On the Equivalence of RLS Implementations of LCMV and GSC Processors

Stefan Werner, *Member, IEEE*, José A. Apolinário, Jr., *Member, IEEE*, and
Marcello L. R. de Campos, *Senior Member, IEEE*

***Abstract*—This letter compares the transients of the constrained recursive least squares (CRLS) algorithm with the generalized sidelobe canceler (GSC) employing the recursive least squares (RLS) algorithm. We prove that the two adaptive implementations are equivalent everywhere regardless of the blocking matrix chosen. This guarantees that algorithm tuning is not affected by the blocking matrix. This result differs from the more restrictive case for transient equivalence of the constrained least mean-square (CLMS) algorithm and the GSC employing the least mean square (LMS) algorithm, for in this case the blocking matrix needs to be unitary.**

***Index Terms*—Beamforming, generalized sidelobe canceler (GSC), linearly constrained adaptive filtering, linearly constrained minimum variance (LCMV).**

I. Introduction

The linearly constrained minimum-variance (LCMV) filter and the generalized sidelobe canceler (GSC) are two alternative structures for implementation of linearly constrained filters which find applications in, for example, beamforming [1], [2] and blind multiuser detection [3]. Several adaptation algorithms have been proposed that estimate the coefficients of the LCMV filter [1]–[10]. Gradient-based adaptive implementations of LCMV filters may suffer from slow convergence due to the correlated nature of the input signal. In these cases, an alternative may be the use of faster algorithms, such as those based on the least squares (LS) solution.

The optimal (LCMV) filter is the one that minimizes the objective function $J_{\mathbf{w}}$ subject to a set of linear constraints, i.e.,

$$\hat{\mathbf{w}} = \arg\min_{\mathbf{w}} J_{\mathbf{w}} \quad \text{subject to} \quad \mathbf{C}^{\mathrm{H}}\mathbf{w} = \mathbf{f} \tag{1}$$

where $\mathbf{w}$ is a vector of coefficients of length N, $\mathbf{C}$ is the $N \times p$ constraint matrix, and $\mathbf{f}$ is the $p \times 1$ gain vector, p being the number of constraints. The most common LCMV filter used in the literature is probably the one minimizing the mean output energy (MOE) objective function, i.e., $J_{\mathbf{w}} = \mathbf{w}^{\mathrm{H}}\mathbf{R}\mathbf{w}$, where $\mathbf{R}$ is the $N \times N$ autocorrelation matrix of the input signal. In the more general minimization problem, a desired signal may be present, and the optimal filter is the one which minimizes the mean-squared error (MSE) subject to a set of p linear constraints given by $\mathbf{C}$ and $\mathbf{f}$.

The GSC processor solves the same problem as the one given in (1) by dividing the filter vector $\mathbf{w}$ into two components operating on orthogonal subspaces

$$\mathbf{w} = \mathbf{F} - \mathbf{B}\mathbf{w}_{\mathrm{GSC}} \tag{2}$$

where $\mathbf{F} = \mathbf{C}(\mathbf{C}^{\mathrm{H}}\mathbf{C})^{-1}\mathbf{f}$ is the constant part of vector $\mathbf{w}$ that satisfies the constraints, i.e., $\mathbf{C}^{\mathrm{H}}\mathbf{F} = \mathbf{f}$. The $N \times (N-p)$ matrix $\mathbf{B}$ of full column-rank is often referred to as the *blocking matrix* and spans the null space of the constraint matrix $\mathbf{C}$, i.e., $\mathbf{B}^{\mathrm{H}}\mathbf{C} = \mathbf{0}$. The $(N-p) \times 1$ vector $\mathbf{w}_{\mathrm{GSC}}$ is given as the solution to the unconstrained problem

$$\hat{\mathbf{w}}_{\mathrm{GSC}} = \arg\min_{\mathbf{w}_{\mathrm{GSC}}} J_{\mathbf{w}} \tag{3}$$

with $\mathbf{w}$ given by (2).

