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Abstract

We establish the equivalence of five classes of hybrid dynamical systems: mixed logical dynamical systems, linear complementarity systems, extended linear complementarity systems, piecewise affine systems and max-min-plus-scaling systems.

Keywords: hybrid systems, complementarity systems, piecewise affine systems, equivalent models

1 Introduction

Hybrid dynamical systems are systems that contain both analog (continuous) and logical (discrete) components. Recently, these systems have received a lot of attention from both the computer science and the control community (Antsaklis et al., 1999; Grossman et al., 1993; Henzinger and Sastry, 1998; Maler, 1997; Alur et al., 1996; Vaandrager and van Schuppen, 1999; Pnueli and Sifakis, 1995; Antsaklis and Nerode, 1998; Morse et al., 1999). As tractable methods to analyze general hybrid systems are not available, several authors have focused on special subclasses of hybrid dynamical systems for which efficient analysis and/or control design techniques are currently being developed. Some examples of such tractable subclasses are: linear complementarity systems (Heemels et al., 1999; van der Schaft and Schumacher, 1996; van der Schaft and Schumacher, 1998; van der Schaft and Schumacher, 2000; Heemels et al., 2000), mixed logical dynamical systems (Bemporad and Morari, 1999; Bemporad et al., 1999), first order linear hybrid systems with saturation (De Schutter, 1999), piecewise affine systems (Sontag, 1981; Sontag, 1996), and so on. In this paper we will show that several of these subclasses are in fact conceptually equivalent. These results enable the transfer of knowledge from one class to another, they show that more applications belong to these classes and moreover, they imply that for the study of a particular hybrid system one can choose the modeling framework that is most suitable.

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2 Classes of hybrid systems

2.1 Mixed logical dynamical systems

In (Bemporad and Morari, 1999) Bemporad and Morari have introduced a class of hybrid systems in which logic, dynamics and constraints are integrated. This lead to a description of the form

$$x(k+1) = Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(k) \tag{1a}$$

$$y(k) = Cx(k) + D_1u(k) + D_2\delta(k) + D_3z(k) \tag{1b}$$

$$E_1x(k) + E_2u(k) + E_3\delta(k) + E_4z(k) \leq e_5, \tag{1c}$$

where $x(k) = [\, x_{\mathrm{r}}{}^T(k)\; x_{\mathrm{b}}{}^T(k)\,]^T \in \mathbb{R}^{n_{\mathrm{r}}} \times \{0,1\}^{n_{\mathrm{b}}}$ is the state of the system at time k, $y(k) = [\, y_{\mathrm{r}}{}^T(k)\; y_{\mathrm{b}}{}^T(k)\,]^T \in \mathbb{R}^{l_{\mathrm{r}}} \times \{0,1\}^{l_{\mathrm{b}}}$ is the output, $u(k) = [\, u_{\mathrm{r}}{}^T(k)\; u_{\mathrm{b}}{}^T(k)\,]^T \in \mathbb{R}^{m_{\mathrm{r}}} \times \{0,1\}^{m_{\mathrm{b}}}$ is the input, and $z(k) \in \mathbb{R}^{r_{\mathrm{r}}}$ and $\delta(k) \in \{0,1\}^{r_{\mathrm{b}}}$ are auxiliary variables. The inequalities (1c) have to be interpreted componentwise. Systems that can be described by the model (1) are called *mixed logical dynamical* (MLD) systems. The time-evolution of the system is determined by solving $\delta(k)$ and $z(k)$ from the linear inequalities (1c) once $x(k)$ and $u(k)$ are specified. Subsequently, this can be used to obtain the new state $x(k+1)$ and the current output $y(k)$. A new input $u(k+1)$ can now be specified after which the cycle is repeated.

Remark 2.1 In the formulation of the time-evolution of the system, it is assumed that for all $x(k)$ with $x_{\mathrm{r}}(k) \in \mathbb{R}^{n_{\mathrm{r}}}$ and $x_{\mathrm{b}}(k) \in \{0,1\}^{n_{\mathrm{b}}}$, all $u(k)$ with $u_{\mathrm{r}}(k) \in \mathbb{R}^{m_{\mathrm{r}}}$ and $u_{\mathrm{b}}(k) \in \{0,1\}^{m_{\mathrm{b}}}$, all $z(k) \in \mathbb{R}^{r_{\mathrm{r}}}$ and all $\delta(k) \in \{0,1\}^{r_{\mathrm{b}}}$ satisfying (1c) it holds that $x(k+1)$ and $y(k)$ determined from (1a)-(1b) are such that $x_{\mathrm{b}}(k+1) \in \{0,1\}^{n_{\mathrm{b}}}$ and $y_{\mathrm{b}}(k) \in \{0,1\}^{l_{\mathrm{b}}}$. This assumption is without loss of generality, since otherwise it is possible to include additional inequalities in (1c) to guarantee this property. Indeed, if $y_{\mathrm{b}}(k) \in \{0,1\}^{l_{\mathrm{b}}}$ is not implied by the equations, we introduce an additional binary variable $\delta_y(k) \in \{0,1\}^{l_{\mathrm{b}}}$ and the inequalities

