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ON THE EQUIVALENCE OF STOCHASTIC COMPLETENESS AND LIOUVILLE AND KHAS'MINSKII CONDITIONS IN LINEAR AND NONLINEAR SETTINGS

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Abstract. Set in the Riemannian enviroment, the aim of this paper is to present and discuss some equivalent characterizations of the Liouville property relative to special operators, which in some sense are modeled after the p-Laplacian with potential. In particular, we discuss the equivalence between the Liouville property and the Khas'minskii condition, i.e. the existence of an exhaustion function which is also a supersolution for the operator outside a compact set. This generalizes a previous result obtained by one of the authors.

1. Introduction

In what follows, let M denote a connected Riemannian manifold of dimension m, with no boundary. We stress that no completeness assumption is required. The relationship between the probabilistic notions of stochastic completeness and parabolicity (respectively the non-explosion and the recurrence of the Brownian motion on M) and function-theoretic properties of M has been the subject of an active area of research in the past few decades. Deep connections with the heat equation, Liouville type theorems, capacity theory and spectral theory have been described, for instance, in the beautiful survey [8]. In [23] and [22], the authors showed that stochastic completeness and parabolicity are also related to weak maximum principles at infinity. This characterization has revealed to be fruitful in investigating many kinds of geometric problems (for a detailed account, see [24]). Among the various conditions equivalent to stochastic completeness, the following two are of prior interest to us:

- [L^∞-*Liouville*] for some (any) $\lambda > 0$, the sole bounded, non-negative, continuous weak solution of $\Delta u - \lambda u \geq 0$ is $u = 0$;
- [*weak maximum principle*] for every $u \in C^2(M)$ with $u^\star = \sup_M u < +\infty$, and for every $\eta < u^\star$,

$$\inf_{\Omega_\eta} \Delta u \leq 0, \qquad \text{where } \Omega_\eta = u^{-1}\{(\eta, +\infty)\}. \tag{1.1}$$

R.Z. Khas'minskii [11] has found the following condition for stochastic completeness. We recall that $w \in C^0(M)$ is called an exhaustion if it has compact sublevels $w^{-1}((-\infty, t])$, $t \in \mathbb{R}$.

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Theorem 1.1 (Khas'minskii test [11])**.** *Suppose that there exists a compact set K and a function $w \in C^0(M) \cap C^2(M \setminus K)$ satisfying for some $\lambda > 0$:*

$$(i) \quad w \text{ is an exhaustion;} \qquad (ii) \quad \Delta w - \lambda w \leq 0 \quad \text{on } M \backslash K.$$

Then M is stochastically complete.

A very similar characterization holds for the parabolicity of M. Namely, among many others, parabolicity is equivalent to:

- every bounded, non-negative continuous weak solution of $\Delta u \geq 0$ on M is constant;
- for every non-constant $u \in C^2(M)$ with $u^\star = \sup_M u < +\infty$, and for every $\eta < u^\star$,

$$\inf_{\Omega_\eta} \Delta u < 0, \qquad \text{where } \ \Omega_\eta = u^{-1}\{(\eta, +\infty)\}. \tag{1.2}$$

Note that the first condition is precisely case $\lambda = 0$ of the Liouville property above. As for Khas'minskii type conditions, it has been proved by M. Nakai [20] and Z. Kuramochi [15] that the parabolicity of M is indeed *equivalent* to the existence of a so-called Evans potential, that is, an exhaustion, harmonic function w defined outside a compact set K and such that $w = 0$ on ∂K. To the best of our knowledge, an analogue of such equivalence for stochastic completeness or for the non-linear case has still to be proved, and this is the starting point of the present work.

With some modifications, it is possible to define the Liouville property, the Khas'minskii test and Evans potentials also for p-Laplacians or other non-linear operators, and the aim of this paper is to prove that in this more general setting the Liouville property is equivalent to the Khas'minskii test, answering in the affirmative a question raised in [26] (question 4.6). After that, a brief discussion on the connection with appropriate definitions of the weak maximum principle is included. The final section will be devoted to the existence of Evans type potentials in the particular setting of radially symmetric manifolds. To fix the ideas, we state the main theorem in the "easy case" of the p-Laplacian and then introduce the more general (and more technical) operators to which our theorem applies. Recall that for a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, the p-Laplacian Δ_p is defined weakly as

$$\int_\Omega \phi \Delta_p u = -\int_\Omega |\nabla u|^{p-2} \langle \nabla u | \nabla \phi \rangle \,, \tag{1.3}$$

where $\phi \in C^\infty_c(\Omega)$ and integration is with respect to the Riemannian measure.