The equivalence of the two optimal processors in (1) and (2) has been shown by numerous authors (e.g., see [11]–[13]). Although the optimal solutions are equivalent, their adaptive implementations are not necessarily identical everywhere, where *everywhere* herein means for all time instants k and for all realizations of the stochastic processes involved. For example, transient equivalence for the least mean-square (LMS) algorithm applied to the GSC processor and the constrained LMS (CLMS) algorithm [1], which is the corresponding LCMV implementation, can be ensured only for the particular case of unitary blocking matrix, i.e., $\mathbf{B}^{\mathrm{H}}\mathbf{B} = \mathbf{I}$ [2]. For a small number of constraints, a unitary blocking matrix can lead to a computationally complex implementation of the GSC structure rendering the total complexity of the filtering operation comparable to that of the recursive LS (RLS) implementation. On the other hand, simple nonunitary blocking matrices may require extra care on algorithm tuning, for nonunitary matrices do not preserve the modes of the correlation function [10].

To our knowledge, no results are available comparing the RLS implementations of the LCMV filter, *viz.*, the constrained RLS (CRLS) algorithm introduced in [4] and the GSC structure employing an unconstrained RLS algorithm. The goal of this letter is to investigate what are the requirements for transient equivalence everywhere when considering the implementations of the CRLS and the GSC–RLS algorithms. In particular, we investigate if the requirement of unitary blocking matrix is also necessary in the case of the RLS implementations. On the

Manuscript received September 20, 2002; revised December 10, 2002. This work was supported in part by CNPq (Brazil), FAPERJ (Brazil), and GETA (Finland). The associate editor coordinating the review of this manuscript and approving it for publication was Dr. See-May Phoong.

S. Werner is with the Signal Processing Laboratory, Helsinki University of Technology, FIN-02015, Espoo, Finland (e-mail: stefan.werner@hut.fi).

J. A. Apolinário, Jr. is with the Departamento de Engenharia Elétrica, Instituto Militar de Engenharia, Rio de Janeiro 22.290-270, Brazil (e-mail: apolin@ieee.org).

M. L. R. de Campos is with the COPPE/Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil, 21945-970 and also with the Centre for Wireless Communications, University of Oulu, Oulu, FIN-90014, Finland (e-mail: campos@lps.ufrj.br).

Digital Object Identifier 10.1109/LSP.2003.818861

contrary, we show that for the RLS implementations to produce identical transient curves, the blocking matrix only needs to be orthogonal to the constraint matrix, which is always the case in any GSC structure. Our argument also provides extra insight to the solution of the constrained LS problem, for it is expected that recursive solutions to a deterministic minimization problem give the same results regardless of the implementation chosen, except for errors introduced by quantization.

II. CRLS ALGORITHM

For the more general case where the desired signal is not necessarily zero, the linearly constrained RLS algorithm to be discussed below uses the weighted LS criterion as objective function $J_{\mathbf{w}}$, resulting in the following optimization problem:

$$\mathbf{w}(k) = \arg\min_{\mathbf{w}} \sum_{i=1}^{k} \lambda^{k-i} |e(i)|^2 \quad \text{s.t.} \quad \mathbf{C}^{\mathrm{H}}\mathbf{w} = \mathbf{f} \tag{4}$$

where the error $e(k)$ is defined as

$$e(k) = d(k) - \mathbf{x}^{\mathrm{H}}(k)\mathbf{w} \tag{5}$$

and λ is the forgetting factor $(0 < \lambda \leq 1)$.

The optimal LS solution, derived in [14], can be split into two terms

$$\mathbf{w}(k) = \mathbf{w}_{uc}(k) + \mathbf{w}_{c}(k) \tag{6}$$

where

$$\mathbf{w}_{uc}(k) = \mathbf{R}^{-1}(k)\mathbf{p}(k) \tag{7}$$

and

$$\mathbf{w}_{c}(k) = \mathbf{R}^{-1}(k)\mathbf{C}\left[\mathbf{C}^{\mathrm{H}}\mathbf{R}^{-1}(k)\mathbf{C}\right]^{-1}\left[\mathbf{f} - \mathbf{C}^{\mathrm{H}}\mathbf{R}^{-1}(k)\mathbf{p}(k)\right]. \tag{8}$$