$$[Cx(k) + D_1u(k) + D_2\delta(k) + D_3z(k)]_{\mathrm{b}} - \delta_y(k) \;\leq\; 0 \tag{2a}$$

$$[-Cx(k) - D_1u(k) - D_2\delta(k) - D_3z(k)]_{\mathrm{b}} + \delta_y(k) \;\leq\; 0, \tag{2b}$$

which sets $\delta_y(k)$ equal to $y_{\mathrm{b}}(k)$. The notation $[\,]_{\mathrm{b}}$ is used to select the last l_{b} rows of the expression (1b), i.e. the rows that correspond to the binary part of $y(k)$. In a similar way, we can deal with $x_{\mathrm{b}}(k+1) \in \{0,1\}^{n_{\mathrm{b}}}$. □

Remark 2.2 In (Bemporad and Morari, 1999) the system matrices in the model (1) were allowed to be time-varying. For sake of simplicity of notation we do not explicitly include the time-dependence of the system matrices for the classes of hybrid systems considered in this paper, i.e. we consider time-invariant systems. Note however that all the results presented in this paper also hold for time-varying systems. □

In (Bemporad and Morari, 1999) it has been shown that the class of MLD systems includes piece-wise linear dynamic systems, linear hybrid systems, finite state machines, linear systems, linear systems with discrete inputs, bilinear systems, etc.

2.2 Piecewise affine systems

Piecewise affine (PWA) systems (Sontag, 1981; Sontag, 1996) are described by

$$\begin{aligned} x(k+1) &= A_i x(k) + B_i u(k) + f_i \\ y(k) &= C_i x(k) + D_i u(k) + g_i \end{aligned} \quad \text{for} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \Omega_i, \tag{3}$$

where Ω_i are convex polyhedra (i.e. given by a finite number of inequalities) in the input/state space. The variables $u(k) \in \mathbb{R}^m$, $x(k) \in \mathbb{R}^n$ and $y(k) \in \mathbb{R}^l$ denote the input, state and output, respectively, at time k. In this model description f_i and g_i are constant vectors.

PWA systems have been studied by several authors (see (van Bokhoven, 1981; Johansson, 1999; Johansson and Rantzer, 1998; Leenaerts and van Bokhoven, 1998; Pettit, 1995; Sontag, 1981; Sontag, 1996; Vandenberghe et al., 1989; van Bokhoven and Leenaerts, 1999) and the references therein) as they form the "simplest" extensions of linear systems and can still approximate any non-linear system arbitrarily close.

2.3 Linear complementarity systems

Linear complementarity (LC) systems are studied in e.g. (Heemels et al., 1999; van der Schaft and Schumacher, 1996; van der Schaft and Schumacher, 1998; Heemels et al., 2000; van der Schaft and Schumacher, 2000). In discrete time these systems are given by the equations

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 u(k) + B_2 w(k) && \text{(4a)} \\ y(k) &= Cx(k) + D_1 u(k) + D_2 w(k) && \text{(4b)} \\ v(k) &= E_1 x(k) + E_2 u(k) + E_3 w(k) + e_4 && \text{(4c)} \\ 0 \leq v(k) &\perp w(k) \geq 0 && \text{(4d)} \end{aligned}$$

in which $u(k) \in \mathbb{R}^m$, $x(k) \in \mathbb{R}^n$ and $y(k) \in \mathbb{R}^l$ denote (again) the input, state and output, respectively, at time k, and where $\perp$ denotes the orthogonality of vectors (i.e. $v(k) \perp w(k)$ means that $v^\top(k) w(k) = 0$). We call $v(k) \in \mathbb{R}^s$ and $w(k) \in \mathbb{R}^s$ the complementarity variables. The dynamics evolves as follows. Once $x(k)$ and $u(k)$ have been specified, one has to solve $w(k)$ and $v(k)$ from (4c)-(4d) (which is a standard Linear Complementarity Problem (Cottle et al., 1992)) after which the new state $x(k+1)$ and the current output $y(k)$ can be computed from (4a)-(4b).