Theorem 1.2. *Let M be a Riemannian manifold and let $p > 1$, $\lambda \geq 0$. Then, the following conditions are equivalent.*

(W) *The weak maximum principle for C^0 holds for Δ_p. That is, for every non-constant $u \in C^0(M) \cap W^{1,p}_{\mathrm{loc}}(M)$ with $u^\star = \sup_M u < \infty$ and for every $\eta < u^\star$ we have*

$$\inf_{\Omega_\eta} \Delta_p u \leq 0 \qquad (< 0 \ \textit{if} \ \lambda = 0) \tag{1.4}$$

weakly on $\Omega_\eta = u^{-1}\{(\eta, +\infty)\}$.

(L) *Every non-negative, $L^\infty \cap W^{1,p}_{\mathrm{loc}}$ solution u of $\Delta_p u - \lambda u^{p-1} \geq 0$ is constant (hence zero if $\lambda > 0$).*

(K) *For every compact K with smooth boundary, there exists an exhaustion $w \in C^0(M) \cap W^{1,p}_{\mathrm{loc}}(M)$ such that*

$$w > 0 \quad \text{on } M \backslash K, \qquad w = 0 \quad \text{on } K, \qquad \Delta_p w - \lambda w^{p-1} \le 0 \quad \text{on } M \setminus K.$$

Up to some minor changes, the implications $(W) \Leftrightarrow (L)$ and $(K) \Rightarrow (L)$ have been shown in [25], Theorem A, where it is also proved that, in (W) and (L), u can be equivalently restricted to the class $C^1(M)$. In this respect, see also [26], Section 2. On the other hand, the second author in [33] has proved that $(L) \Rightarrow (K)$ when $\lambda = 0$. The proof developed in this article covers both the case $\lambda = 0$ and $\lambda > 0$, is easier and more straightforward and, above all, does not depend on some features which are typical of the p-Laplacian.

2. Definitions and main theorems

Notational conventions. We set $\mathbb{R}^+ = (0, +\infty)$, $\mathbb{R}^+_0 = [0, +\infty)$, and $\mathbb{R}^-$, $\mathbb{R}^-_0$ accordingly; for a function u defined on some set Ω, $u^\star = \mathrm{esssup}_\Omega u$ and $u_\star = \mathrm{essinf}_\Omega u$; we will write $K \Subset \Omega$ whenever the set K has compact closure in Ω; $\mathrm{Lip}_{\mathrm{loc}}(M)$ denotes the class of locally Lipschitz functions on M; with $u \in \mathrm{H\ddot{o}l}_{\mathrm{loc}}(M)$ we mean that, for every $\Omega \Subset M$, $u \in C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1]$ possibly depending on Ω. Finally, we will adopt the symbol $Q \doteq \ldots$ to define the quantity Q as $\ldots$.

In order for our techniques to work, we will consider quasilinear operators of the following form. Let $A : TM \to TM$ be a Carathéodory map, i. e., if $\pi : TM \to M$ is the bundle projection, $\pi \circ A = \pi$. Moreover every representation $\tilde{A}$ of A in local charts satisfies

- $\tilde{A}(x, \cdot)$ continuous for a.e. $x \in M$,
- $\tilde{A}(\cdot, v)$ measurable for every $v \in \mathbb{R}^m$.

Note that every continuous bundle map satisfies these assumptions. Furthermore, let $B : M \times \mathbb{R} \to \mathbb{R}$ be of Carathéodory type, that is, $B(\cdot, t)$ is measurable for every fixed $t \in \mathbb{R}$, and $B(x, \cdot)$ is continuous for a.e. $x \in M$. We shall assume that there exists $p > 1$ such that, for each fixed open set $\Omega \Subset M$, the following set of assumptions $\mathscr{S}$ is met:

(A1) $\qquad \langle A(X)|X\rangle \ge a_1|X|^p \quad \forall\, X \in TM,$

(A2) $\qquad |A(X)| \le a_2|X|^{p-1} \quad \forall X \in TM,$

(Mo) $\qquad A$ is strictly monotone, i.e. $\langle A(X) - A(Y)|X - Y\rangle_p \ge 0$ for every $x \in M,\ X, Y \in T_xM,$ with equality if and only if $X = Y$,

(B1) $\qquad |B(x,t)| \le b_1 + b_2|t|^{p-1} \quad$ for $t \in \mathbb{R}$,

(B2) $\qquad$ for a.e. $x,\ B(x, \cdot)$ is monotone non-decreasing,

(B3) $\qquad$ for a.e. $x,\ B(x,t)t \ge 0$,

where a_1, a_2, b_1, b_2 are positive constants possibly depending on Ω. As explained in Remark 4.2, we could state our main theorem relaxing condition B1 to

(B1+) $$|B(x,t)| \le b(t) \quad \text{for } t \in \mathbb{R}$$

for some positive and finite function b. However for the moment we assume B1 to avoid some complications in the notation, and explain later how to extend our result to this more general case.