$\mathbf{R}(k)$ is the $N \times N$ *deterministic* correlation matrix, and $\mathbf{p}(k)$ is the $N \times 1$ *deterministic* cross-correlation vector, defined respectively as

$$\mathbf{R}(k) = \sum_{i=0}^{k} \lambda^{k-i}\mathbf{x}(i)\mathbf{x}^{\mathrm{H}}(i) \tag{9}$$

$$\mathbf{p}(k) = \sum_{i=0}^{k} \lambda^{k-i}\mathbf{x}(i)d^{*}(i). \tag{10}$$

The coefficient vector $\mathbf{w}_{uc}(k)$ is independent of the constraints, whereas $\mathbf{w}_{c}(k)$ depends on the constraints imposed by $\mathbf{C}^{\mathrm{H}}\mathbf{w}(k) = \mathbf{f}$.

After lengthy but straightforward manipulations, an RLS update of the coefficient vector in (6), referred to as the CRLS algorithm, can be derived [4]

$$\mathbf{w}(k) = \mathbf{w}(k-1) + e^{*}(k)\boldsymbol{\kappa}(k) - \lambda e^{*}(k)\boldsymbol{\Gamma}(k)\boldsymbol{\ell}(k) \tag{11}$$

where $\boldsymbol{\kappa}(k)$ is the $N \times 1$ *gain vector*

$$\boldsymbol{\kappa}(k) = \mathbf{R}^{-1}(k)\mathbf{x}(k) \tag{12}$$

and $\boldsymbol{\Gamma}(k)$, $\boldsymbol{\ell}(k)$, and and $\boldsymbol{\Psi}(k)$ are auxiliary matrices with dimensions $N \times p$, $N \times 1$, and $p \times p$, respectively

$$\boldsymbol{\Gamma}(k) = \mathbf{R}^{-1}(k)\mathbf{C} \tag{13}$$

$$\boldsymbol{\ell}(k) = \frac{1}{\lambda}\frac{\boldsymbol{\Psi}^{-1}(k-1)\mathbf{C}^{\mathrm{H}}\boldsymbol{\kappa}(k)}{1 - \mathbf{x}^{\mathrm{H}}(k)\boldsymbol{\Gamma}(k-1)\boldsymbol{\Psi}^{-1}(k-1)\mathbf{C}^{\mathrm{H}}\boldsymbol{\kappa}(k)} \tag{14}$$

$$\boldsymbol{\Psi}(k) = \mathbf{C}^{\mathrm{H}}\boldsymbol{\Gamma}(k) = \mathbf{C}^{\mathrm{H}}\mathbf{R}^{-1}(k)\mathbf{C}. \tag{15}$$

For the equivalence study to be carried out in Section IV, we will make use of (11)–(15). For efficient recursive implementations of the matrices $\boldsymbol{\Gamma}(k)$ and $\boldsymbol{\Psi}(k)$, see [4].

III. GSC–RLS ALGORITHM

Many implementations of linearly constrained adaptive filters utilize the advantages of the GSC model [15], mainly because this model employs unconstrained adaptation algorithms that have been extensively studied in the literature. The RLS recursions for the GSC structure become

$$\mathbf{x}_{\mathrm{GSC}}(k) = \mathbf{B}^{\mathrm{H}}\mathbf{x}(k) \tag{16}$$

$$e_{\mathrm{GSC}}(k) = \mathbf{F}^{\mathrm{H}}\mathbf{x}(k) - \mathbf{w}_{\mathrm{GSC}}^{\mathrm{H}}(k)\mathbf{x}_{\mathrm{GSC}} - d(k) \tag{17}$$

$$\boldsymbol{\kappa}_{\mathrm{GSC}}(k) = \mathbf{R}_{\mathrm{GSC}}^{-1}(k)\mathbf{x}_{\mathrm{GSC}}(k) \tag{18}$$

$$\mathbf{w}_{\mathrm{GSC}}(k) = \mathbf{w}_{\mathrm{GSC}}(k-1) + e_{\mathrm{GSC}}^{*}(k)\boldsymbol{\kappa}_{\mathrm{GSC}}(k) \tag{19}$$

where $e_{\mathrm{GSC}}(k)$ is the *a priori* error, and $\boldsymbol{\kappa}_{\mathrm{GSC}}(k)$ is the gain vector. The inverse of the correlation matrix $\mathbf{R}_{\mathrm{GSC}}^{-1}(k)$ can be updated recursively in a standard way by using the *matrix inversion* lemma [15]. In the next section, we will compare the recursion in (19) with that of the CRLS algorithm in (11) in order to prove that both implementations yield the same transient solution everywhere, regardless of the blocking matrix $\mathbf{B}$ used.