2.4 Extended linear complementarity systems

In (De Schutter and De Moor, 1999; De Schutter, 1999; De Schutter and van den Boom, 2000) we have shown that several types of hybrid systems can be modeled as extended linear complementarity (ELC) systems:

$$\begin{aligned} & x(k+1) = Ax(k) + B_1 u(k) + B_2 d(k) && \text{(5a)} \\ & y(k) = Cx(k) + D_1 u(k) + D_2 d(k) && \text{(5b)} \\ & E_1 x(k) + E_2 u(k) + E_3 d(k) \leq e_4 && \text{(5c)} \\ & \sum_{i=1}^{p} \prod_{j \in \phi_i} \left(e_4 - E_1 x(k) - E_2 u(k) - E_3 d(k) \right)_j = 0 && \text{(5d)} \end{aligned}$$

where $x(k) \in \mathbb{R}^n$ is the state at time k, $y(k) \in \mathbb{R}^l$ is the output, $u(k) \in \mathbb{R}^m$ is the input, and $d(k) \in \mathbb{R}^r$ is an auxiliary variable. Note that condition (5d) is equivalent to

$$\prod_{j \in \phi_i} \left(e_4 - E_1 x(k) - E_2 u(k) - E_3 d(k)\right)_j = 0 \quad \text{for each } i \in \{1, 2, \ldots, p\} \tag{6}$$

due to the inequality conditions (5c). This implies that (5c) – (5d) can be considered as a system of linear inequalities (i.e. (5c)) where there are p groups of linear inequalities (one group for each index set ϕ_i) such that in each group at least one inequality should hold with equality. The motion of an ELC system can be determined as follows. Given the current input $u(k)$ and the current state $x(k)$, first the system (5c) – (5d) is solved (this is an Extended Linear Complementarity Problem (De Schutter and De Moor, 1995)). This yields $d(k)$. Next the new state $x(k+1)$ and the output $y(k)$ can be determined from (5a) – (5b).

Remark 2.3 Note the following differences between LC systems and ELC systems. For ELC systems inequalities of the form (1c) can be incorporated directly, whereas in LC systems these inequalities have to be transformed into a (void) complementarity condition by using slack variables (see also the proof of Proposition 3.1). For LC systems products consisting of more than 2 factors (such as e.g. $u_1(k)u_2(k)u_3(k) = 0$) are not allowed (directly) while in ELC systems products of 3 or more factors are possible. □

In (De Schutter and De Moor, 1999; De Schutter, 1999; De Schutter and van den Boom, 2000) we have shown that the class of ELC systems encompasses max-plus-linear systems (Baccelli et al., 1992), first order linear hybrid systems subject to saturation (De Schutter, 1999), and *unconstrained* max-min-plus-scaling systems (which will be introduced in the next section).

2.5 Max-min-plus-scaling systems

In (De Schutter and van den Boom, 2000) we have introduced a class of discrete event systems that can be modeled using the operations maximization, minimization, addition and scalar multiplication. Expressions that are built using these operations are called max-min-plus-scaling (MMPS) expressions.

Definition 2.4 (Max-min-plus-scaling expression) *A max-min-plus-scaling expression f of the variables $x_1, x_2, \ldots, x_n$ is defined by the grammar*[1]

$$f := x_i | \alpha | \max(f_k, f_l) | \min(f_k, f_l) | f_k + f_l | \beta f_k \tag{7}$$

with $i \in \{1, 2, \ldots, n\}$, $\alpha \in \mathbb{R}$, and where f_k and f_l are again MMPS expressions.

Some examples of MMPS expressions are $x_1 + 2x_2 - 3$, $\max(\min(x_1, -3x_2), x_1 + 8x_3)$, and $x_1 - 5\max(x_1 - 4x_2 + 7x_3, x_1 - \min(x_1 - x_2 - x_3, \max(x_1 - 6x_2, x_1 + x_2 - x_3)))$. Note that the min operation is in fact not explicitly needed in (7) since we have $\min(f_k, f_l) = -\max(-f_k, -f_l)$.

Now we consider systems that can be described by state space equations of the following form:

$$x(k+1) = \mathcal{M}_x(x(k), u(k), d(k)) \tag{8a}$$

[1]The symbol | stands for OR and the definition is recursive; so an MMPS expression is a single variable, a constant, the maximum, minimum or sum of two MMPS expressions, or a scalar multiple of an MMPS expression.

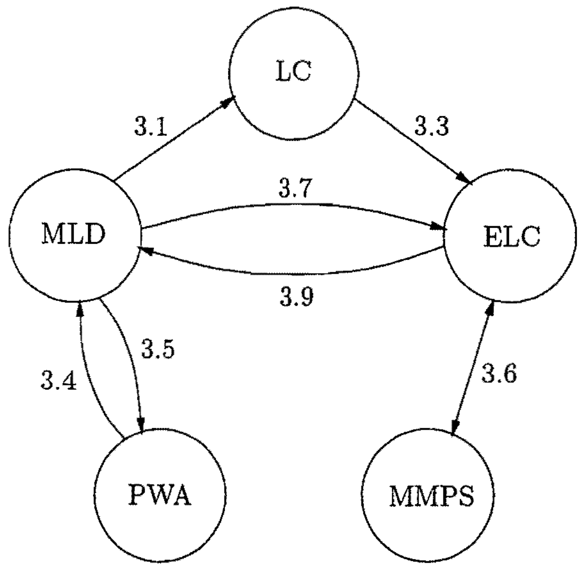

Figure 1: Graphical representation of the links between the classes of hybrid systems considered in this paper. An arrow going from class A to class B means that A is a subset of B. The number next to each arrow corresponds to the proposition that states this relation and that specifies the conditions, if any, under which the relation holds.