We define the operators $\mathcal{F}, \mathcal{A}, \mathcal{B} : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^\star$ by setting

$$\begin{array}{llll} \mathcal{A}: & u & \longmapsto & \left[\phi \in W^{1,p}(\Omega) \longmapsto \int_\Omega \langle A(\nabla u) | \nabla \phi \rangle \right], \\ \mathcal{B}: & u & \longmapsto & \left[\phi \in W^{1,p}(\Omega) \longmapsto \int_\Omega B(x, u(x))\phi \right], \\ \mathcal{F} \doteq \mathcal{A} + \mathcal{B}. & & & \end{array} \tag{2.1}$$

With these assumptions, it can be easily verified that both $\mathcal{A}$ and $\mathcal{B}$ map to continuous linear functionals on $W^{1,p}(\Omega)$ for each fixed $\Omega \Subset M$. We define the operators $L_\mathcal{A}$, $L_\mathcal{F}$ according to the distributional equality:

$$\int_M \phi L_\mathcal{A} u \doteq -\langle \mathcal{A}(u), \phi \rangle, \qquad \int_M \phi L_\mathcal{F} u \doteq -\langle \mathcal{F}(u), \phi \rangle$$

for every $u \in W^{1,p}_{\mathrm{loc}}(M)$ and $\phi \in C^\infty_c(M)$, where $\langle , \rangle$ is the duality. In other words, in the weak sense

$$L_\mathcal{F} u = \mathrm{div}(A(\nabla u)) - B(x, u) \qquad \forall\ u \in W^{1,p}_{\mathrm{loc}}(M).$$

Example 2.1. The p-Laplacian defined in (1.3), corresponding to the choices $A(X) \doteq |X|^{p-2}X$ and $B(x,t) \doteq 0$, satisfies all the assumptions in $\mathscr{S}$ for each $\Omega \Subset M$. Another admissible choice of B is $B(x,t) \doteq \lambda |t|^{p-2}t$, where $\lambda \geq 0$. For such a choice,

$$L_\mathcal{F} u = \Delta_p u - \lambda |u|^{p-2} u \tag{2.2}$$

is the operator of Theorem 1.2. However, we stress that in $\mathscr{S}$ we require no homogeneity condition either on A or on B.

Example 2.2. More generally, as in [25] and [29], for each function $\varphi \in C^0(\mathbb{R}^+_0)$ such that $\varphi > 0$ on $\mathbb{R}^+$, $\varphi(0) = 0$, and for each symmetric, positive definite 2-covariant continuous tensor field $h \in \Gamma(\mathrm{Sym}_2(TM))$, we can consider differential operators of type

$$L_{\varphi,h} u \doteq \mathrm{div}\left(\frac{\varphi(|\nabla u|)}{|\nabla u|} h(\nabla u, \cdot)^\sharp \right),$$

where $\sharp$ is the musical isomorphism. Due to the continuity and the strict positivity of h, conditions (A1) and (A2) in $\mathscr{S}$ can be rephrased as

$$a_1 t^{p-1} \leq \varphi(t) \leq a_2 t^{p-1}. \tag{2.3}$$

Furthermore, if $\varphi \in C^1(\mathbb{R}^+)$, a sufficient condition for (Mo) to hold is given by

$$\frac{\varphi(t)}{t} h(X,X) + \left(\varphi'(t) - \frac{\varphi(t)}{t} \right) \langle Y | X \rangle\, h(Y,X) > 0 \tag{2.4}$$

for every X, Y with $|X| = |Y| = 1$. The reason why it implies the strict monotonicity can be briefly justified as follows: for $L_{\varphi,h}$, (Mo) is equivalent to requiring

$$\frac{\varphi(|X|)}{|X|} h(X, X-Y) - \frac{\varphi(|Y|)}{|Y|} h(Y, X-Y) > 0 \qquad \text{if} \ \ X \neq Y. \tag{2.5}$$

In the non-trivial case when X and Y are not proportional, the segment $Z(t) = Y + t(X-Y)$, $t \in [0,1]$, does not pass through zero so that

$$F(t) = \frac{\varphi(|Z|)}{|Z|} h(Z, Z')$$

is C^1. Condition (2.4) implies that $F'(t) > 0$. Hence, integrating we get $F(1) > F(0)$, that is, (2.5). We observe that, if h is the metric tensor, the strict monotonicity is satisfied whenever φ is strictly increasing on $\mathbb{R}^+$ even without any differentiability assumption on φ.

Example 2.3. Even more generally, if A is of class C^1, a sufficient condition for the monotonicity of A has been considered in [1], Section 5 (see the proof of Theorem 5.3). Indeed, the authors required that, for every $x \in M$ and every $X \in T_xM$, the differential of the map $A_x : T_xM \to T_xM$ at the point $X \in T_xM$ is positive definite as a linear endomorphism of $T_X(T_xM)$. This is the analogue, for Riemannian manifolds, of Proposition 2.4.3 in [28].

We recall the concept of subsolutions and supersolutions for $L_{\mathcal{F}}$.

