads and boundary conditions. By requiring all loads to be vertical, it provides explicit control over the 3D shape of a funicular network in compression through projected form and force diagrams describing horizontal equilibrium in the system. As a consequence, TNA produces results in the form of heightfields over a two-dimensional diagram of forces. This method has been investigated further by Vouga et al. [15], Liu et al. [16], de Goes et al. [17], Panozzo et al. [18] related to the design and construction of self-supporting surfaces. The fundamental principle of these methods is to allow separating horizontal and vertical equilibrium by requiring all loads to be vertical, or at least parallel, and perpendicular to the planes of projection of the form and force diagrams. Therefore, these methods cannot easily account for non-parallel applied load cases.

Extensions of graphic statics to 3D

Since the 19th century, several methods have been developed to extend graphic statics to three dimensions. The works of Föppl [8], who used projective geometry to analyze three-dimensional trusses, and Schrems and Kotnik [19], who suggested a force-pair technique, are examples of such efforts. These methods are limited to the analysis of determinate system of forces and fail to preserve the intuitive aspect of graphic statics.

3D reciprocal diagrams

Rankine [20] suggested that the equilibrium of polyhedral frames can be described using a reciprocal polyhedron of forces (see Section 2). In the same year, and in response to Rankine's proposition, Maxwell [3] provided a geometric procedure to construct 3D reciprocal diagrams for a specific case, which is only suitable for self-stressed, structurally determinate systems (see Appendix), and pointed out the complexity of the construction for more general cases. The problem was never investigated further until Akbarzadeh et al. [21] recently visualized and thereby clarified the contents of Rankine's dense proposition.

Simplicial structures

The geometrical relationships between the reciprocal figures described by Rankine are very similar to the geometrical relationships in orthogonal dual structures in Poincaré duality theorem [22]. However, Poincaré duality is defined for n-manifold triangulated space, whereas the reciprocal diagrams discussed in this paper are not limited to triangulated/tetrahedralized space. There is a large body of research in the field of computer graphics, mathematics, and engineering that emphasizes the use of triangulated dual structures and their constructing algorithms. For further readings on these topics, we refer readers to de Goes et al. [23] and Mullen et al. [24].

1.2. Objectives and contributions

The ultimate goal of our research is to create a fully three-dimensional graphical method for the design of spatial structural systems with non-parallel applied external loads that preserves the intuitive and explicit control over both form and forces provided by (traditional) two-dimensional graphic statics through geometrically linked diagrams. In this paper, as a first step, we introduce a subset of this method for designing spatial frames in pure compression (or tension). This method assumes external loads are applied only on the boundaries of the form diagram. The selfweight of the frames is thus ignored.

In Section 2, we briefly discuss Rankine's original proposition, and provide a short proof.

In Section 3, we define polyhedral form and force diagrams, and describe the topological and geometrical requirements of their reciprocal relationship in a structural context. We furthermore discuss the additional geometric requirements related to the intended compression-only or tension-only nature of the structural systems investigated in this paper.

In Section 4, we discuss a procedure for constructing a pair of convex, reciprocal polyhedral diagrams from a given polyhedron with convex cells and planar faces.

In Section 5, we use this procedure to find spatial frames in pure compression by starting from a given force distribution represented by a convex polyhedral force diagram. We furthermore discuss how the force distribution and the geometry of the form can change by exploiting the geometric degrees of freedom of the force diagram.

2. Principle of equilibrium of polyhedral frames

Rankine [20], in *Principle of equilibrium of polyhedral frames*, stated that forces acting on a point, perpendicular and proportional to the areas of the faces of a closed polyhedron, are in equilibrium. This proposition contains only two short paragraphs that we include here, as they appeared in the *Philosophical Magazine*:

> "If planes diverging from a point or line be drawn normal to the lines of resistance of the bars of a polyhedral frame, then the faces of a polyhedron whose edges lie in those diverging planes (in such a manner that those faces, together with the diverging planes which contain their edges, form a set of contiguous diverging pyramids or wedges) will represent, and be normal to, a system of forces which, being applied to the summits of the polyhedral frame, will balance each other – each such force being applied to the summit of meeting of the bars whose lines of resistance are normal to the set of diverging planes that enclose that face of the polyhedron of forces which represents and is normal to the force in question. Also the areas of the diverging planes will represent the stresses along the bars to whose lines of resistance they are respectively normal.
>
> It is obvious that the polyhedron of forces and the polyhedral frame are reciprocally related as follows: their numbers of edges are equal, and their corresponding pairs of edges perpendicular to each other; and the number of faces in each polyhedron is equal to the number of summits in the other."

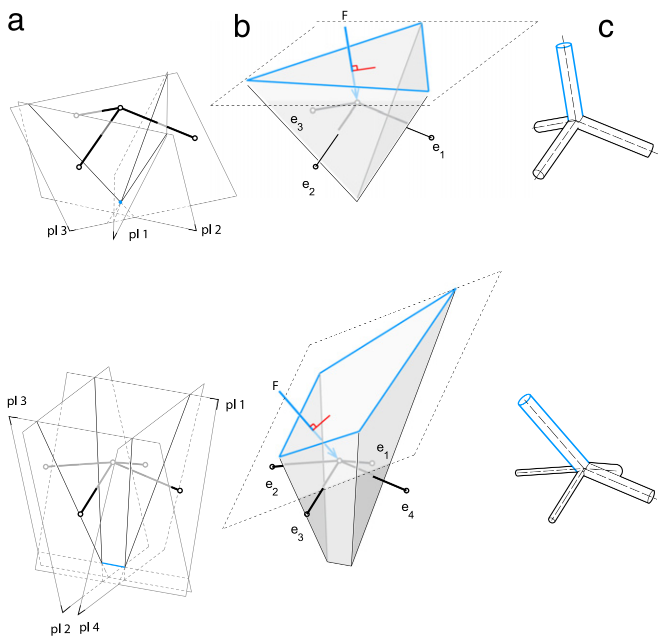


Fig. 1. Two polyhedral frames with their reciprocal, polyhedral force diagrams: (a) the intersection of planes perpendicular to bars diverging from a point (top), or a line (bottom), form an open polyhedron; (b) the plane normal to the direction of an additional applied force closes the polyhedron and creates equilibrium; and (c) the pipe diagram represents the magnitude of force, each calculated from the area of the corresponding (perpendicular) face in the force diagram.

Fig. 1 depicts the application of Rankine's theory to two polyhedral frames, consisting of three and four bars respectively [21]. Fig. 1(a) shows the planes that are perpendicular to the bars of the frames, and diverge from a point (top) or line (bottom). In both cases, the planes form an open polyhedron. A face perpendicular to an additional force (for example, an applied load) closes the polyhedron (Fig. 1(b)) and completes the force equilibrium for the node. The areas of the faces of the closed polyhedron are proportional to the magnitudes of the forces in the corresponding bars of the frame. Each choice of closing face thus results in a different distribution of forces in the frame (Fig. 1(c)).

To the knowledge of the authors, Rankine never proved his theory. Therefore, we provide here a short proof based on the divergence theorem [25,26].