$$y(k) = \mathcal{M}_y(x(k), u(k), d(k)) \tag{8b}$$

together with the constraint

$$\mathcal{M}_c(x(k), u(k), d(k)) \leq c, \tag{8c}$$

where $\mathcal{M}_x$, $\mathcal{M}_y$ and $\mathcal{M}_c$ are MMPS expressions in terms of the state $x(k)$ at time k, the input $u(k)$ and the auxiliary variables $d(k)$, which are all real-valued. Systems that can be described by models of the form (8) will be called constrained MMPS systems. If the inequalities (8c) are absent, we speak of unconstrained MMPS systems. The dynamics of an MMPS can be determined by solving (8c) for $d(k)$ when the current state $x(k)$ and input $u(k)$ are given. Next, $x(k+1)$ and $y(k)$ can be computed from (8a) and (8b).

The model (8a) – (8b) is a generalized framework that encompasses several special subclasses of hybrid and discrete-event systems such as max-plus-linear discrete event systems (Baccelli et al., 1992), max-min-plus systems (Gunawardena, 1994; Olsder, 1994), and max-plus-polynomial systems (De Schutter and van den Boom, 2000).

3 The equivalence of MLD, LC, ELC, PWA and MMPS systems

In this section we show that MLD, LC, ELC, PWA and MMPS systems are equivalent (although in some cases this requires additional assumptions). Figure 1 shows how these equivalences are proved in this paper.

Proposition 3.1 *Every MLD system can be written as an LC system.*

Proof: To rewrite (1) as an LC system we have to realize that $\delta(k)$ and $z(k)$ follow from the inequality constraints (1c) once $x(k)$ and $u(k)$ have been specified. To rephrase the condition $\delta(k) \in \{0,1\}^{r_b}$ in complementarity terms, we note that $\delta_i(k) \in \{0,1\}$ is equivalent to $0 \leq \delta_i(k) \perp 1 - \delta_i(k) \geq 0$. By introducing the auxiliary vector $v_1(k)$ this gives in vector notation $v_1(k) = e - \delta(k)$ together with $0 \leq \delta(k) \perp v_1(k) \geq 0$, where e denotes the vector for which all entries are equal to one. Next the inequality constraints in (1c) are modeled by introducing the auxiliary vectors $w_2(k)$ and $v_2(k)$. Define $v_2(k) = e_5 - E_1x(k) - E_2u(k) - E_3\delta(k) - E_4z(k)$. It is clear that $v_2(k) \geq 0$ implies the existence of an $w_2(k)$ (take $w_2(k) = 0$) such that

$$0 \leq v_2(k) \perp w_2(k) \geq 0. \tag{9}$$

Vice versa, if (9) is satisfied, it is obvious that $v_2(k) \geq 0$. Since $w_2(k)$ does not influence any other relation, it follows that $v_2(k) \geq 0$ can be replaced by (9) (see also Remark 2.3). Combining all the relations obtained so far yields the system description

$$\begin{aligned} x(k+1) &= Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(k) && \text{(10a)}\\ y(k) &= Cx(k) + D_1u(k) + D_2\delta(k) + D_3z(k) && \text{(10b)}\\ \underbrace{\begin{pmatrix} v_1(k) \\ v_2(k) \end{pmatrix}}_{=:v(k)} &= \begin{pmatrix} e \\ e_5 - E_1x(k) - E_2u(k) - E_4z(k) \end{pmatrix} + \begin{pmatrix} -I & 0 \\ -E_3 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} \delta(k) \\ w_2(k) \end{pmatrix}}_{=:w(k)} && \text{(10c)}\\ 0 &\leq v(k) \perp w(k) \geq 0, && \text{(10d)} \end{aligned}$$

where I denotes the identity matrix. Hence, in case additional auxiliary variables (like $z(k)$ in (1a)-(1b) and $d(k)$ in (5a)-(5b)) are allowed in the right-hand sides of (4a)-(4c), the proof would be complete. However, the description of LC systems is more special in the sense that only complementarity variables $w(k)$ are allowed in the right-hand sides of (4a)-(4b). Hence, $w(k)$ in (4a)-(4b) must include $z(k)$ in some way. This can be achieved by splitting $z(k)$ in its "positive" and "negative part" as

$$z(k) := z^+(k) - z^-(k) \tag{11}$$

with

$$z^+(k) = \max(0, z(k)) \text{ and } z^-(k) = \max(0, -z(k)). \tag{12}$$

In complementarity terms this leads to

$$z(k) = z^+(k) - z^-(k) \text{ with } 0 \leq z^+(k) \perp z^-(k) \geq 0. \tag{13}$$