IV. EQUIVALENCE OF CRLS AND GSC–RLS IMPLEMENTATIONS

Our goal in this section is to investigate under what circumstances the transients of the CRLS and the GSC-RLS algorithms are identical everywhere.

We will study the coefficient-vector evolution defined as

$$\Delta\mathbf{w}(k) = \mathbf{w}(k) - \mathbf{w}(k-1). \tag{20}$$

Equations (11)–(14) give us the coefficient-vector evolution for the CRLS algorithm as

$$\Delta\mathbf{w}(k) = e^{*}(k)\left\{\mathbf{I} - \mathbf{R}^{-1}(k)\mathbf{C}\left[\mathbf{C}^{\mathrm{H}}\mathbf{R}^{-1}(k)\mathbf{C}\right]^{-1}\mathbf{C}^{\mathrm{H}}\right\} \cdot\mathbf{R}^{-1}(k)\mathbf{x}(k). \tag{21}$$

For the GSC-RLS algorithm, with $\mathbf{w}(k) = \mathbf{F} - \mathbf{B}\mathbf{w}_{\mathrm{GSC}}(k)$, (19) gives us

$$\begin{aligned}\Delta\mathbf{w}(k) &= \mathbf{F} - \mathbf{B}\mathbf{w}_{\mathrm{GSC}}(k) - \mathbf{w}(k-1)\\ &= \mathbf{F} - \mathbf{B}\left[\mathbf{w}_{\mathrm{GSC}}(k-1) + e_{\mathrm{GSC}}^{*}(k)\mathbf{R}_{\mathrm{GSC}}^{-1}(k)\mathbf{x}_{\mathrm{GSC}}(k)\right]\\ &\quad - \mathbf{w}(k-1)\\ &= e^{*}(k)\mathbf{B}\left[\mathbf{B}^{\mathrm{H}}\mathbf{R}(k)\mathbf{B}\right]^{-1}\mathbf{B}^{\mathrm{H}}\mathbf{x}(k)\\ &= e^{*}(k)\left\{\mathbf{B}\left[\mathbf{B}^{\mathrm{H}}\mathbf{R}(k)\mathbf{B}\right]^{-1}\mathbf{B}^{\mathrm{H}}\mathbf{R}(k)\right\}\mathbf{R}^{-1}(k)\mathbf{x}(k)\end{aligned} \tag{22}$$

where $\mathbf{w}(k-1) = \mathbf{F} - \mathbf{B}\mathbf{w}_{\mathrm{GSC}}(k-1)$ was used together with $\mathbf{R}_{\mathrm{GSC}}(k) = \mathbf{B}^{\mathrm{H}}\mathbf{R}(k)\mathbf{B}$ and $e^*(k) = -e^*_{\mathrm{GSC}}(k)$. In order for (21) and (22) to be identical, it is required that the following matrix equality holds:

$$\mathbf{B}\left[\mathbf{B}^{\mathrm{H}}\mathbf{R}(k)\mathbf{B}\right]^{-1}\mathbf{B}^{\mathrm{H}}\mathbf{R}(k) + \mathbf{R}^{-1}(k)\mathbf{C}\left[\mathbf{C}^{\mathrm{H}}\mathbf{R}^{-1}(k)\mathbf{C}\right]^{-1}\mathbf{C}^{\mathrm{H}} = \mathbf{I}. \quad (23)$$

In addition, the initialization of both schemes (CRLS and GSC–RLS) should also be equivalent, which means that $\mathbf{R}^{-1}_{\mathrm{GSC}}(0) = [\mathbf{B}^{\mathrm{H}}\mathbf{R}(0)\mathbf{B}]^{-1}$ and $\mathbf{w}(0) = \mathbf{F} - \mathbf{B}\mathbf{w}_{\mathrm{GSC}}(0)$. As a consequence, $e^*(0) = -e^*_{\mathrm{GSC}}(0)$ holds. Therefore, equivalence of the CRLS and the GSC–RLS processors using the correct initialization is ensured by induction with the following lemma.