Let P be a closed polyhedron in $\mathbb{R}^3$ with enclosing faces f_i and volume V, and $\vec{v}$ an arbitrary, constant vector field in $\mathbb{R}^3$. Each face, furthermore, has area A_i and normal vector $\vec{n}_i$. According to the divergence theorem, the total outward flux of $\vec{v}$ through the closed surface of the polyhedron is equal to the divergence of $\vec{v}$ over the enclosed region, which is zero since $\vec{v}$ is constant

$$\oiint_A \vec{v} \cdot \vec{n}\, \mathrm{d}A = \iiint_V \mathrm{div}\vec{v}\, \mathrm{d}V = 0.$$

For the polyhedron, we can thus write

$$\oiint_A \vec{v} \cdot \vec{n}\, \mathrm{d}A = \vec{v} \cdot \sum_i A_i \vec{n}_i = 0.$$

This means the sum of all area-weighted normals $\sum_i A_i \vec{n}_i$ of the polyhedron must be zero, since $\vec{v}$ is arbitrary. Therefore, if the forces $\vec{F}_i$ applied to a point in space are perpendicular to the faces of a polyhedron, and their magnitudes proportional to the areas of the faces, the sum of these forces must be zero, leaving the point in equilibrium

$$\sum_i \vec{F}_i = \sum_i A_i \vec{n}_i = 0.$$

3. Reciprocal, convex, polyhedral form and force diagrams

A polyhedral diagram consists of vertices, edges, faces, and cells. The faces can be bounded or unbounded, and cells can be open or closed, as seen in Fig. 2. Two diagrams are reciprocal if certain topological and geometrical requirements are fulfilled.

3.1. Duality

The diagrams are required to be dual. Two diagrams are dual if the following statements are true:

- Each edge e_i of the form diagram corresponds to one and only one face $f_i{}^*$ of the force diagram (Fig. 2(b)).
- Each vertex v_i in the form diagram corresponds to a closed polyhedral cell $p_i{}^*$ in the force diagram (Fig. 2(c)).
- Each open/closed polyhedral cell p_i of the form diagram corresponds to one and only one vertex $v_i{}^*$ of the force diagram (Fig. 2(d)).
- Each bounded/unbounded face f_i in the form diagram corresponds to one and only one edge $e_i{}^*$ in the force diagram (Fig. 2(e)).

A direct result of these requirements is that the number of edges in one diagram is equal to the number of faces in the other, and that the number of vertices is equal to the number of cells (Fig. 2(b)–(e)). Fig. 3(a), (b) illustrates these topological relationships. The elements of the form diagram are labeled with lower case letters, and the elements of the force diagram tagged with an asterisk ($*$).

3.2. Planarity and perpendicularity

If, in addition, all faces are planar, and all edges perpendicular to their dual faces, the diagrams are reciprocal. The force diagram then represents the structural equilibrium of the system of forces represented by the form diagram, with the force in each edge of the form diagram $\left|\vec{F}_{e_i}\right|$ equal to the area of its dual face $A_{f_i'}$.

Note that due to the presence of external loads and reaction forces, the form diagram has both bounded and unbounded faces and open and closed cells. The force diagram, on the other hand, has only bounded faces and closed cells. The outside faces of the force diagram correspond to the external forces. All other faces represent the internal forces of the form.

The closed cell formed by the outside faces represents global equilibrium of all external forces. Each internal cell p_i' represents the equilibrium of its dual node v_i.

Fig. 3(a), (b) illustrates the reciprocal relationships between form and force diagrams in 3D. The elements of the reciprocal force diagram suffixed with an apostrophe (').

3.3. Convexity

In this paper we focus on the equilibrium of compression or tension-only structures. A system of forces is in pure compression (or tension), if its force polyhedron is a proper cell decomposition of three-dimensional space. This constraint is analogous to the conditions for equilibrium of spider webs described by Ash et al. [27]. A force polyhedron is a proper cell decomposition of space, if the following statements are true:

- every point in space belongs at least to one cell;
- the cells have disjoint interiors;
- the decomposition is face to face; that is, every face of one cell is a complete face of another cell.

4. Constructing reciprocal polyhedrons

In this section, we describe a straightforward algorithm for constructing a pair of reciprocal diagrams from a given polyhedron representing either the form or force diagram of a structural

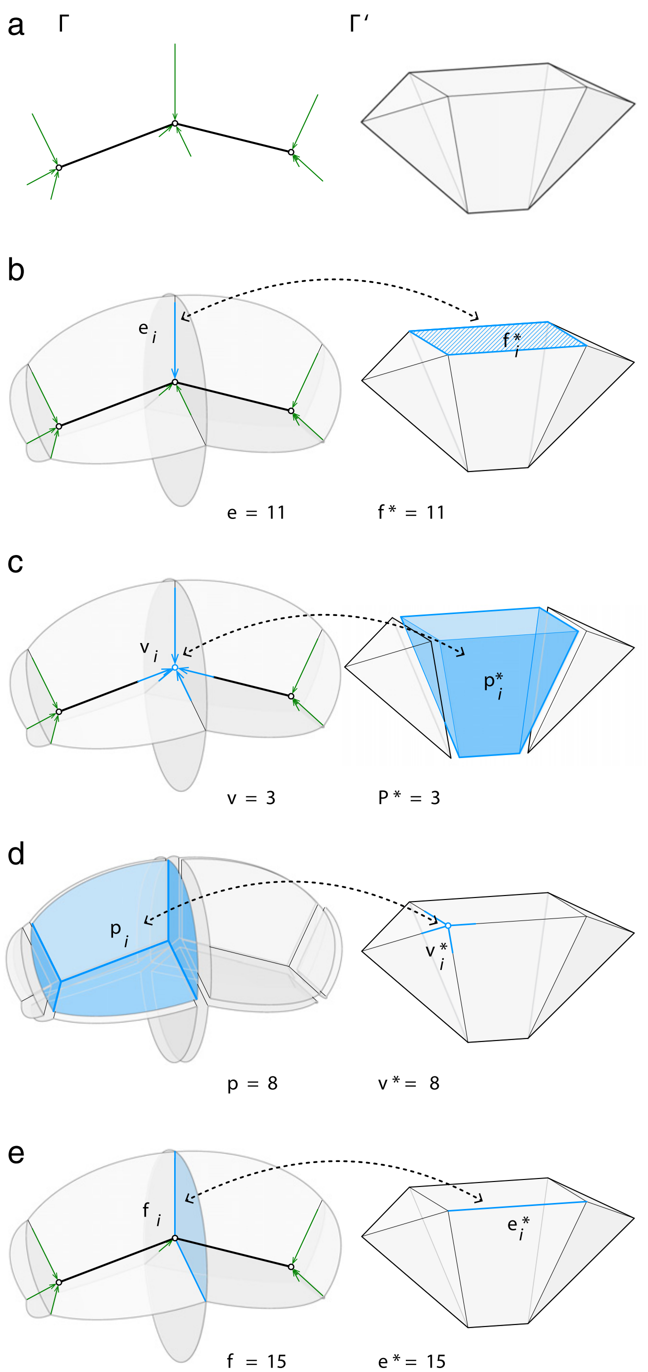


Fig. 2. Topological relationships between the form diagram Γ and the force diagram Γ' in 3D: (a) Γ and its reciprocal Γ'; (b) edge e_i of Γ and its corresponding face $f_i{}^*$ of Γ'; (c) closed cell $p_i{}^*$ of Γ' representing the equilibrium of a node v_i of Γ; (d) open cell p_i of Γ and its corresponding vertex $v_i{}^*$ of Γ'; and, (e) open face f_i of Γ and its corresponding edge $e_i{}^*$ of Γ'.

system. For simplicity of the explanation we assume here the given diagram is the force diagram.

As depicted in Fig. 4, the algorithm consists of three main sections: (1) constructing the force diagram from a given geometric representation and extracting its topology (see Section 4.1); (2) generating the topology of the form diagram (see Section 4.2); and (3) imposing perpendicularity (see Section 4.3).