Definition 2.4. We say that $u \in W^{1,p}_{\mathrm{loc}}(M)$ solves $L_{\mathcal{F}}u \geq 0$ (resp. ≤ 0, $= 0$) weakly on M if, for every non-negative $\phi \in C^\infty_c(M)$, $\langle \mathcal{F}(u), \phi\rangle \leq 0$ (resp., ≥ 0, $= 0$). Explicitly,

$$\int_M \langle A(\nabla u)|\nabla\phi\rangle + \int_M B(x,u)\phi \leq 0 \quad (\text{resp., } \geq 0, = 0).$$

Solutions of $L_{\mathcal{F}}u \geq 0$ (resp., ≤ 0, $= 0$) are called (weak) subsolutions (resp., supersolutions, solutions) for $L_{\mathcal{F}}$.

Remark 2.5. When defining solutions of $L_{\mathcal{F}}u = 0$, we can drop the requirement that the test function ϕ be non-negative. This can be easily seen by splitting ϕ into its positive and negative parts and using a density argument.

Remark 2.6. Note that, since B is Carathéodory, (B3) implies that $B(x, 0) = 0$ a.e. on M. Therefore, the constant function $u = 0$ solves $L_{\mathcal{F}}u = 0$. Again by (B3), positive constants are supersolutions.

Following [25] and [26], we present the analogues of the L^∞-Liouville property and the Khas'minskii property for the non-linear operators constructed above.

Definition 2.7. Let M be a Riemannian manifold, and let $\mathcal{A}, \mathcal{B}, \mathcal{F}$ be as above.

- We say that the L^∞-Liouville property (L) for L^∞ (respectively, $\mathrm{H\ddot{o}l}_{\mathrm{loc}}$) functions holds for the operator $L_{\mathcal{F}}$ if every $u \in L^\infty(M) \cap W^{1,p}_{\mathrm{loc}}(M)$ (respectively, $\mathrm{H\ddot{o}l}_{\mathrm{loc}}(M) \cap W^{1,p}_{\mathrm{loc}}(M)$) is essentially bounded, satisfying $u \geq 0$, and $L_{\mathcal{F}}u \geq 0$ is constant.
- We say that the Khas'minskii property (K) holds for $L_{\mathcal{F}}$ if, for every pair of open sets $K \Subset \Omega \Subset M$ with Lipschitz boundary, and every $\varepsilon > 0$, there exists an exhaustion function
$$w \in C^0(M) \cap W^{1,p}_{\mathrm{loc}}(M)$$
such that
$$\begin{aligned} &w > 0 \ \text{ on } M\backslash K, && w = 0 \ \text{ on } K,\\ &w \leq \varepsilon \ \text{ on } \Omega\backslash K, && L_{\mathcal{F}}w \leq 0 \text{ on } M\backslash K. \end{aligned}$$
A function w with such properties will be called a Khas'minskii potential relative to the triple (K, Ω, ε).
- A Khas'minskii potential w relative to some triple (K, Ω, ε) is called an Evans potential if $L_{\mathcal{F}}w = 0$ on $M\backslash K$. The operator $L_{\mathcal{F}}$ has the Evans property (E) if there exists an Evans potential for every triple (K, Ω, ε).

The main result in this paper is the following.

Theorem 2.8. *Let M be a Riemannian manifold, and let A, B satisfy the set of assumptions $\mathscr{S}$, with* (B1+) *instead of* (B1)*. Define $\mathcal{A}, \mathcal{B}, \mathcal{F}$ as in* (2.1)*, and $L_\mathcal{A}, L_\mathcal{F}$ accordingly. Then, conditions (L) for* $\mathrm{Höl}_{\mathrm{loc}}$*, (L) for L^∞ and (K) are equivalent.*

Remark 2.9. It should be observed that if $L_\mathcal{F}$ is homogeneous, as in (2.2), the Khas'minskii condition simplifies considerably as in (K) of Theorem 1.2. Indeed, the fact that δw is still a supersolution for every $\delta > 0$, and the continuity of w, allow us to get rid of Ω and ε.

Next, in Section 5 we briefly describe in which way (L) and (K) are related to the concepts of weak maximum principle and parabolicity. Such a relationship has been deeply investigated in [24], [25], whose ideas and proofs we will follow closely. With the aid of Theorem 2.8, we will be able to prove Theorem 2.12. To state it, we shall restrict ourselves to a particular class of potentials $B(x,t)$, those of the form $B(x,t) = b(x)f(t)$ with

$$\begin{aligned} &b, b^{-1} \in L^\infty_{\mathrm{loc}}(M), \quad b > 0 \text{ a.e. on } M;\\ &f \in C^0(\mathbb{R}), \quad f(0) = 0, \quad f \text{ is non-decreasing on } \mathbb{R}. \end{aligned} \tag{2.6}$$

Clearly, B satisfies (B1+), (B2) and (B3). As for A, we require (A1) and (A2), as before.