Lemma 1: For $\mathbf{B}^{\mathrm{H}}\mathbf{C} = \mathbf{0}$, if $\mathbf{R}^{-1}(k)$ exists and is symmetric, if rank$(\mathbf{B}) = N - p$, and if rank$(\mathbf{C}) = p$, (23) holds true.

Proof: Define matrices $\bar{\mathbf{B}} = \mathbf{R}^{\mathrm{H}/2}(k)\mathbf{B}$ and $\bar{\mathbf{C}} = \mathbf{R}^{-1/2}(k)\mathbf{C}$, where $\mathbf{R}(k) = \mathbf{R}^{1/2}(k)\mathbf{R}^{\mathrm{H}/2}(k)$. The left-hand side of (23) becomes $\bar{\mathbf{B}}(\bar{\mathbf{B}}^{\mathrm{H}}\bar{\mathbf{B}})^{-1}\bar{\mathbf{B}}^{\mathrm{H}} + \bar{\mathbf{C}}(\bar{\mathbf{C}}^{\mathrm{H}}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}^{\mathrm{H}}$, and it remains to show that this addition of matrices equals identity. For this purpose, let us introduce the matrix $\mathbf{T} = [\bar{\mathbf{C}}\ \bar{\mathbf{B}}]$. $\mathbf{T}$ is a full-rank $(N \times N)$ matrix, and, consequently, $\mathbf{T}^{-1}$ exists. We have

$$\mathbf{T}^{\mathrm{H}}\mathbf{T} = \begin{bmatrix} \bar{\mathbf{C}}^{\mathrm{H}}\bar{\mathbf{C}} & \bar{\mathbf{C}}^{\mathrm{H}}\bar{\mathbf{B}} \\ \bar{\mathbf{B}}^{\mathrm{H}}\bar{\mathbf{C}} & \bar{\mathbf{B}}^{\mathrm{H}}\bar{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{C}}^{\mathrm{H}}\bar{\mathbf{C}} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{B}}^{\mathrm{H}}\bar{\mathbf{B}} \end{bmatrix} \quad (24)$$

where the relation $\bar{\mathbf{B}}^{\mathrm{H}}\bar{\mathbf{C}} = \mathbf{0}$ was used. We have

$$(\mathbf{T}^{\mathrm{H}}\mathbf{T})^{-1} = \begin{bmatrix} (\bar{\mathbf{C}}^{\mathrm{H}}\bar{\mathbf{C}})^{-1} & \mathbf{0} \\ \mathbf{0} & (\bar{\mathbf{B}}^{\mathrm{H}}\bar{\mathbf{B}})^{-1} \end{bmatrix}. \quad (25)$$

Therefore

$$\begin{aligned}\mathbf{T}(\mathbf{T}^{\mathrm{H}}\mathbf{T})^{-1}\mathbf{T}^{\mathrm{H}} &= \mathbf{T}\mathbf{T}^{-1}\mathbf{T}^{-\mathrm{H}}\mathbf{T}^{\mathrm{H}} = \mathbf{I} \\ &= [\bar{\mathbf{C}}\ \bar{\mathbf{B}}]\begin{bmatrix} (\bar{\mathbf{C}}^{\mathrm{H}}\bar{\mathbf{C}})^{-1} & \mathbf{0} \\ \mathbf{0} & (\bar{\mathbf{B}}^{\mathrm{H}}\bar{\mathbf{B}})^{-1} \end{bmatrix}\begin{bmatrix} \bar{\mathbf{C}}^{\mathrm{H}} \\ \bar{\mathbf{B}}^{\mathrm{H}} \end{bmatrix} \\ &= \bar{\mathbf{C}}(\bar{\mathbf{C}}^{\mathrm{H}}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}^{\mathrm{H}} + \bar{\mathbf{B}}(\bar{\mathbf{B}}^{\mathrm{H}}\bar{\mathbf{B}})^{-1}\bar{\mathbf{B}}^{\mathrm{H}} = \mathbf{I}\end{aligned}$$

which concludes the proof.