4.1. Force diagram topology

The first step is to determine the topology of the force diagram. The topology of a polyhedron can be described with a winged-edge

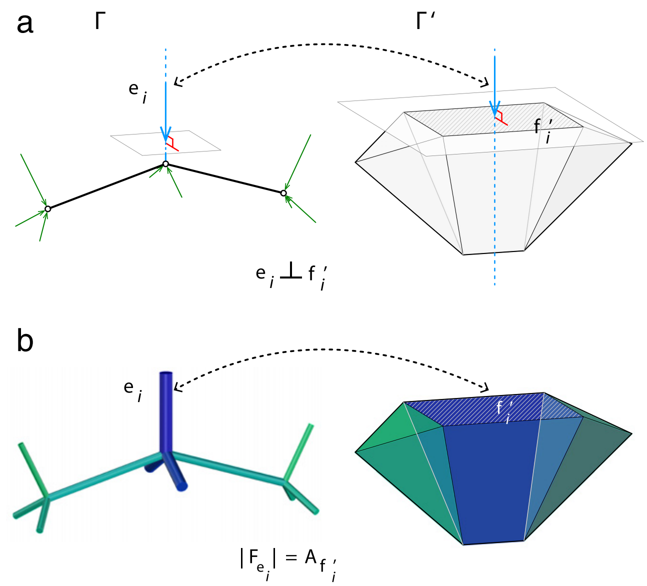


Fig. 3. Reciprocal relationships between the form diagram Γ and the force diagram Γ' in 3D: (a) edge e_i of Γ and its corresponding face f_i' of Γ'; and, (b) piped representation of the magnitudes of the equilibrated forces $|F_{e_i}|$ proportional to the areas of the corresponding faces $A_{f_i'}$ of Γ'.

data structure (WED) [28]. With common CAD modeling software, it is possible to represent a polyhedron by a wireframe model that consists of edges and vertices, or a boundary representation model that consists of connected surfaces or mesh elements (Fig. 5).

A wireframe model is essentially a set of connected lines. Its connectivity graph can be easily determined by identifying all unique vertices among the start and the end points of the lines, and assigning a pair of connected vertices to each line (Fig. 5(a)). The faces of the input geometry are not directly represented by the wireframe and should be detected from the connectivity of vertices and edges using an algorithm that can recognize all possible planar faces in the model.

Boundary-representation (BREP) models already contain the information of the faces (Fig. 5(b)). This input therefore simplifies the construction of the WED, since no face finding is required. Note that the faces of BREP models are not necessarily planar. An algorithm for planarizing its faces can be found in [29], for example.

From the vertices, edges, and faces of the input model, we construct the WED and find all internal cells and one external cell, as seen in Fig. 6.

4.2. Form diagram topology

The connectivity of the form diagram follows immediately from the adjacency graph of the polyhedral cells of the force diagram. We can, therefore, construct a topologically correct form diagram by connecting the centroids of adjacent cells, v_r^*, v_i^*, v_j^* in the force diagram (Fig. 7(a)). Each internal cell adjacent to the external cell is furthermore connected to the centroids of its external faces (Fig. 7(b)).

4.3. Form diagram perpendicularity

So far, we constructed a polyhedral frame that has the topology of the desired final form diagram. However, the edges of this polyhedral frame are generally not perpendicular to the faces of the force diagram.

We impose perpendicularity through an iterative procedure in which all iterations consist of two steps. A similar algorithm is used as the one for 2D force diagrams described by Rippmann et al. [30].

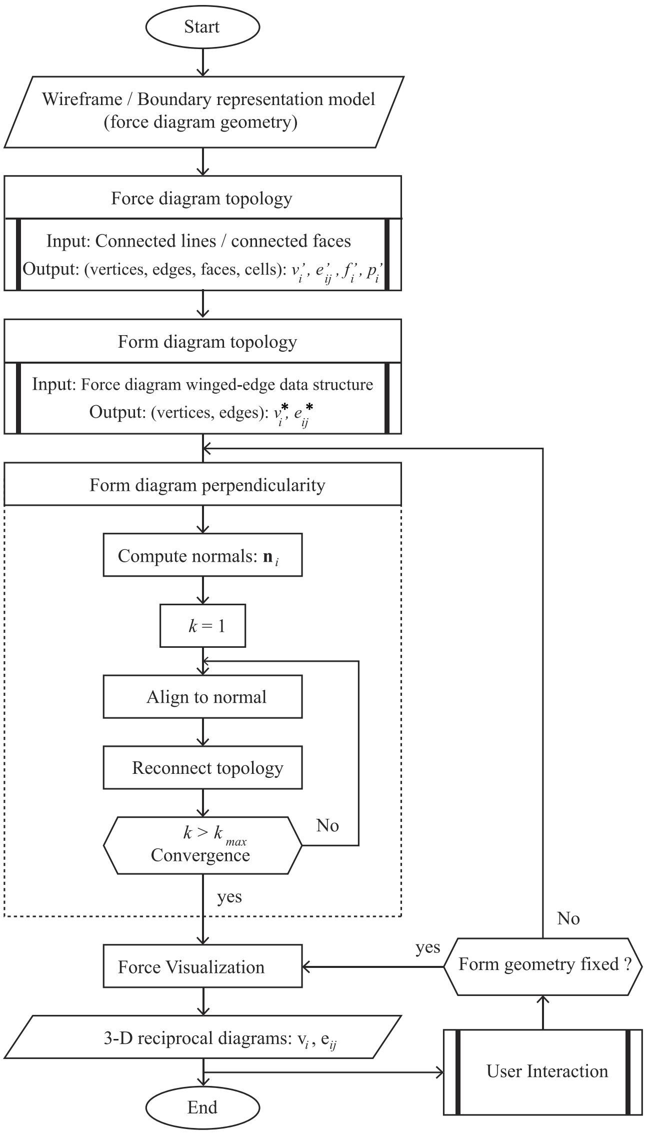


Fig. 4. Form-finding flowchart representing multiple stages of the form-finding algorithm.

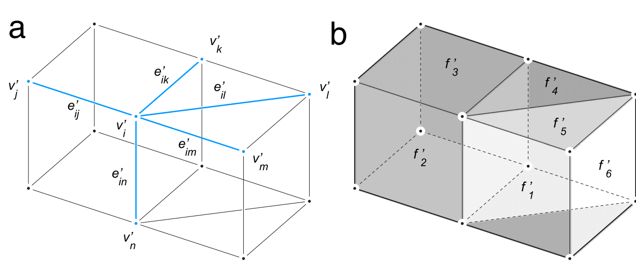


Fig. 5. (a) Wireframe model of a force diagram including the connectivity information of edges and vertices; and (b) boundary representation model of the force diagram consisting of connected faces.

At the start, we compute the normal vectors of the faces of each polyhedron of the force diagram obtained in Section 4.1. Then, in the first step of each iteration, we rotate the edges of the polyhedral frame around their mid point, found in Section 4.2, such that they become parallel to the normal vectors of the faces of the force diagram. This requires the edges of the polyhedral frame to become disconnected. Therefore, in the second step of each iteration, we reconnect the edges, which then results in a polyhedral frame that is 'slightly more perpendicular' to the faces of the force diagram.

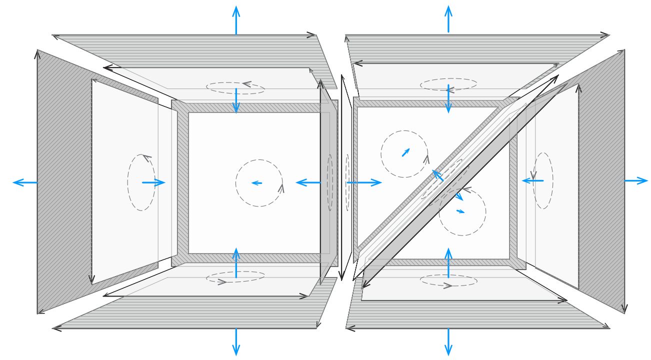


Fig. 6. Visualization of the winged-edge data structure of the force diagram.