Definition 2.10. Let A, B be as above, define $\mathcal{A}, \mathcal{B}, \mathcal{F}$ as in (2.1) and $L_\mathcal{A}, L_\mathcal{F}$ accordingly.

(W) We say that $b^{-1}L_\mathcal{A}$ satisfies the weak maximum principle for C^0 functions if, for every $u \in C^0(M) \cap W^{1,p}_{\mathrm{loc}}(M)$ such that $u^\star < +\infty$, and for every $\eta < u^\star$,

$$\inf_{\Omega_\eta} b^{-1}L_\mathcal{A} u \le 0 \qquad \text{weakly on } \ \Omega_\eta = u^{-1}\{(\eta, +\infty)\}.$$

(W_{pa}) We say that $b^{-1}L_\mathcal{A}$ is parabolic if, for every non-constant $u \in C^0(M) \cap W^{1,p}_{\mathrm{loc}}(M)$ such that $u^\star < +\infty$, and for every $\eta < u^\star$,

$$\inf_{\Omega_\eta} b^{-1}L_\mathcal{A} u < 0 \qquad \text{weakly on } \ \Omega_\eta = u^{-1}\{(\eta, +\infty)\}.$$

- We say that $\mathcal{F}$ is of type 1 if, in the potential $B(x,t)$, the factor $f(t)$ satisfies $f > 0$ on $\mathbb{R}^+$. Otherwise, when $f = 0$ on some interval $[0, T]$, $\mathcal{F}$ is said to be of type 2.

Remark 2.11. $\inf_{\Omega_\eta} b^{-1}L_\mathcal{A} u \le 0$ weakly means that, for every $\varepsilon > 0$, there exists $0 \le \phi \in C^\infty_c(\Omega_\eta)$, $\phi \not\equiv 0$ such that

$$-\langle \mathcal{A}(u), \phi\rangle \ < \varepsilon \int b\phi.$$

Similarly, with $\inf_{\Omega_\eta} b^{-1}L_\mathcal{A} u < 0$ weakly we mean that there exist $\varepsilon > 0$ and $0 \le \phi \in C^\infty_c(\Omega_\eta)$, $\phi \not\equiv 0$ such that $-\langle \mathcal{A}(u), \phi\rangle \ < -\varepsilon \int b\phi$.

Theorem 2.12. *Under the assumptions* (2.6) *for $B(x,t) = b(x)f(t)$, and* (A1), (A2) *for A, the following properties are equivalent:*

- *The operator $b^{-1}L_\mathcal{A}$ satisfies (W).*

- *Property* (L) *holds for some (hence any) operator* $\mathcal{F}$ *of type* 1.
- *Property* (K) *holds for some (hence any) operator* $\mathcal{F}$ *of type* 1.

Furthermore, under the same assumptions, the next equivalence holds:

- *The operator* $b^{-1}L_{\mathcal{A}}$ *is parabolic.*
- *Property* (L) *holds for some (hence any) operator* $\mathcal{F}$ *of type* 2.
- *Property* (K) *holds for some (hence any) operator* $\mathcal{F}$ *of type* 2.

In Section 6, we address the question as to whether (W), (K), (L) are equivalent to the Evans property (E). Indeed, it should be observed that, in Theorem 2.8, no growth control on B as a function of t is required at all. On the contrary, as we will see, the validity of the Evans property forces some precise upper bound for its growth. To better grasp what we shall expect, we will restrict ourselves to the case of radially symmetric manifolds. For the statements of the main results, we refer the reader directly to Section 6 covering the situation.

3. Technical tools

In this section we introduce some technical tools, such as the obstacle problem, that will be crucial to the proof of our main theorems. In doing so, a number of basic results from literature are recalled. We have decided to add a full proof to those results for which we have not found any reference converting the situation at hand. Our aim is to keep the paper basically self-contained and to give the non-expert reader interested in this topic a brief overview of the standard technical tricks. Throughout this section, we will always assume that the assumptions in $\mathscr{S}$ are satisfied, if not explicitly stated. First, we state some basic results on subsolutions-supersolutions such as the comparison principle, which follows from the monotonicity of A and B.

Proposition 3.1. *Assume w and s are a supersolution and a subsolution defined on Ω. If $\min\{w-s,0\}\in W_0^{1,p}(\Omega)$, then $w\geq s$ a.e. in Ω.*

Proof. This theorem and its proof, which follows quite easily using the right test function in the definition of a supersolution, are standard in potential theory. For a detailed proof see [1], Theorem 4.1. □

Next, we observe that A, B satisfy all the assumptions for the subsolution-supersolution method in [14] to be applicable.

