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On the equivalence of a countable disjoint class of sets of positive measure and a weaker condition than total σ-finiteness of measures

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Let (X,S) be a measurable space and S be a σ-algebra of subsets of X. A nonempty class M is said to be a class of null sets if $M \subset S$, M is closed under countable unions of sets and $E \cap F \in M$ whenever $E \in M$ and $F \in S$. It is possible to show that such concepts as absolute continuity, singularity and independence of measures can be studied simply by classes of null sets and that similar results can be obtained under the condition that each disjoint subclass of $S - M$ is countable, denoted $(S - M)C$. If (X,S,μ) is a measure space then $M = \{E \in S : \mu(E) = 0\}$ is a class of null sets of S and $S - M$ the class of all sets of positive measure. We say that a measure μ has the property σ if there exists a sequence of totally finite measures $\{\mu_n\}_{n=1}^{\infty}$ on S such that $\mu(E) = \sum_{n=1}^{\infty} \mu_n(E)$ for all $E \in S$. This property of measures is weaker than total σ-finiteness of measures. The main result of the present paper is as follows: Let (X,S,μ) be a measure space and $M = \{E \in S : \mu(E) = 0\}$. Then $(S - M)C$ if and only if μ has the property σ.

1.

Throughout this paper S is a σ-algebra of sets and (X,S,μ) is a measure space. We shall say a measure μ has the property σ, if there

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exists a sequence of totally finite measures $\{\mu_n\}_{n=1}^{\infty}$ defined on S such that $\mu(E) = \sum_{n=1}^{\infty} \mu_n(E)$ for all $E \in S$. We note that each totally σ-finite measure μ has the property σ. The purpose of this paper is to prove the following Theorem.

THEOREM. *Each class of disjoint sets of positive measure is countable if and only if* μ *has the property* σ.

Next we give an example of a measure μ which is not σ-finite but has the property σ.

EXAMPLE. Let X be a nonempty set. Suppose $S = \{\phi, X\}$, and $\mu(X) = +\infty$, $\mu(\phi) = 0$. Put $\mu_n(X) = 1$, $\mu_n(\phi) = 0$ for $n = 1,2,3,\ldots$, then $\mu(E) = \sum_{n=1}^{\infty} \mu_n(E)$ for all $E \in S$.

2.

Let (X,S) be a measurable space. A nonempty class N of sets, where $N \subset S$ is called a class of null sets of S

(i) if $E \in N$ and $F \in S$, then $E \cap F \in N$, and

(ii) if $E_n \in N$, $n = 1,2,3,\ldots$, then $\bigcup_{n=1}^{\infty} E_n \in N$.

We note, if (X,S,μ) is a measure space, that the nonempty class M of all those measurable sets E for which $\mu(E) = 0$ is a class of null sets of S and $S - M$ is the class of all measurable sets E of positive measure. The notation $(S - M)C$ indicates that each subclass of disjoint sets of $S - M$ is countable. The symmetric difference of two sets E and F is denoted by $E \Delta F$ and is defined by

$$E \Delta F = (E - F) \cup (F - E) = (E \cap F^c) \cup (F \cap E^c) .$$

The following Theorem is due to T. Neubrunn [4].

THEOREM 1. *Let* (X,S) *be a measurable space and* N *be a class of null sets of* S. *Suppose* $(S - N)C$ *and let* P *be a property of measurable sets* E, *and be preserved under the formation of countable unions of disjoint sets. If at least one set* E *in* $S - N$ *has the*

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property P *, then there exists a maximal set* M *in* $S - N$ *with the property* P *such that the conditions* $E \in S$ *,* E *has the property* P *,* $E \subset M^c$ *imply* $E \in N$ *.*

Let (X,S,μ) be a measure space. A set E in S is called an atom if $\mu(E) > 0$, and if $F \in S$ such that $F \subset E$, then either we have $\mu(E-F) = 0$ or $\mu(F) = 0$.

LEMMA 1. *Let* (X,S,μ) *be a measure space. If* E_1 *and* E_2 *are atoms, then either* $\mu(E_1 \Delta E_2) = 0$ *or* $\mu(E_1 \cap E_2) = 0$ *.* (cf. [*1*, p. 308]).

All other concepts are used as in [3].

3.