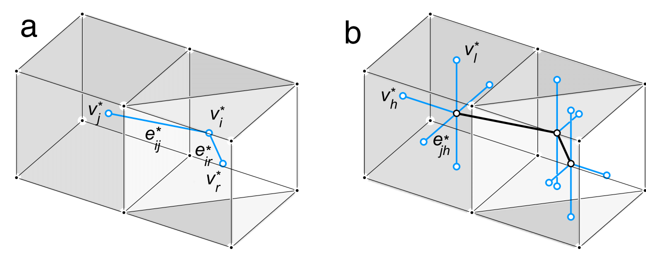


Fig. 7. (a) The internal vertices and edges of the topological polyhedral frame are constructed by connecting the interiors of adjacent polyhedral cells; and (b) connecting the internal vertices to the external faces completes the polyhedral frame's topology.

The procedure is repeated until all edges are perpendicular to their reciprocal faces up to a chosen tolerance (Fig. 9).

Fig. 9(a)–(c) show the different steps of the first iteration for edges e^*_{ij}, and e^*_{ir}. Note the use of an asterisk (*) as suffix at this point, indicating that the diagrams are not yet reciprocal, but merely topologically dual. Fig. 9(d) shows the edges at the end of the iterative procedure, at which point they are perpendicular to corresponding faces f_j' and f_i' of the force diagram, up to a given tolerance.

Finally, we visualize the distribution of forces by adding thickness to the edges of the form diagram, proportional to the area of the reciprocal faces in the force diagram. Fig. 10 shows the four stages of the 'form diagram perpendicularity' algorithm.

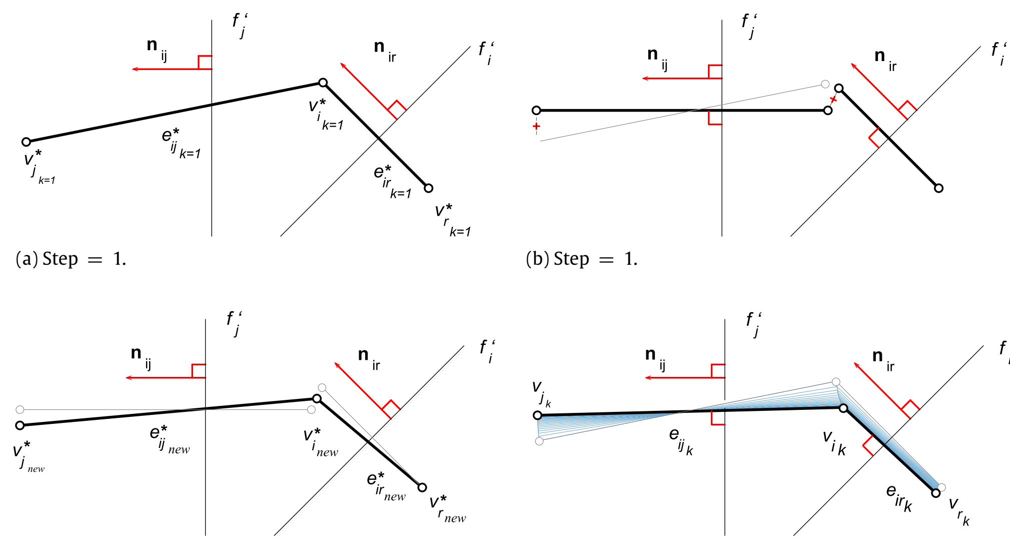
Although a proof of convergence is not provided in this study, convergence was not a concern in any of the presented examples. Fig. 8 gives an overview of the required number of iterations and computing times for the structures in Figs. 11 and 12.

5. Exploring structural forms in pure compression or tension

In this section, we focus on the equilibrium of spatial forms in pure compression. A spatial frame is only guaranteed to be in pure compression, if its force diagram is a *proper cell decomposition of space* as defined in Section 3.3. A wide range of structural forms can be explored by aggregating (convex) polyhedral force cells. Fig. 11 depicts multiple examples of cell aggregations and their reciprocal form diagrams. Each example in Fig. 11 consists of four drawings in two columns. The left column represents the aggregation of multiple polyhedral cells and the resulting force diagram. The right column represents the reciprocal structural form using a bar-node representation, and a form diagram representation in which the forces in the members are visualized by the thickness of the pipes. For readability of the diagrams, and particularly to clarify the reciprocal relationship between them, a color gradient (from blue to red) is used to visualize the proportional range (from minimum to maximum) of force magnitudes in the edges of the form diagram, and thus of the areas of the corresponding faces in the force diagram.

Fig.	*Force Diagram*				*Form Diagram*				*Number of Iterations*	*Computing Time (s)*
	v'	*e'*	*f'*	*p'*	*p*	*f*	*e*	*v*		
12. a	13	36	38	13	13	36	38	13	271	0.218
12. b	25	55	43	12	25	55	43	12	265	0.298
12. c	39	98	84	24	39	98	84	24	212	1.043
12. d	24	40	54	24	24	40	54	24	265	0.281
12. e	16	42	40	13	16	42	40	13	130	1.54
12. f	97	219	162	43	97	219	162	43	299	5.327
13	44	147	176	72	44	147	176	72	300	6.56

Fig. 8. The number of iterations and computing times of the form finding process of the examples in Figs. 11 and 12.



(a) Step = 1.

(b) Step = 1.

(c) Step = 1.

(d) Step = *k*.

Fig. 9. (a) Computing normals of the faces; (b) aligning edges with their corresponding normals of the faces of the force diagram; (c) reconnecting geometry; and (d) progression and the end of the iterative process.

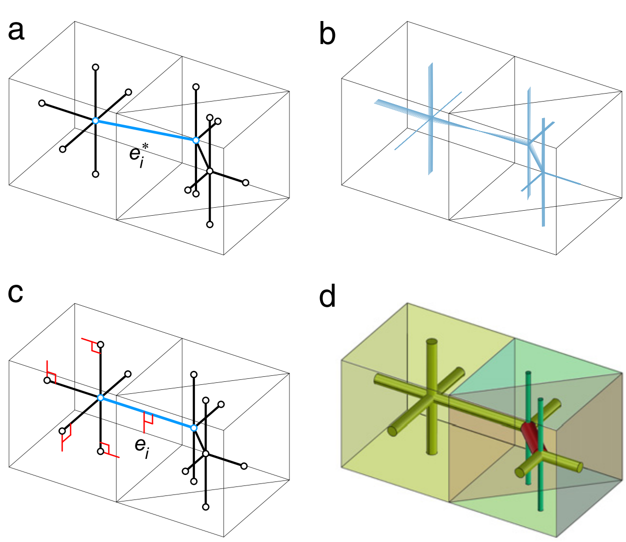


Fig. 10. (a) Polyhedral frame, with a topology of the form diagram that is not perpendicular to the force diagram; (b) imposing perpendicularity; (c) reciprocal polyhedral frame as form diagram; and (d) visualization of the force distribution in the form diagram.

Fig. 11(a) shows a force diagram resulting from the aggregation of tetrahedral cells that converge to a point. This force diagram is structurally reciprocal to a form diagram that is a compression-only 'surface' network of forces, a *thrust network*, subjected to non-parallel applied forces. In Fig. 11(b), an aggregation of 5-sided polyhedra that converge to a line results in a force diagram that is reciprocal to another type of thrust network. Fig. 11(c) represents an example of an aggregation of 5-sided and 6-sided polyhedra in two layers. The resulting force diagram is reciprocal to a spatial system of forces in a double-layered compression structure. Fig. 11(d) shows a radial aggregation of tetrahedra stacked in multiple layers. This force diagram is reciprocal to a tubular system of forces in compression. Scaling tetrahedra while aggregating them in Fig. 11(e) can describe a force diagram that is structurally reciprocal to a 3D branching, structural form. More spatially complex structural forms can also emerge by aggregating space-packing polyhedra such as the aggregation of a regular and a truncated tetrahedral cells (Fig. 11(f)). This force diagram is structurally reciprocal to a cellular form diagram.