Theorem 3.2 ([14], Theorems 4.1, 4.4 and 4.7)**.** *Let $\phi_1,\phi_2\in L^\infty_{\mathrm{loc}}\cap W^{1,p}_{\mathrm{loc}}$ be, respectively, a subsolution and a supersolution for $L_{\mathcal{F}}$ on M, and suppose that $\phi_1\leq\phi_2$ a.e. on M. Then, there is a solution $u\in L^\infty_{\mathrm{loc}}\cap W^{1,p}_{\mathrm{loc}}$ of $L_{\mathcal{F}}u=0$ satisfying $\phi_1\leq u\leq\phi_2$ a.e. on M.*

A fundamental property is the strong maximum principle, which follows from the next Harnack inequality

Theorem 3.3 ([28], Theorems 7.1.2, 7.2.1 and 7.4.1)**.** *Let $u\in W^{1,p}_{\mathrm{loc}}(M)$ be a non-negative solution of $L_{\mathcal{A}}u\leq 0$. Let the assumptions in $\mathscr{S}$ be satisfied. Fix a relatively compact open set $\Omega\Subset M$.*

(i) *Suppose that $1<p\leq m$, where $m=\dim M$. Then, for every ball $B_{4R}\subset\Omega$ and for every $s\in(0,(p-1)m/(m-p))$, there exists a constant C depending*

on R, on the geometry of B_{4R}, on m and on the parameters a_1, a_2 in $\mathscr{S}$ such that

$$\|u\|_{L^s(B_{2R})} \le C\Big(\mathrm{essinf}_{B_{2R}} u\Big).$$

(i) *Suppose that $p > m$. Then, for every ball $B_{4R} \subset \Omega$, there exists a constant C depending on R, on the geometry of B_{4R}, on m and on the parameters a_1, a_2 in $\mathscr{S}$ such that*

$$\mathrm{esssup}_{B_R} u \le C\Big(\mathrm{essinf}_{B_R} u\Big).$$

In particular, for every $p > 1$, each non-negative solution u of $L_{\mathcal{A}} u \le 0$ on M is such that either $u = 0$ on M or $\mathrm{essinf}_\Omega u > 0$ *for every relatively compact set Ω.*

Remark 3.4. We now write a few words to comment on the Harnack inequalities quoted from [28]. In our assumptions $\mathscr{S}$, the functions $\bar a_2, \bar a, b_1, b_2, b$ in Chapter 7, (7.1.1) and (7.1.2) and the function a in the monotonicity inequality (6.1.2) can be chosen to be identically zero. Thus, in Theorems 7.1.2 and 7.4.1 the quantity $k(R)$ is zero. This gives no non-homogeneous term in the Harnack inequality, which is essential for us. For this reason, we cannot weaken (A2) to

$$|A(X)| \le a_2|X|^{p-1} + \bar a$$

locally on Ω, since the presence of non-zero $\bar a$ implies that $k(R) > 0$. It should be observed that Theorem 7.1.2 is only stated for $1 < p < m$, but, as observed at the beginning of Section 7.4, the proof can be adapted to cover the case $p = m$.

Remark 3.5. In the rest of the paper, we will only use the fact that either $u \equiv 0$ or $u > 0$ on M, that is, the strong maximum principle. It is worth observing that, for the operators $L_{\mathcal{A}} = L_{\varphi,h}$ described in Example 2.2, very general strong maximum principles for C^1 or $\mathrm{Lip}_{\mathrm{loc}}$ solutions of $L_{\varphi,h} u \le 0$ on Riemannian manifolds have been obtained in [27] (see Theorem 1.2 when h is the metric tensor and Theorems 5.4 and 5.6 for the general case). In particular, if h is the metric tensor the sole requirements

$$\varphi \in C^0(\mathbb{R}^+_0), \quad \varphi(0) = 0, \quad \varphi > 0 \text{ on } \mathbb{R}^+, \quad \varphi \text{ in strictly increasing on } \mathbb{R}^+ \tag{3.1}$$

are enough for the strong maximum principle to hold for C^1 solutions of $L_\varphi u \le 0$. Hence, for instance for L_φ, the two-sided bound (2.3) on φ can be weakened to any bound ensuring that the comparison and strong maximum principles hold, the subsoluton-supersolution method is applicable and the obstacle problem has a solution. For instance, besides (3.1), the requirement

$$\varphi(0) = 0, \quad a_1 t^{p-1} \le \varphi(t) \le a_2 t^{p-1} + a_3 \tag{3.2}$$

is enough for Theorems 3.1 and 3.2, and it also suffices for the obstacle problem to admit a unique solution, as the reader can infer from the proof of Theorem 3.11.

Remark 3.6. Regarding the above observation, if φ is merely continuous, then even solutions of $L_\varphi u = 0$ are not expected to be C^1, or even $\mathrm{Lip}_{\mathrm{loc}}$. Indeed, in our assumptions the optimal regularity for u is (locally) some Hölder class; see Theorem 3.7. If $\varphi \in C^1(\mathbb{R}^+)$ is more regular, then we can avail ourselves of the regularity result in [32] to go even beyond the C^1 class. Indeed, under the assumptions

$$\gamma(k+t)^{p-2} \le \min\left(\varphi'(t), \frac{\varphi(t)}{t}\right) \le \max\left(\varphi'(t), \frac{\varphi(t)}{t}\right) \le \Gamma(k+t)^{p-2},$$

for some $k \geq 0$ and some positive constants $\gamma \leq \Gamma$, then each solution of $L_\varphi u = 0$ is in some class $C^{1,\alpha}$ on each relatively compact set Ω, where $\alpha \in (0,1)$ may depend on Ω. When h is not the metric tensor, the condition on φ and h is more complicated, and we refer the reader to [25] (in particular, see (0.1) (v), (vi) p. 803).