As a consequence of Lemma 1, and (21) and (22), we can conclude that the necessary requirement for equivalent transients of the CRLS and the GSC-RLS algorithms is that $\mathbf{B}^{\mathrm{H}}\mathbf{C} = \mathbf{0}$, which holds true in any GSC structure. This is a looser requirement than the transient equivalence of the CLMS and GSC–LMS algorithms, which in addition to $\mathbf{B}^{\mathrm{H}}\mathbf{C} = \mathbf{0}$, requires $\mathbf{B}$ to be unitary. For reasons of computational complexity and robustness, the result just presented serves as an indication that implementing the unconstrained form of the RLS algorithm may be preferable, either using the Householder transform as described in [10] or in a GSC structure.

As the overall complexity is a function of filtering and coefficient updating, implementation of a nonunitary sparse blocking matrix together with a fast RLS algorithm may be an interesting alternative for constrained adaptive filters.

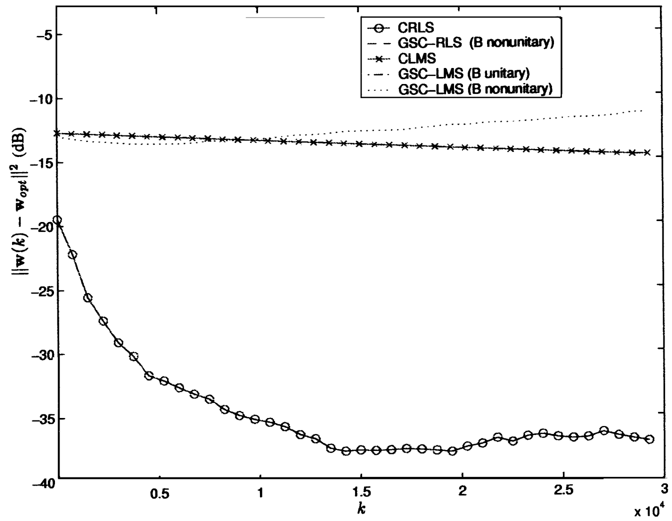

Fig. 1. Coefficient-error vector as a function of the iteration k for a beamforming application using derivative constraints.

V. Simulations

In this section, the equivalence of the CRLS and GSC–RLS algorithms is investigated in a beamforming application where the desired signal is set to zero, i.e., $d(k) = 0$. A uniform linear array with $M = 12$ antennas with element spacing equal to half wave-length was used in a system with $K = 5$ users, where the signal of one user is of interest, and the other four are treated as interferers. The desired signal had an SNR of 15 dB, and two of the interfering users had 20 dB while the other two had 25 dB. For a more complete description of the setup, see [10].

A second-order derivative constraint matrix [16] was used, giving a total of three constraints. The GSC implementation used a nonunitary blocking matrix constructed through a sequence of sparse matrices as presented in [17] rendering an implementation of the multiplication $\mathbf{B}\mathbf{x}(k)$ of low computational complexity.

The CRLS and the GSC–RLS algorithms used $\lambda = 0.99$. Fig. 1 shows the evolution of coefficient-error norm for the CRLS and the GSC–RLS algorithms. Fig. 1 also plots the results for the CLMS and the GSC-LMS algorithms. As can be seen from the figure, the CLMS and the GSC-LMS algorithms only become identical when using the unitary blocking matrix, whereas the CRLS and the GSC–RLS algorithms are identical for the nonunitary blocking matrix.

VI. Conclusion

This letter presents theoretical results linking transient behavior of the constrained RLS algorithm and the GSC structure with the RLS algorithm. We showed that contrary to the LMS algorithm, in the case of the RLS algorithm, transient behavior can always be ensured identical in the two forms of implementation provided only that the blocking matrix and the constraint matrix span orthogonal subspaces. This result facilitates algorithm tuning, for it establishes that the constrained algorithm behaves exactly like its unconstrained counterpart in transient as well as in steady state. This confirms intuition, for both implementations solve the same LS problem exactly. The result presented herein may favor the utilization of the unconstrained counterpart of the CRLS algorithm, for it facilitates the choice of various versions of the RLS algorithm optimized with respect to computational complexity and robustness.