By collecting all equations, replacing $z(k)$ by $z^+(k) - z^-(k)$, setting $v_3(k) = z^-(k)$ and $v_4(k) = z^+(k)$ we obtain the LC system

$$\begin{aligned} x(k+1) &= Ax(k) + B_1u(k) + [B_2 \ 0 \ B_3 \ -B_3]w(k) && \text{(14a)}\\ y(k) &= Cx(k) + D_1u(k) + [D_2 \ 0 \ D_3 \ -D_3]w(k) && \text{(14b)}\\ \underbrace{\begin{pmatrix} v_1(k) \\ v_2(k) \\ v_3(k) \\ v_4(k) \end{pmatrix}}_{=:v(k)} &= \begin{pmatrix} 0 \\ -E_1 \\ 0 \\ 0 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ -E_2 \\ 0 \\ 0 \end{pmatrix} u(k) + \begin{pmatrix} -I & 0 & 0 & 0 \\ -E_3 & 0 & -E_4 & E_4 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{pmatrix} \underbrace{\begin{pmatrix} \delta(k) \\ w_2(k) \\ z^+(k) \\ z^-(k) \end{pmatrix}}_{=:w(k)} + \begin{pmatrix} e \\ e_5 \\ 0 \\ 0 \end{pmatrix} && \text{(14c)} \end{aligned}$$

$$0 \ \leq \ v(k) \perp w(k) \geq 0. \tag{14d}$$

$\square$

Remark 3.2 We would like to emphasize that if additional real auxiliary variables would be allowed in the right-hand sides of (4a)-(4c) (like $z(k)$ in (1a)-(1b) and $d(k)$ in (5a)-(5b)) a description like (10) is easily obtained. However, in the LC formulation this is not possible.

Determining the evolution of an LC system requires solving a Linear Complementarity Problem (LCP) (Cottle et al., 1992) of the form $0 \leq v(k) = q(k) + Mw(k) \perp w(k) \geq 0$ each time step. The matrix M is fixed here, while the vector $q(k)$ is time-varying and determined by $x(k)$ and $u(k)$ only. Many efficient techniques are available for solving LCPs (Cottle et al., 1992).

The splitting of the variable $z(k)$ in its positive and negative part and the translation of the inequality constraints in the MLD model as complementarity conditions may seem artificial and may not result in the most efficient models. In many cases such tricks can be avoided (see e.g. the example in Section 4), but they are convenient to show that "MLD $\subseteq$ LC" holds in general. $\square$

Proposition 3.3 *Every LC system can be written as an ELC system.*

Proof: It can easily be verified that (4) can be rewritten as

$$x(k+1) = Ax(k) + B_1u(k) + B_2 \underbrace{w(k)}_{=d(k)} \tag{15a}$$

$$y(k) = Cx(k) + D_1u(k) + D_2w(k) \tag{15b}$$

$$-E_1x(k) - E_2u(k) - E_3w(k) \leq e_4 \tag{15c}$$

$$-w(k) \leq 0 \tag{15d}$$

$$(e_4 + E_1x(k) + E_2u(k) + E_3w(k))_j(w(k))_j = 0. \tag{15e}$$

Hence, the sets ϕ_i contain typically two elements (see also Remark 2.3). To be specific, $\phi_i = \{i, i+s\}$ for $i = 1, 2, \ldots, s$, where s is the dimension of $w(k)$. $\square$

A PWA system of the form (3) is called (completely) well-posed, if (3) is uniquely solvable in $x(k+1)$ once $x(k)$ and $u(k)$ are specified. The following result proven in (Bemporad and Morari, 1999) can now be stated.

Proposition 3.4 *Every well-posed PWA system can be rewritten as an MLD system assuming that the set of feasible states and inputs is bounded*[2].

The reverse statement has been established in (Bemporad et al., 1999) under the condition that the MLD system is completely well-posed, which means that $x(k+1)$, $y(k)$, $\delta(k)$ and $z(k)$ are uniquely specified by (1), when $x(k)$ and $u(k)$ are given (Bemporad and Morari, 1999, Def. 1).

[2]In rewriting a PWA system as an MLD model the authors of (Bemporad and Morari, 1999) replace strict inequalities like $x(k) < 0$ by $x(k) \leq -\varepsilon$ for some $\varepsilon > 0$ (typically the machine precision) and assume that $-\varepsilon < x(k) < 0$ cannot occur due to the finite number of bits used for representing real numbers. See (Bemporad and Morari, 1999) for more details and Section 4 for an example. Maybe one should be more careful in stating the inclusion in this proposition and replace "rewritten" by "approximately rewritten."