Part of the regularity properties that we need are summarized in the following theorem.

Theorem 3.7. *Let the assumptions in $\mathscr{S}$ be satisfied.*

(i) [[18]*, Theorem* 4.8] *If u solves $L_{\mathcal{F}}u \leq 0$ on some open set Ω, then there exists a representative in $W^{1,p}(\Omega)$ which is lower semicontinuous.*

(ii) [[16]*, Theorem* 1.1*, p.* 251] *If $u \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$ is a bounded solution of $L_{\mathcal{F}}u = 0$ on Ω, then there exists $\alpha \in (0,1)$ depending on the geometry of Ω, on the constants in $\mathscr{S}$ and on $\|u\|_{L^\infty(\Omega)}$ such that $u \in C^{0,\alpha}(\Omega)$. Furthermore, for every $\Omega_0 \Subset \Omega$, there exists $C = C(\gamma, \mathrm{dist}(\Omega_0, \partial\Omega))$ such that*
$$\|u\|_{C^{0,\alpha}(\Omega_0)} \leq C.$$

Remark 3.8. As for (i), it is worth observing that, in our assumption, both b_0 and $\mathbf{a}$ in the statement of [18], Theorem 4.8 are identically zero. Although we will not need the following properties, it is worth noting that any u solving $L_{\mathcal{F}}u \leq 0$ has a Lebesgue point everywhere and is also p-finely continuous (where finite).

Next, this simple elliptic estimate for locally bounded supersolutions is useful:

Proposition 3.9. *Let u be a bounded solution of $L_{\mathcal{F}}u \leq 0$ on Ω. Then, for every relatively compact, open set $\Omega_0 \Subset \Omega$ there is a constant $C > 0$ depending on p, Ω, Ω_0 and on the parameters in $\mathscr{S}$ such that*
$$\|\nabla u\|_{L^p(\Omega_0)} \leq C(1 + \|u\|_{L^\infty(\Omega)}).$$

Proof. Given a supersolution u, the monotonicity of B assures that for every positive constant c $u + c$ is also a supersolution, so without loss of generality we may assume that $u_\star \geq 0$. Thus, $u^\star = \|u\|_{L^\infty(\Omega)}$.

Shortly, with $\|\cdot\|_p$ we denote the L^p norm on Ω and with C we denote a positive constant depending on p, Ω and on the parameters in $\mathscr{S}$, that may vary from place to place. Let $\eta \in C^\infty_c(\Omega)$ be such that $0 \leq \eta \leq 1$ on Ω and $\eta = 1$ on Ω_0. Then, we use the non-negative function $\phi = \eta^p(u^\star - u)$ in the definition of a supersolution to obtain, after some manipulation and from (A1), (A2) and (B3),

$$a_1 \int_\Omega \eta^p |\nabla u|^p \quad \leq \quad pa_2 \int_\Omega |\nabla u|^{p-1}\eta^{p-1}(u^\star - u)|\nabla\eta| + \int_\Omega \eta^p B(x,u)u^\star. \tag{3.3}$$

Using (B1), the integral involving B is roughly estimated as follows:

$$\int_\Omega \eta^p B(x,u)u^\star \leq |\Omega|(b_1 u^\star + b_2(u^\star)^p) \leq C(1+u^\star)^p, \tag{3.4}$$

where the last inequality follows by applying Young's inequality on the first addendum. As for the term involving $|\nabla\eta|$, using $(u^\star - u) \leq u^\star$ and again Young's inequality $|ab| \leq |a|^p/(p\varepsilon^p) + \varepsilon^q|b|^q/q$, we obtain

$$\begin{aligned} pa_2 \int_\Omega \big(|\nabla u|^{p-1}\eta^{p-1}(u^\star - u)|\nabla\eta|\big) \quad &\leq \quad pa_2 \int_\Omega \big(|\nabla u|^{p-1}\eta^{p-1}\big)\big(u^\star|\nabla\eta|\big) \\ &\leq \quad \tfrac{a_2}{\varepsilon^p}\left\|\eta\nabla u\right\|_p^p + \tfrac{a_2 p\varepsilon^q}{q}\left\|\nabla\eta\right\|_p^p (u^\star)^p. \end{aligned} \tag{3.5}$$