ACKNOWLEDGMENT

M. L. R. de Campos would like to thank Nokia Foundation for a Nokia Visiting Fellowship awarded.

REFERENCES

[1] O. L. Frost III, "An algorithm for linearly constrained adaptive array processing," *Proc. IEEE*, vol. 60, pp. 926–935, Aug. 1972.

[2] L. J. Griffiths and C. W. Jim, "An alternative approach to linearly constrained adaptive beamforming," *IEEE Trans. Antennas Propagat.*, vol. AP-30, pp. 27–34, Jan. 1982.

[3] M. Honig, U. Madhow, and S. Verdú, "Blind multiuser detection," *IEEE Trans. Inform. Theory*, vol. 41, pp. 944–960, July 1995.

[4] L. S. Resende, J. M. T. Romano, and M. G. Bellanger, "A fast least-squares algorithm for linearly constrained adaptive filtering," *IEEE Trans. Signal Processing*, vol. 44, pp. 1168–1174, May 1996.

[5] J. A. Apolinário, Jr., S. Werner, and P. S. R. Diniz, "Constrained normalized adaptive filters for CDMA mobile communications," in *Proc. Eur. Signal Processing Conf.*, vol. 4, Rhodes, Greece, 1998, pp. 2053–2056.

[6] M. L. R. de Campos, S. Werner, J. A. Apolinário, Jr., and T. I. Laakso, "Constrained quasi-Newton algorithm for CDMA mobile communications," in *Proc. Int. Telecommunications Symp.*, São Paulo, Brazil, Aug. 1998, pp. 371–376.

[7] M. L. R. de Campos, S. Werner, and J. A. Apolinário, Jr., "Householder-transform constrained LMS algorithms with reduced-rank updating," in *Proc. ICASSP*, Phoenix, AZ, 1999, pp. 1857–1960.

[8] J. A. Apolinário, Jr., M. L. R. de Campos, and C. P. Bernal, "The constrained conjugate-gradient algorithm," *IEEE Signal Processing Lett.*, vol. 7, pp. 351–354, Dec. 2000.

[9] M. L. R. de Campos, J. A. Apolinário, Jr., and P. S. R. Diniz, "The constrained affine projection algorithm—development and convergence issues," in *Proc. 1st Balkan Conf. Signal Processing, Communications, Circuits and Systems*, Istanbul, Turkey, May 2000.

[10] M. L. R. de Campos, S. Werner, and J. A. Apolinário, Jr., "Constrained adaptation algorithms employing Householder transformation," *IEEE Trans. Signal Processing*, vol. 50, pp. 2187–2195, Sept. 2002.

[11] C. W. Jim, "A comparison of two LMS constrained optimal structures," *Proc. IEEE*, vol. 65, pp. 1730–1731, Dec. 1977.

[12] D. H. Johnson and D. E. Dudgeon, *Array Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1993.

[13] B. R. Breed and J. Strauss, "A short proof of the equivalence of LCMV and GSC beamforming," *IEEE Signal Processing Lett.*, vol. 9, pp. 168–169, June 2002.

[14] M. T. Schiavoni and M. G. Amin, "A linearly constrained minimization approach to adaptive linear phase and notch filters," in *Proc. 20th Southeastern Symp. System Theory*, Philadelphia, PA, Mar. 1988, pp. 682–685.

[15] S. Haykin, *Adaptive Filter Theory*. Englewood Cliffs, NJ: Prentice-Hall, 1996.

[16] Y. Chu and W.-H. Fang, "A novel wavevelet-based generalized sidelobe canceller," *IIEEE Trans. Antennas Propagat.*, vol. 47, pp. 1485–1494, Sept. 1999.

[17] C.-Y. Tseng and L. J. Griffiths, "A systematic procedure for implementing the blocking matrix in decomposed form," in *Proc. 22nd Asilomar Conf. Signals, System and Computers*, vol. 2, 1988, pp. 808–812.