Proposition 3.5 *A completely well-posed MLD system can be rewritten as a PWA system.*

Proposition 3.6 *The classes of (constrained) MMPS and ELC systems coincide.*

Proof: Consider a constrained MMPS system of the form (8). Let us now show that this system can be recast as an ELC system. This will be done by showing that each of the 6 basic constructions for MMPS expressions fit in the ELC framework:

- Expressions of the form $f = x_i$, $f = \alpha$, $f = f_k + f_l$ and $f = \beta f_k$ (or their combinations) result in linear equations of the form (5a) - (5b) or in inequalities of the form (5c)[3].
- An expression of the form $f = \max(f_k, f_l) = -\min(-f_k, -f_l)$ can be rewritten as

  $$\begin{aligned} &f - f_k \geq 0 \\ &f - f_l \geq 0 \\ &(f - f_k)(f - f_l) = 0 \ , \end{aligned}$$

  which is an expression of the form (5c) - (5d).

This implies that by introducing additional dummy variables if necessary, any MMPS expression can be recast as an ELC expression. Furthermore, if is easy to verify that two or more ELC systems can be combined into one large ELC system. As a consequence, every MMPS system can be rewritten as an ELC system.

Now we consider an ELC system of the form (5) and we show that it can be written in the form (8). Clearly, (5a) and (5b) are MMPS expressions (albeit without max or min) of the form (8a) and (8b), respectively. Furthermore, since by (5c) we have

$$\big(e_4 - E_1x(k) - E_2u(k) - E_3d(k)\big)_j \geq 0 \qquad \text{for each } j \tag{16}$$

and since the sum of nonnegative numbers is 0 if and only if each of the numbers is equal to 0, the condition (5d) can be rewritten as

$$\prod_{j \in \phi_i} \big(e_4 - E_1x(k) - E_2u(k) - E_3d(k)\big)_j = 0 \qquad \text{for } i = 1, 2, \ldots, p$$

(cf. (6)), or equivalently:

$$\forall i \in \{1, 2, \ldots, p\} : \exists j \in \phi_i \text{ such that } \big(e_4 - E_1x(k) - E_2u(k) - E_3d(k)\big)_j = 0 \ .$$

If we combine this with (16) we obtain

$$\min_{j \in \phi_i} \big(e_4 - E_1x(k) - E_2u(k) - E_3d(k)\big)_j = 0 \quad \text{for } i = 1, 2, \ldots, p, \tag{17}$$

which are all MMPS constraints of the form (8c). The conditions in (16) for which j does not belong to some ϕ_i can be bundled as

$$\min_{j \in \Psi} \big(e_4 - E_1x(k) - E_2u(k) - E_3d(k)\big)_j \geq 0$$

[3]Recall that a condition of the form $\alpha = \beta$ is equivalent to the conditions $\alpha \leq \beta$ and $-\alpha \leq -\beta$.

where $\Psi = \{j \in \{1,2,\ldots,q\} \,|\, \forall i \in \{1,2,\ldots,p\} : j \notin \phi_i\}$ and where q is the dimension of the vector e_4. This condition can be rewritten as

$$\max_{j\in\Psi}\big(E_1x(k) + E_2u(k) + E_3d(k) - e_4\big)_j \leq 0 \tag{18}$$

which is again an MMPS constraint of the form (8c). So the constraints (5c) – (5d) are equivalent to the MMPS constraints (17) – (18). Hence, every ELC system can be written as an MMPS system. □

Proposition 3.7 *Every MLD system can be rewritten as an ELC or an MMPS system.*

Proof: If we make an abstraction of the range of the variables then (1a) – (1c) coincide with (5a) – (5c) with $d(k) = [\delta^T(k) \;\; z^T(k)]^T$. Furthermore, a condition of the form $\delta_i(k) \in \{0,1\}$ is equivalent to the ELC conditions $-\delta_i(k) \leq 0$, $\delta_i(k) \leq 1$, $\delta_i(k)(1-\delta_i(k)) = 0$. So every MLD system can be rewritten as an ELC system and thus also as an MMPS system (by Proposition 3.6). □

Remark 3.8 Note that the condition $\delta_i(k) \in \{0,1\}$ is also equivalent to the MMPS constraint $\max(-\delta_i(k), \delta_i(k)-1) = 0$ or $\min(\delta_i(k), 1-\delta_i(k)) = 0$. □

Proposition 3.9 *Consider an ELC (or an MMPS) system. If the set $\mathcal{X}$ of feasible inputs, states and auxiliary variables of the ELC system is bounded, then the ELC system can be rewritten as an MLD system.*