These examples clearly demonstrate the potential of using the aggregation of force polyhedra to explore a wide range of compression or tension only forms for a variety of loading conditions in three dimensions.

As a final example, Fig. 12 depicts a complex branching structure with 72 nodes and 176 branches. Consequently, the force diagram has 72 convex cells with a total of 176 faces. The top faces of the force diagram are horizontal, and therefore represent a set of vertical applied loads. These loads could, for example, correspond to the weight of a floor slab supported by the tree. The four large vertical faces on the sides represent the horizontal reaction forces at the four corners at the top, and the horizontal faces at the bottom represent the vertical reaction forces on the ground.

6. Manipulating the force diagram

Once the reciprocal form and force diagrams are found, the designer can manipulate the geometry of the force diagram and consequently the form diagram in the two possible ways: manipulations that preserve the geometry of the form, and only redistribute the force magnitudes in the form diagram; and

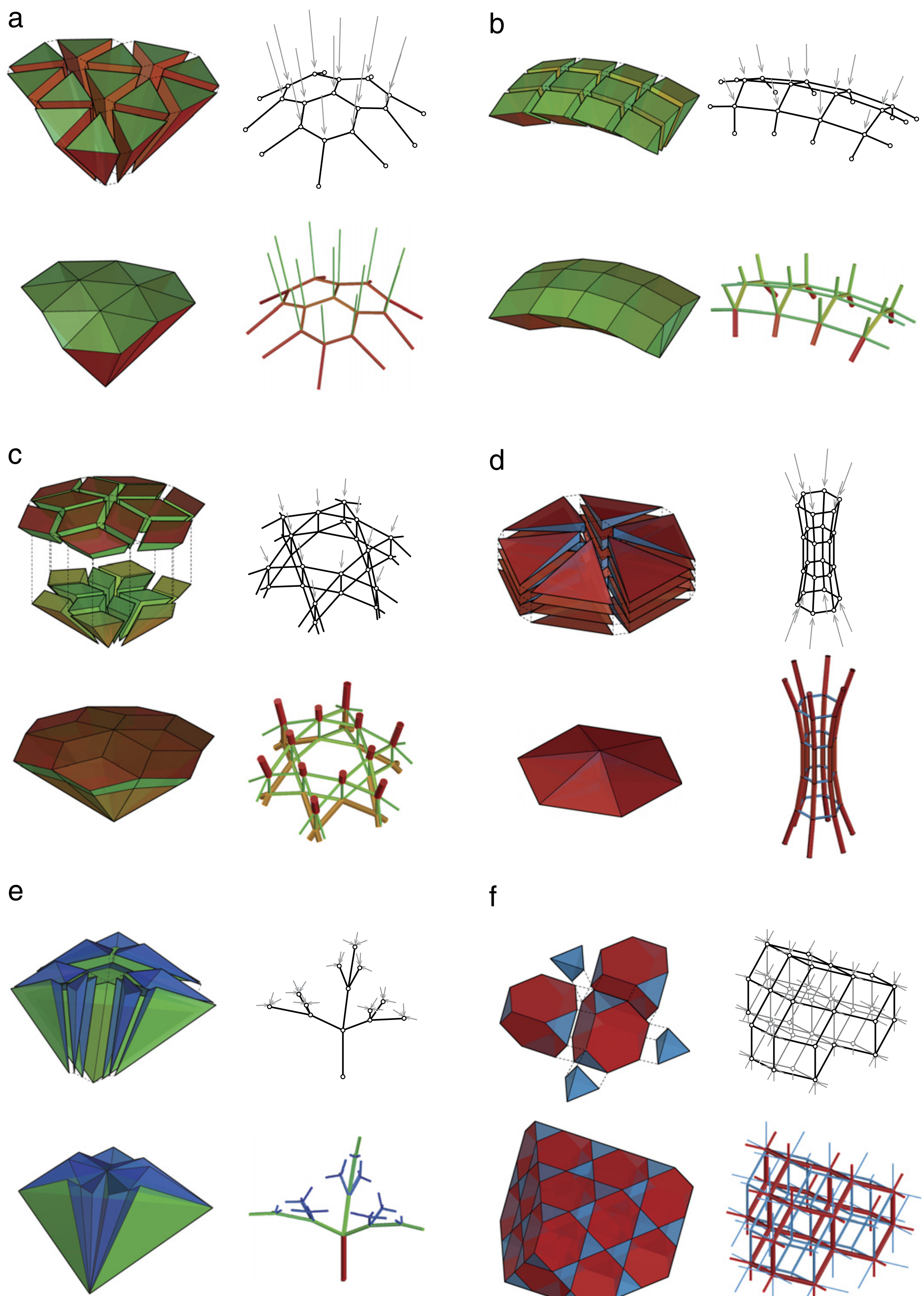


Fig. 11. Six examples of the design of spatial systems of forces by aggregation of polyhedral force diagram cells. Aggregation of (a) tetrahedral cells converging to a point; (b) 5-sided polyhedra converging to a curved line; (c) 5-sided and 6-sided polyhedra in two layers; (d) tetrahedra stacked radially in multiple layers; (e) tetrahedra in a fractal-like pattern; (f) regular and truncated tetrahedra. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

manipulations that change both the geometry and the distribution of the forces in the form diagram.

6.1. Manipulating the force and preserving the geometry of the form diagram

For statically indeterminate systems of forces, the designer can modify the areas of the faces of the force diagram without changing the direction of their normal vector. Such a modification changes the magnitude of the internal and external forces without changing the geometry of the form diagram. One possible method to change the area is to move faces along their normal vector direction. This transformation preserves the reciprocal relationship to a form diagram with fixed geometry. Each cell of the force diagram is adjacent to at least one other cell, and, therefore, shares a face with that adjacent cell. There are only two types of topologically different faces in the force diagram that a user can select: the *local*