LEMMA 2. *Let* (X,S,μ) *be a measure space. If* μ *has the property* σ *then* $(S - M)C$ *.*

Proof. Since μ has the property σ , there exists a sequence of totally finite measures $\{\mu_n\}_{n=1}^{\infty}$ such that $\mu(E) = \sum_{n=1}^{\infty} \mu_n(E)$, whenever $E \in S$. Then there exists a probability measure μ' equivalent to μ . It is sufficient to put $\mu'(E) = \sum_{n=1}^{\infty} 2^{-n} \mu_n(E) [\mu_n(X)]^{-1}$, whenever $E \in S$. It is well known that $(S - M')C$. Since $\mu \equiv \mu'$ i.e., $M = M'$ we have $(S - M)C$.

COROLLARY. *If* (X,S,μ) *is a measure space and* μ *has the property* σ *then the class* A *of all atoms in* S *is countable.*

Proof. According to Lemma 1 for any two sets $A,B \in A$ we have either $A \Delta B \in M$ and the set $A \cap B$ represents an atom or $A \cap B \in M$ and then the sets $A - B$ and $B - A$ represent two disjoint atoms i.e., two disjoint sets in $S - M$. Each class of disjoint sets in $S - M$ is countable according to Lemma 2.

LEMMA 3. *Let* (X,S,μ) *be a measure space. If there are no atoms in* S *and* μ *is not* σ*-finite, then there exists in* $S - M$ *an uncountable class of disjoint sets of finite measure.*

Proof. There exists at least one set E of positive and finite measure such that $E \subset X$ and $E \neq X$, since X is not an atom. Now,

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suppose $(S - M)C$ and let P be the following property of measurable sets G ..

$P : G$ is a countable union of disjoint sets of finite measure from $S - M$.

There exists at least one measurable set with the above property, E itself. Evidently the property P is preserved under the formation of unions of countable disjoint sets. According to Theorem 1 there exists a maximal measurable set M with property P such that the conditions $E \in S$, $E \subset M^c$, imply $E \in M$. Otherwise M^c would be an atom. Since M is a countable union of disjoint sets of finite measure from $S - M$ and $X = M \cup M^c$ then μ is a σ-finite measure. This is a contradiction.

LEMMA 4. *Let (X,S,μ) be a measure space. If there are no atoms in S and μ is not σ-finite, then μ has not the property σ .*

Proof. According to Lemma 3 there exists an uncountable class of disjoint sets of positive and finite measure μ , say $A = \{A_t, t \in T\}$. On the contrary let us assume that μ has the property σ , then $\mu(A_t) = \sum_{n=1}^{\infty} \mu_n(A_t)$ for $t \in T$. Then, there exists a positive integer n_0 such that $\mu_{n_0}(A_t) > 0$ for an uncountable set T' of indices, where $t \in T' \subset T$. This contradicts the total finiteness of the measure μ_{n_0} . Thus μ has not the property σ .

REMARK 1. We note, if μ has the property σ , then also μ_E has the property σ , where $\mu_E(F) = \mu(E \cap F)$, whenever $F \in S$.

LEMMA 5. *Let (X,S,μ) be a measure space and let $(S - M)C$. Then μ has the property σ .*

Proof. (i) Suppose, there are no atoms in S . Then we prove that μ is a σ-finite measure. On the contrary if μ is not σ-finite, then according to Lemma 3 the class of all disjoint sets of positive measure μ is uncountable. This is a contradiction with $(S - M)C$. Therefore μ is a σ-finite measure, hence μ has the property σ .

(ii) Let us admit that there are also atoms but of finite μ-measure only. Then the class $\mathcal{A}$ of all disjoint atoms is countable, i.e. $\mathcal{A} = \{A_1, A_2, A_3, \ldots\}$. Put $A = \bigcup_{i=1}^{\infty} A_i$, $A_i \in \mathcal{A}$, $i = 1,2,3,\ldots$, and put $\lambda(F) = \mu(F \cap A)$ and $\nu(F) = \mu(F \cap A^c)$, for all $F \in S$. Then λ and ν are two σ-finite measures. Evidently $\mu = \lambda + \nu$ on S and μ is a σ-finite measure, hence μ has the property σ.