Proof: Consider an ELC system that can be modeled by (5). Clearly, (5a) and (5b) fit the MLD framework. Recall that (5d) is equivalent to

$$\prod_{j\in\phi_i}\big(e_4 - E_1x(k) - E_2u(k) - E_3d(k)\big)_j = 0 \tag{19}$$

for $i = 1,2,\ldots,p$. Now we consider a set ϕ_i for some i. By introducing auxiliary variables $\alpha_j(k) \in \mathbb{R}$ and $\delta_j(k) \in \{0,1\}$, we find that condition (19) in combination with (5c) is equivalent to

$$\big(E_1x(k) + E_2u(k) + E_3d(k)\big)_j + \delta_j(k)\alpha_j(k) = \big(e_4\big)_j \qquad \text{for each } j \in \phi_i \tag{20}$$

$$\alpha_j(k) \geq 0 \qquad \text{for each } j \in \phi_i \tag{21}$$

$$\delta_j(k) \in \{0,1\} \qquad \text{for each } j \in \phi_i \tag{22}$$

$$\sum_{j\in\phi_i}\delta_j(k) \leq m_i - 1, \tag{23}$$

where m_i is the number of elements in ϕ_i. Note that (23) implies that at least one of the $\delta_j(k)$'s equals 0, so that equality is reached in (20) for at least one index j with $\delta_j(k)\alpha_j(k) = 0$, which implies that (19) holds. Equations (21) – (23) fit the MLD framework. However, (20) does not fit the MLD framework due to the occurrence of the bilinear slack term $\delta_j(k)\alpha_j(k)$. Now we use a reasoning similar to the one used in (Bemporad and Morari, 1999) to transform (20) into a system of equations that fit the MLD framework. First we define

$$M_j = \max_{(x(k),u(k),d(k))\in\mathcal{X}}\big(e_4 - E_1x(k) - E_2u(k) - E_3d(k)\big)_j$$

for each $j \in \phi_i$. Note that the boundedness of $\mathcal{X}$ implies that all the M_j's are finite. Furthermore, the definition of M_j implies that $0 \leq \delta_j(k)\alpha_j(k) \leq M_j$. Note that we may assume without loss of generality that the following constraint is added to the system (20) – (23):

$$\alpha_j(k) \leq M_j \qquad \text{for each } j \in \phi_i. \tag{24}$$

This clearly holds if $\delta_j(k) = 1$, and for $\delta_j(k) = 0$ the value of $\alpha_j(k)$ does not influence (20) – (23) and hence we may take any arbitrary value for $\alpha_j(k)$ (e.g. satisfying (24)). Now we introduce extra variables $v_j(k)$ such that

$$0 \leq v_j(k) \leq M_j\delta_j(k) \qquad \text{for each } j \in \phi_i \tag{25}$$

$$\alpha_j(k) - M_j(1 - \delta_j(k)) \leq v_j(k) \leq \alpha_j(k) \qquad \text{for each } j \in \phi_i. \tag{26}$$

Note that these conditions do not contain any bilinear terms and that they fit in the MLD framework. Let us now show that the conditions (25) – (26) imply that $v_j(k) = \delta_j(k)\alpha_j(k)$.

- If $\delta_j(k) = 0$, then (25) implies that $v_j(k) = 0$. Hence, $v_j(k) = 0 = \delta_j(k)\alpha_j(k)$. Moreover, for $\delta_j(k) = 0$ (26) leads to $\alpha_j(k) - M_j \leq v_j(k) \leq \alpha_j(k)$, which holds since $0 \leq \alpha_j(k) \leq M_j$ and thus $\alpha_j(k) - M_j \leq 0 = v_j(k) \leq \alpha_j(k)$.
- If $\delta_j(k) = 1$, then (26) leads to $v_j(k) = \alpha_j(k)$. Hence, $v_j(k) = \alpha_j(k) = \delta_j(k)\alpha_j(k)$. Furthermore, in case $\delta_j(k) = 1$ (25) results in $0 \leq v_j(k) \leq M_j$, which also holds due to $v_j(k) = \alpha_j(k)$, (21) and (24).

Hence, (25) – (26) imply that $v_j(k) = \delta_j(k)\alpha_j(k)$ for each $j \in \phi_i$. As a consequence, (20) is equivalent to

$$\big(E_1x(k) + E_2u(k) + E_3d(k)\big)_j + v_j(k) = \big(e_4\big)_j \qquad \text{for each } j \in \phi_i \tag{27}$$

with the additional conditions (24) – (26). The conditions (24) – (27) all fit in the MLD framework. As a consequence, the ELC system can be rewritten as an MLD system. □

Corollary 3.10 *Every LC system (4) for which E_3 is a P-matrix*[4] *can be written as an unconstrained MMPS system.*

Proof: See (van Bokhoven and Leenaerts, 1999, Thm. 2). □

4 Example

To demonstrate the equivalences proven above, we consider an example taken from (Bemporad and Morari, 1999) which is given as follows:

$$x(k+1) = \begin{cases} 0.8x(k) + u(k) & \text{if } x(k) \geq 0 \\ -0.8x(k) + u(k) & \text{if } x(k) < 0 \end{cases} \tag{28}$$

with $x(k), u(k) \in \mathbb{R}$.