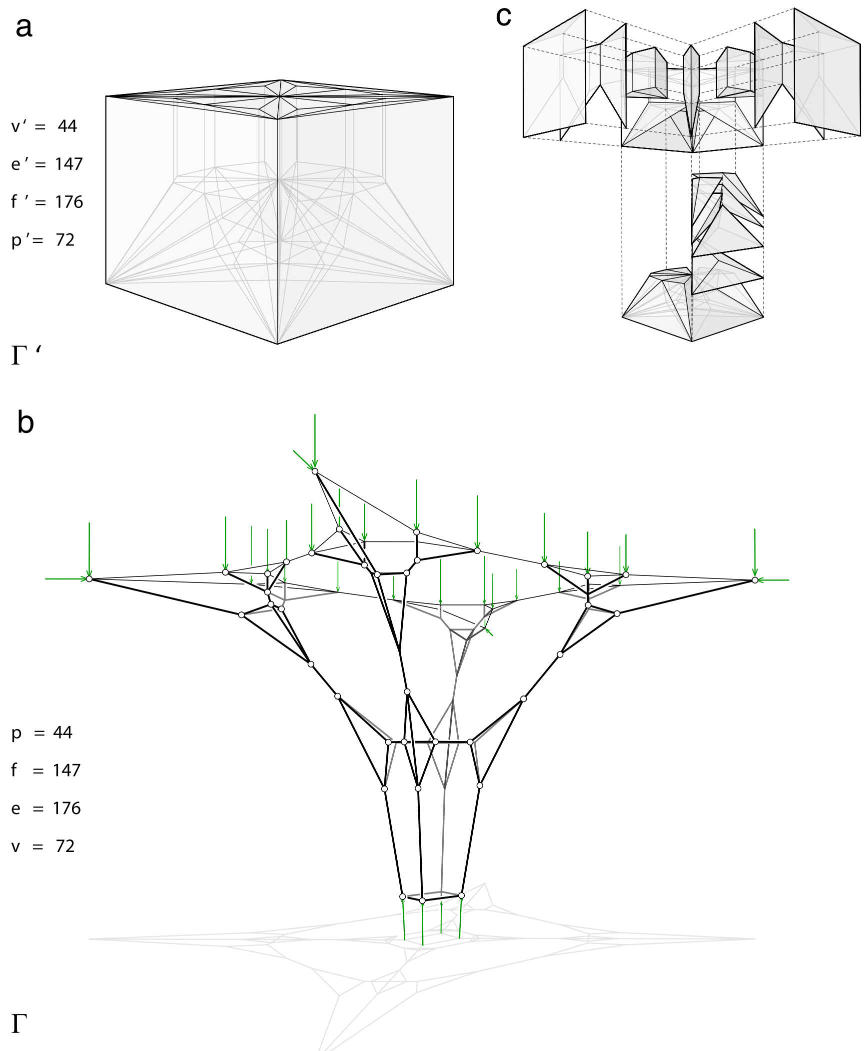


Fig. 12. A complex 3D branching structure designed using 3D form and force diagrams: (a) the force diagram consisting of 72 closed, convex cells; and (b) the reciprocal form diagram; and (c) exploded axon of the force polyhedrons representing various groups of polyhedral cells of the force diagram.

face, and the *global face*. Local face is a face that shares edges only with the faces of a single cell. The change of the area of the local face alters the geometry of a single cell in the force diagram and therefore, changes the force distribution in a single node in the reciprocal form diagram.

Fig. 13(a) shows the process of changing the area of a local face. The user can move the selected face along its normal direction to change its area. This change alters the area of the adjacent faces only for that single polyhedral force cell. The force magnitudes are adapted accordingly in the form diagram and visualized by the thickness of the pipes.

A global face is a face that shares at least an edge with a face of an adjacent cell. Any change in area of the selected face, therefore, affects the area of the adjacent faces in multiple polyhedral cells of the force diagram. This thus changes the force magnitudes in several nodes of the form diagram. Fig. 13(b) shows the process of selecting and changing the area of a global face. As illustrated, the user moves the face along its normal vector, which causes the motion of its adjacent faces in the neighboring cell. As a result, the change in the area of the global face not only affects the area of the faces of a single cell, but also alters the area of the faces of its adjacent cells.

Another possible manipulation that preserves the geometry of the form is to globally scale the force diagram. This increases or decreases the overall magnitude of the forces in the form diagram proportionally. Fig. 13(c) illustrates the process of selecting a vertex and changing the area of the faces in the force diagram. If the vertex moves (in any direction) the faces of the polyhedron will no longer stay planar. Therefore, the only possible transformation that changes the area of the faces in the force diagram and preserves the geometry of the form diagram is scaling. As illustrated in Fig. 13(c), the magnitudes of the forces are decreased or increased globally.

6.2. Manipulating the force and changing the geometry of the form diagram

Selecting and moving a vertex of the force diagram results in a polyhedron with non-planar faces (Fig. 14). If such a manipulation is induced by the user, the faces of the force diagram must be planarized again prior to finding the reciprocal form diagram. An iterative approach can be used to planarize the faces similar to the algorithm presented by Rippmann and Block [29]. Since this manipulation changes the direction of the normal vector of the faces connected to the selected vertex, the geometry of the form

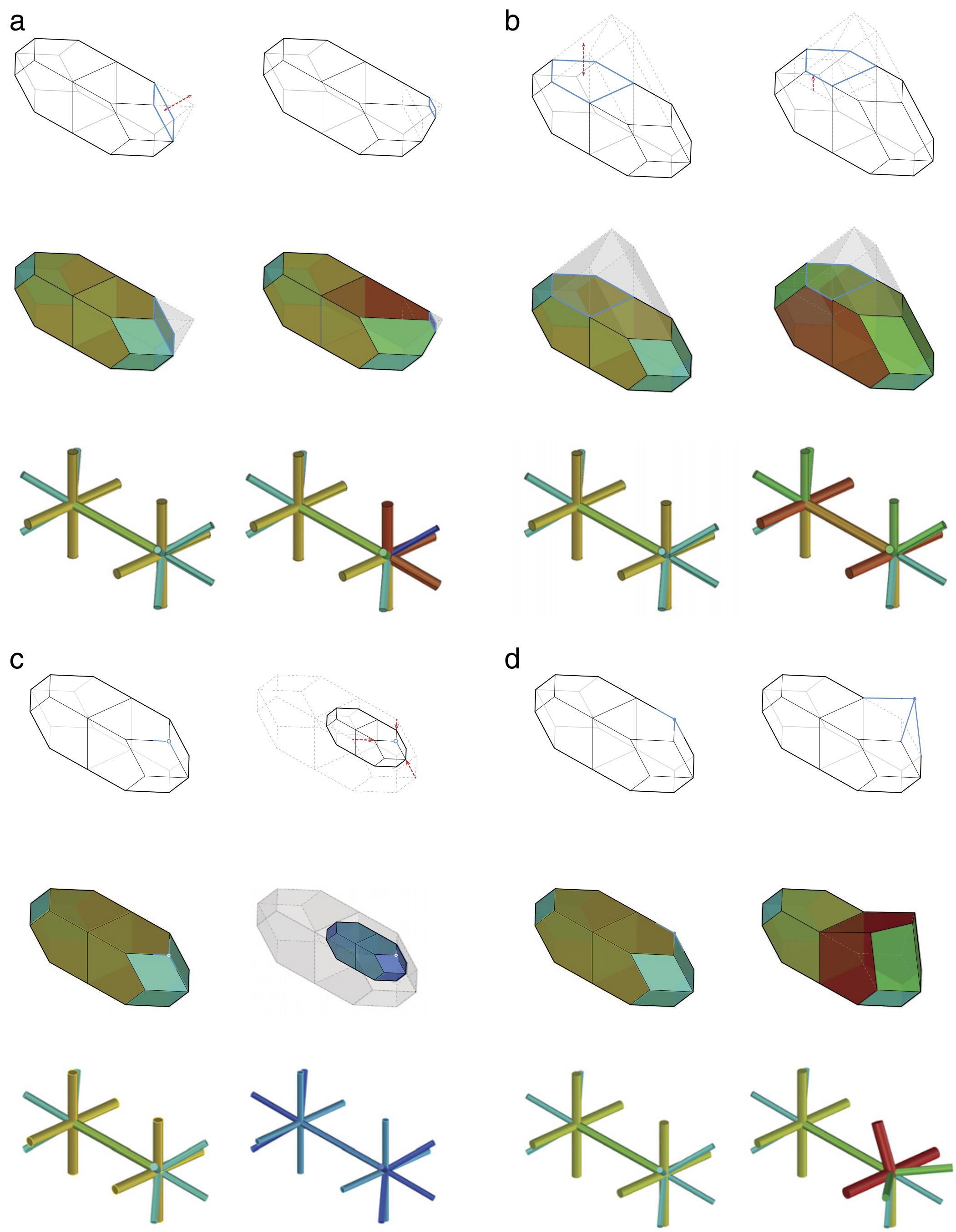


Fig. 13. Manipulations of the force diagram: (a) moving a local face along its normal, (b) moving a global face along its normal, (c) global scaling of the diagram with respect to a vertex, and (d) free movement of a vertex.

diagram will no longer be kept the same. Fig. 13(d) represents the force distribution and the geometry change of the form and the force diagrams before and after moving a vertex of the force diagram.

7. Discussions and future work

This paper discussed the basis for a three-dimensional graphical method for the design of spatial frames that preserves the intuitive and explicit control provided by traditional two-dimensional graphic statics through the use of form and force diagrams.

We have described the properties of reciprocal form and force polyhedrons and defined the requirements that guarantee compression- or tension-only equilibrium.