(iii) Suppose now that there exist atoms B such that $\mu(B) = +\infty$. The class of all such disjoint atoms is countable since $\mathcal{A} \subset S - M$. Put $A = \bigcup_{i=1}^{\infty} A_i$, $A_i \in \mathcal{A}$, $i = 1,2,3,\ldots$ Let us define

$$\mu_i(E) = \begin{cases} 1, & \text{if } \mu(E \cap A_i) = \mu(A_i), \\ 0, & \text{if } \mu(E \cap A_i) \neq \mu(A_i), \end{cases}$$

whenever $E \in S$. We prove μ_i is a measure for each positive integer i. Fix i, then evidently $\mu_i \geqq 0$ and $\mu_i(\phi) = 0$. Let $\left\{E_k\right\}_{k=1}^{\infty}$ be a sequence of disjoint sets from S. Put $E = \bigcup_{k=1}^{\infty} E_k$. If $\mu(E \cap A_i) = \mu(A_i)$ then $\mu_i(E) = 1$. Then from $\mu(E \cap A_i) = \sum_{k=1}^{\infty} \mu(E_k \cap A_i)$ follows that $\sum_{k=1}^{\infty} \mu(E_k \cap A_i) = \mu(A_i)$. We prove that there is exactly one positive integer k_0 such that $\mu(E_{k_0} \cap A_i) = \mu(A_i)$. On the contrary suppose there are at least two such integers, say k_1 and k_2 then $\mu(E_{k_1} \cap A_i) > 0$ and $\mu(E_{k_2} \cap A_i) > 0$, since $E_{k_1} \cap E_{k_2} = \phi$ for $k_1 \neq k_2$ and $E_{k_1} \cap A_i \subset A_i$, $E_{k_2} \cap A_i \subset A_i$. The latter two relations contradict the assumption that A_i is an atom. Therefore $\mu_i(E_{k_0}) = 1$, and $\mu_i(E_k) = 0$ for all $k \neq k_0$. Then we have $\mu_i(E) = \sum_{k=1}^{\infty} \mu_i(E_k)$. Further $\mu_i(E) = 0$, if $\mu(E \cap A_i) \neq \mu(A_i)$, then $\mu(E \cap A_i) = 0$, since $\mu(A_i) = +\infty$ and A_i is an atom. This implies $\mu(E_k \cap A_i) = 0$ for $k = 1,2,3,\ldots$, hence $\mu(E_k \cap A_i) \neq \mu(A_i)$ and we have $\mu_i(E_k) = 0$ for $k = 1,2,3,\ldots$, this implies $\mu_i(E) = \sum_{k=1}^{\infty} \mu_i(E_k)$.

Therefore μ_i is a σ-additive set function, hence μ_i is a measure.

Let n be a positive integer, put $\nu_n = \sum_{i=1}^{n} \mu_i$. Evidently ν_n is a totally finite measure on S . Now we prove $\mu(E \cap A) = \sum_{n=1}^{\infty} \nu_n(E)$, whenever $E \in S$. Evidently $\mu(E \cap A) = \sum_{i=1}^{\infty} \mu(E \cap A_i)$. If $E \in S$ is such a set that there exists at least one index i_0 that $\mu(E \cap A_{i_0}) = \mu(A_{i_0})$, then $\mu(E \cap A) = +\infty$ and then for all $n \geqq i_0$ we have $\nu_n(E) \geqq 1$, therefore $\mu(E \cap A) = \sum_{n=1}^{\infty} \nu_n(E)$. If $\mu(E \cap A_i) \neq \mu(A_i)$ for $i = 1,2,3,\ldots,$ then $\mu(E \cap A_i) = 0$ for all positive integers i and also $\mu_i(E) = 0$. The latter condition implies $\nu_n(E) = 0$, for $n = 1,2,3,\ldots,$ therefore $\mu(E \cap A) = \sum_{n=1}^{\infty} \nu_n(E)$. Hence μ_A has the property σ .

Let $E \in S$, then $\mu(E) = \nu(E) + \lambda(E)$, where $\nu(E) = \mu(E \cap A)$ and $\lambda(E) = \mu(E \cap A^c)$. Then λ is a σ-finite measure, since it is either (i) or (ii), therefore $\lambda(E) = \sum_{n=1}^{\infty} \lambda_n(E)$.

Finally, if $\mu_{2n-1}(E) = \nu_n(E)$ and $\mu_{2n}(E) = \lambda_n(E)$, for $n = 1,2,3,\ldots,$ whenever $E \in S$ then $\mu(E) = \sum_{n=1}^{\infty} \mu_n(E)$. This completes the proof.

Now from Lemmas 2 and 5 immediately follows,

THEOREM 2. *Let* (X,S,μ) *be a measure space. Then* $(S - M)C$ *if and only if* μ *has the property* σ .

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