In (Bemporad and Morari, 1999) additional assumptions are needed to rewrite this PWA system into the MLD framework. First of all, one assumes that the state $x(k)$ is bounded

[4] A matrix $M \in \mathbb{R}^{n \times n}$ is said to be a P-matrix if all its principal minors are positive (Cottle et al., 1992).

by $m \leq x(k) \leq M$ for all times k. Second, one replaces the strict inequality $x(k) < 0$ by $x(k) \leq -\varepsilon$, where $\varepsilon > 0$ is a small number (typically the machine precision). Under these conditions one shows that (28) is equivalent to

$$x(k+1) = -0.8x(k) + u(k) + 1.6z(k) \tag{29a}$$

together with the linear inequalities

$$\begin{aligned}
-m\delta(k) &\leq x(k) - m && \text{(29b)}\\
-(M+\varepsilon)\delta(k) &\leq -x(k) - \varepsilon && \text{(29c)}\\
z(k) &\leq M\delta(k) && \text{(29d)}\\
z(k) &\geq m\delta(k) && \text{(29e)}\\
z(k) &\leq x(k) - m(1-\delta(k)) && \text{(29f)}\\
z(k) &\geq x(t) - M(1-\delta(k)) && \text{(29g)}
\end{aligned}$$

and the condition $\delta(k) \in \{0, 1\}$.

One can easily verify that (28) can be rewritten as the (unconstrained) MMPS model

$$x(k+1) = -0.8x(k) + 1.6\max(0, x(k)) + u(k) \ , \tag{30}$$

as the LC formulation

$$\begin{aligned}
x(k+1) &= -0.8x(k) + u(k) + 1.6z(k) && \text{(31a)}\\
w(k) &= -x(k) + z(k) && \text{(31b)}\\
0 &\leq w(k) \perp z(k) \geq 0 \ , && \text{(31c)}
\end{aligned}$$

and as the ELC representation

$$\begin{aligned}
x(k+1) &= -0.8x(k) + u(k) + 1.6d(k) && \text{(32a)}\\
-d(k) &\leq 0 && \text{(32b)}\\
x(k) - d(k) &\leq 0 && \text{(32c)}\\
0 &= \big(x(k) - d(k)\big)\big(-d(k)\big) && \text{(32d)}
\end{aligned}$$

without having to impose any assumptions as was done in (Bemporad and Morari, 1999).

Note that we only need one max-operator in (30) and one complementarity pair in (31). If we would transform the MLD system (29) into e.g. the LC model as indicated by the equivalence proof of Proposition 3.1, this would require seven complementarity pairs. Hence, it is clear that the proofs only show that the system representations are conceptually equivalent, but do not result in the most efficient models.

5 Conclusions and topics for future research

We have shown the equivalence of five classes of hybrid systems: mixed logical dynamical systems, linear complementarity systems, extended linear complementarity systems, piecewise affine systems and max-min-plus-scaling systems. For some of the transformations additional conditions like boundedness of the state and input variables or well-posedness had to be made.

An important topic for future research is a further investigation of the links between these different subclasses of hybrid systems. These links enable the transfer of techniques for analysis and synthesis from one class of hybrid systems to another. Moreover, it is interesting to study which modeling framework is most appropriate for solving specific control problems related to e.g. well-posedness, controllability and stability of hybrid dynamical systems. From a computational point of view, one might pose the question which canonical representation leads to the most efficient numerical algorithms for obtaining and analyzing control strategies.

As a specific example, consider model predictive control (MPC) for MLD systems (Bemporad and Morari, 1999) (using mixed integer quadratic programming) or for (unconstrained) MMPS systems (De Schutter and van den Boom, 2000) (using the Extended Linear Complementarity Problem and non-linear programming with real-valued variables). The main difference between MPC for MLD systems and MPC for ELC (and MMPS) systems is that in the latter case all optimization variables are real-valued, which may ease the computational burden. In fact, under certain additional assumptions the MPC problem for MMPS systems can be recast as a convex optimization problem (De Schutter and van den Boom, 2000). Hence, the links revealed in this paper give the possibility to choose between several modeling frameworks, which all indicate different directions for solving the (related) optimization/control problems. The development of efficient algorithms for MPC for these generic classes of hybrid systems will be a topic for future research.

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