We have provided a straightforward algorithm for constructing a pair of reciprocal polyhedral diagrams from a given polyhedron representing either the form or force diagram of a structural system.

We have shown how systems of forces in pure compression (or tension) can be generated by constructing force diagrams through the aggregation of convex polyhedral cells representing the equilibrium of individual nodes in the system.

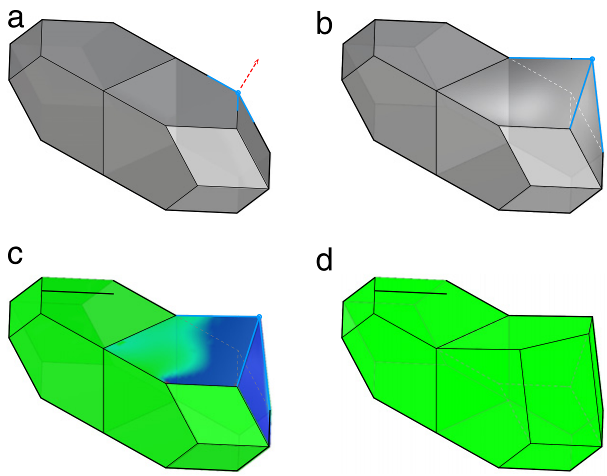


Fig. 14. (a) Selecting a vertex of a force diagram; (b) moving the vertex in the 3D space; (c) curvature analysis of the force diagram with non-planar faces; and (d) curvature analysis of the force diagram after planarization.

Finally, we have discussed the effects of transformations of the force diagram on the geometry of the form diagram and the distribution of forces in it.

It is clear that many aspects of the proposed methodology require further exploration to establish a complete theory of reciprocal polyhedra and their interpretation as form and force diagrams for the design of spatial frames. Such explorations could be concerned with the development of different algorithms for constructing reciprocal diagrams or an investigation of the conditions for existence of those diagrams in a structural context.

Some practical considerations could also provide interesting directions for further research. It would, for example, be interesting to investigate how tension and compression can be combined while preserving the legibility of the force polyhedron.

Another aspect that has not been addressed in this paper is the application of geometric constraints on both diagrams. These constraints are necessary to impose boundary conditions on the form diagram such as the location of the lines of action of applied loads and locations of the supports. They are also necessary to control the size and orientation of specific faces of the force diagram such that they match the orientation and magnitude of external loads defined in the form diagram.

Appendix. Maxwell's reciprocity in 3D

Rankine did not provide a method by which a polyhedral frame and its reciprocal force diagram may be constructed. Maxwell proposed to address this problem in a purely geometrical manner, and stated some of the properties of reciprocal figures and the condition of their existence [3]. According to Maxwell's (geometric) definition, reciprocal figures both consist, solely, of closed polyhedra such that:

- each figure is made up of closed polyhedra with planar faces;
- every point of intersecting lines in one figure is represented by a closed polyhedron in the other; and
- each face in both figures belongs to two and only two polyhedra.

According to Maxwell, the simplest figure that fits this definition, and for which, thus, a reciprocal can be found, is the group of tetrahedral cells resulting from five points in space (Fig. A.15(a)). These five points are connected with ten lines, which form ten triangular faces making up five tetrahedra. Each face of this figure is shared by only two tetrahedra (Fig. A.15(b)). Note that each of the four inner tetrahedra shares a face with the outer tetrahedron.

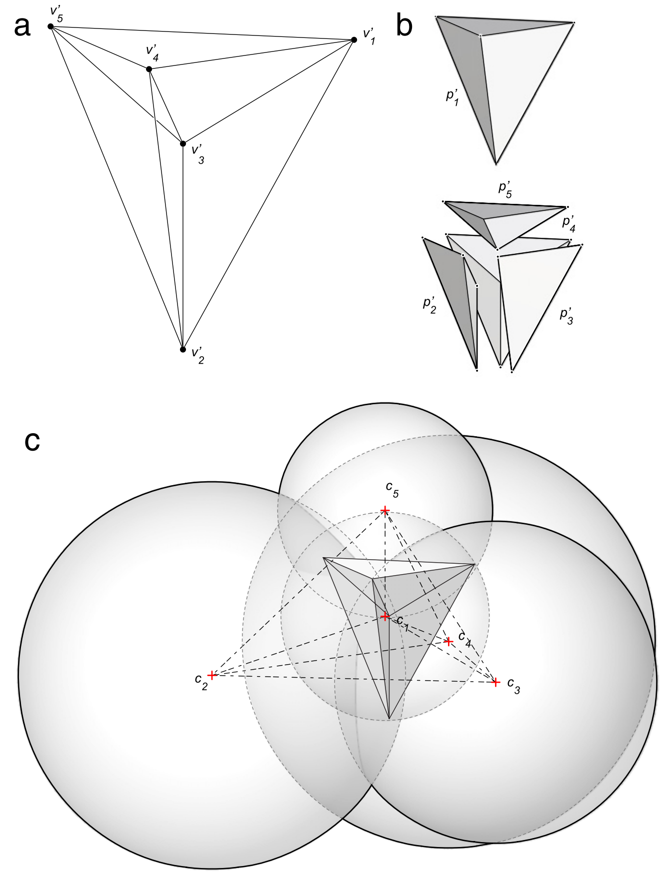


Fig. A.15. (a) Five points in space connected by ten lines; (b) ten triangular faces and five tetrahedra, one external cell and four internal tetrahedral cells; and (c) connecting the centers of the five circumscribing spheres results in the reciprocal figure visualized with dashed lines.

The reciprocal of this figure can be found through strictly geometrical operations. Indeed, connecting the centers of the circumscribing spheres of each tetrahedron results in a figure in which the edges are perpendicular to the faces of the original figure (Fig. A.15(c)). On the difficulty of constructing reciprocal polyhedra, Maxwell says the following:

> "It is manifest that the mechanical problem may be solved, though the reciprocal figure cannot be constructed owing to the condition of all the sides of a face lying in a plane not being fulfilled, or owing to a face belonging to more than two cells. Hence, the mechanical interest of reciprocal figures in space rapidly diminishes with their complexity."

Maxwell, furthermore, points out that these reciprocal figures are the same as the reciprocal figures of Rankine's. He states that, indeed, if we call one the *force* figure and the other the *form* figure, the mechanical interpretation of the relationship between these figures is that the area of a face in the force figure represents the magnitude of force in the line perpendicular to that face in the form figure such that the entire system is in equilibrium. For instance, in Fig. A.16(a), the area of the face that is made up by the three vertices v'_2, v'_4 and v'_5, is proportional to the magnitude of the force in edge e_{21}. The node v_1 is reciprocal to the tetrahedron p'_1 and edge e_{52} is reciprocal and thus perpendicular to the face f'_5 (Fig. A.16(b)–(e)).

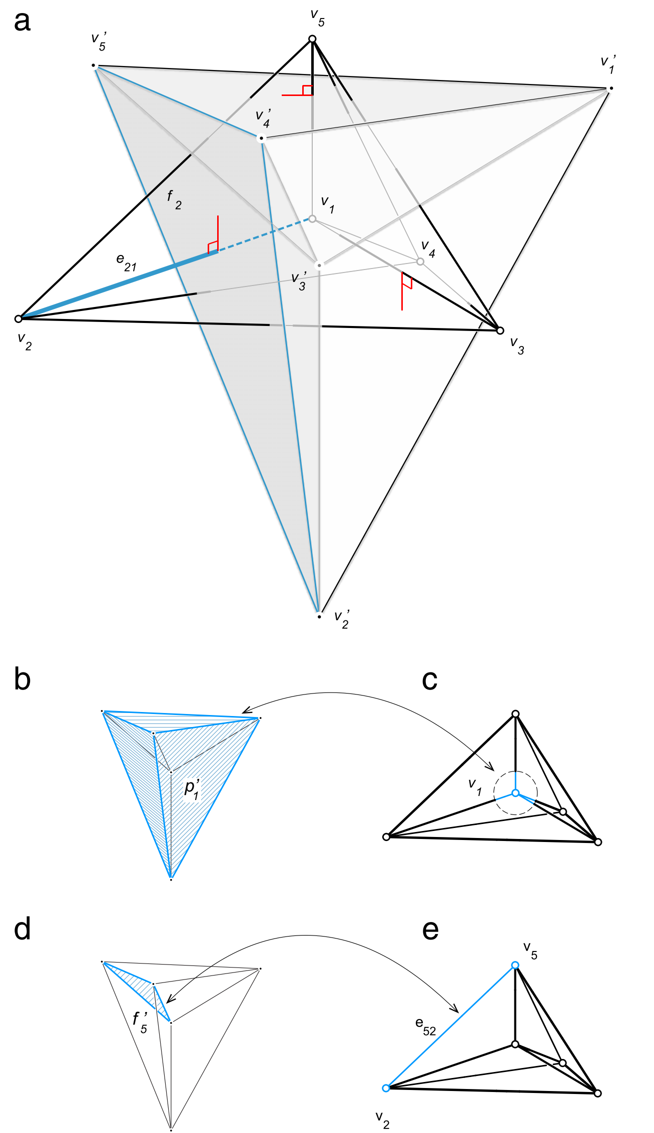


Fig. A.16. (a) Reciprocal figures in 3D: the area of the face between vertices v_2', v_4', and v_5', in the force figure is proportional to the magnitude of the force in edge e_{21} in the form figure; (b) and (c) a node in one figure represents a closed tetrahedron in the other, and vice versa; (d) and (e) each edge in one figure corresponds to a face in the other figure, and vice versa.

References

[1] Varignon P. Nouvelle mécanique ou statique. Paris: Claude Jombert; 1725.
[2] Culmann K. Die graphische statik. Zürich: Verlag Meyer und Zeller; 1864.
[3] Maxwell J. On reciprocal figures, frames and diagrams of forces. Phil Mag 1864; 27:250–61.
[4] Cremona L. Graphical statics: two treatises on the graphical calculus and reciprocal figures in graphical statics. Oxford: Clarendon Press; 1890.
[5] Van Mele T, Rippmann M, Lachauer L, Block P. Geometry-based understanding of structures. J Int Assoc Shell Spat Struct 2012;53:285–95.
[6] Beghini L, Carrion J, Beghini A, Mazurek A, Baker W. Structural optimization using graphic statics. Struct Multidiscip Optim 2013;49:351–66.
[7] Fivet C, Zastavni D. Constraint-based graphic statics: new paradigms of computer-aided structural equilibrium design. J Int Assoc Shell Spat Struct 2013;54:271–80.
[8] Föppl A. Das fachwerk im raume. Leipzig: Verlag von B.G. Teubner; 1892.
[9] Pirard A. La statique graphique. Liége, Paris: Vaillant-Carmanne / Dunod; 1967.
[10] Rankine M. A manual of applied mechanics. London: R. Griffin; 1858.
[11] Möbius A. Über eine besondere art dualer verhältnisse zwischen figuren im raume. Crelle's J 1833;X:317–41. Berline.
[12] Wolfe W. Graphical analysis: a text book on graphic statics. New York: McGraw-Hill Book Company, Inc.; 1921.
[13] Williams CJK. Defining and designing curved flexible tensile surface structures. In: Gregory JA, editor. The mathematics of surfaces. Oxford: Clarendon Press; 1986. p. 143–77.
[14] Block P, Ochsendorf J. Thrust network analysis: a new methodology for three-dimensional equilibrium. J Int Assoc Shell Spat Struct 2007;48:167–73.
[15] Vouga E, Höbinger M, Wallner J, Pottmann H. Design of self-supporting surfaces. ACM Trans Graph 2012;31:87:1–87:11. \relax$\@@underline\hbox{http://dx.doi.org/10.1145/2185520.2185583}\mathsurround\z@$\relax. URL: http://doi.acm.org/10.1145/2185520.2185583.
[16] Liu Y, Pan H, Snyder J, Wang W, Guo B. Computing self-supporting surfaces by regular triangulation. ACM Trans Graph 2013;32:92:1–92:10. \relax$\@@underline\hbox{http://dx.doi.org/10.1145/2461912.2461927}\mathsurround\z@$\relax. URL: http://doi.acm.org/10.1145/2461912.2461927.
[17] de Goes F, Alliez P, Owhadi H, Desbrun M. On the equilibrium of simplicial masonry structures. ACM Trans Graph 2013;32:93:1–93:10. \relax$\@@underline\hbox{http://dx.doi.org/10.1145/2461912.2461932}\mathsurround\z@$\relax. URL: http://doi.acm.org/10.1145/2461912.2461932.
[18] Panozzo D, Block P, Sorkine-Hornung O. Designing unreinforced masonry models. ACM Trans Graph—SIGGRAPH 2013 2013;32:91:1–91:12. \relax$\@@underline\hbox{http://dx.doi.org/10.1145/2461912.2461958}\mathsurround\z@$\relax.
[19] Schrems M, Kotnik T. Statically motivated formfinding based on extended graphical statics. In: Stouffs R, Janssen P, Roudavski S, Tuncer B, editors. Open systems: proceedings of the 18th international conference on computer-aided architectural design research in Asia. (CAADRIA 2013), Singapore: Department of Architecture, NUS; 2013. p. 843.
[20] Rankine M. Principle of the equilibrium of polyhedral frames. Phil Mag 1864; 27:92.
[21] Akbarzadeh M, Van Mele T, Block P. Equilibrium of spatial structures using 3-d reciprocal diagrams. In: Obrebski J, Tarczewski R, editors. Proceedings of IASS symposium 2013, beyond the limits of man. Poland: Wroclaw University of Technology; 2013.
[22] Munkrez J. Elements of algebraic topology. Boston: Addison-Wesley; 1993.
[23] de Goes F, Memari P, Mullen P, Desbrun M. Weighted triangulations for geometry processing. ACM Trans Graph 2014;33:28:1–28:13. \relax$\@@underline\hbox{http://dx.doi.org/10.1145/2602143}\mathsurround\z@$\relax. URL: http://doi.acm.org/10.1145/2602143.
[24] Mullen P, Memari P, de Goes F, Desbrun M. Hot: hodge-optimized triangulations. In: ACM SIGGRAPH 2011 papers. New York, NY, USA: ACM; 2011. p. 103:1–103:12. http://dx.doi.org/10.1145/1964921.1964998. URL: http://doi.acm.org/10.1145/1964921.1964998.
[25] Gauss CF. Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum, methodo nova tractata. In: Werke. 1813. p. 279–86.
[26] Stokes GG. Mathematical and physical papers. Cambridge: Cambridge Univ. Press; 1905.
[27] Ash P, Bolker E, Crapo H, Whiteley W. Convex polyhedra, Dirichlet tessellations, and spider webs. In: Senechal M, Fleck G, editors. Shaping space: a polyhedral approach. Boston: Birkhäuser; 1988. (Chapter 17).
[28] Baumgart B. A polyhedron representation for computer vision, in: AFIPS (Ed.), Proceedings national computer conference, 1975, p. 589–96.
[29] Rippmann M, Block P. Rethinking structural masonry: unreinforced, stone-cut shells. Proc ICE—Constr Mater 2013;166:378–89. \relax$\@@underline\hbox{http://dx.doi.org/10.1680/coma.12.00033}\mathsurround\z@$\relax.
[30] Rippmann M, Lachauer L, Block P. Interactive vault design. Int J Space Struct 2012;27:219–30. http://dx.doi.org/10.1260/0266-3511.27.4.219.