Choosing ε such that $a_2\varepsilon^{-p} = a_1/2$, inserting (3.4) and (3.5) into (3.3) and rearranging we obtain

$$\frac{a_1}{2}\,\|\eta\nabla u\|_p^p \leq C\Big[1 + (1 + \|\nabla\eta\|_p^p)(u^\star)^p\Big].$$

Since $\eta = 1$ on Ω_0 and $\|\nabla\eta\|_p \leq C$, taking the p-root the desired estimate follows. □

Remark 3.10. We observe that when $B \neq 0$ we cannot apply the technique of [9], Lemma 3.27 to get a Caccioppoli type inequality for bounded, non-negative supersolutions. The reason is that subtracting a positive constant from a supersolution does not yield, for general $B \neq 0$, a supersolution. It should be stressed that, however, when $p \leq m$ a refined Caccioppoli inequality for a supersolution has been given in [18], Theorem 4.4.

Now, we fix our attention on the obstacle problem. There are many references regarding this subject (for example see [18], Chapter 5 or [9], Chapter 3 in the case $B = 0$). As often happens, notation can be quite different from one reference to another. Here we try to adapt the conventions used in [9], and for the reader's convenience we also sketch some of the proofs.

First of all, we give some definitions. Given a function $\psi : \Omega \to \mathbb{R} \cup \pm\infty$ and given $\theta \in W^{1,p}(\Omega)$, we define the closed convex set

$$\mathcal{K}_{\psi,\theta} \doteq \{f \in W^{1,p}(\Omega) \mid \ f \geq \psi \ \text{ a.e. and } \ f - \theta \in W_0^{1,p}(\Omega)\}.$$

Loosely speaking, θ determines the boundary condition for the solution u, while ψ is the "obstacle" function. Most of the time, obstacle and boundary functions coincide, and in this case we use the convention $\mathcal{K}_\theta \doteq \mathcal{K}_{\theta,\theta}$. We say that $u \in \mathcal{K}_{\psi,\theta}$ solves the obstacle problem if for every $\varphi \in \mathcal{K}_{\psi,\theta}$,

$$\langle\mathcal{F}(u), \varphi - u\rangle \ \geq 0. \tag{3.6}$$

Note that for every non-negative $\phi \in C_c^\infty(\Omega)$ the function $\varphi = u+\phi$ belongs to $\mathcal{K}_{\psi,\theta}$, and this implies that the solution to the obstacle problem is always a supersolution. Note also that if we choose $\psi = -\infty$, we get the standard Dirichlet problem with Sobolev boundary value θ for the operator $\mathcal{F}$. In fact, in this case any test function $\phi \in C_c^\infty(\Omega)$ verifies $u\pm\phi \in \mathcal{K}_{\psi,\theta}$, and so the inequality in (3.6) becomes an equality. Next, we address the solvability of the obstacle problem.

Theorem 3.11. *Under the assumptions $\mathscr{S}$, if Ω is relatively compact and $\mathcal{K}_{\psi,\theta}$ is non-empty, then there exists a unique solution to the relative obstacle problem.*

Proof. The proof is basically the same if we assume $B = 0$, as in [9], Appendix 1. In particular, it is an application of the Stampacchia theorem; see for example Corollary III.1.8 in [13]. To apply the theorem, we shall verify that $\mathcal{K}_{\psi,\theta}$ is closed and convex, which follows straightforwardly from its very definition, and that $\mathcal{F} : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^\star$ is weakly continuous, monotone and coercive. Monotonicity is immediate by properties (Mo), (B2). To prove that $\mathcal{F}$ is weakly continuous, we take a sequence $u_i \to u$ in $W^{1,p}(\Omega)$. By using (A2) and (B1), we deduce from (2.1) that

$$|\langle\mathcal{F}(u_i), \phi\rangle| \leq \Big((a_2 + b_2)\,\|u_i\|_{W^{1,p}(\Omega)}^{p-1} + b_1|\Omega|^{\frac{p-1}{p}}\Big)\,\|\phi\|_{W^{1,p}(\Omega)}\,.$$

Hence the $W^{1,p}(\Omega)^\star$ norm of $\{\mathcal{F}(u_i)\}$ is bounded. Since $W^{1,p}(\Omega)^\star$ is reflexive, we can extract from any subsequence a weakly convergent sub-subsequence $\mathcal{F}(u_k) \rightharpoonup z$

in $W^{1,p}(\Omega)^\star$, for some z. From $u_k \to u$ in $W^{1,p}(\Omega)$, by the Riesz theorem we get (up to a further subsequence) $(u_k, \nabla u_k) \to (u, \nabla u)$ pointwise on Ω, and since the maps

$$X \longmapsto A(X), \qquad t \longmapsto B(x,t)$$

are continuous, then necessarily $z = \mathcal{F}(u)$. Since this is true for every weakly convergent subsequence $\{\mathcal{F}(u_k)\}$, we deduce that the whole $\mathcal{F}(u_i)$ converges weakly to $\mathcal{F}(u)$. This proves the weak continuity of $\mathcal{F}$.

Coercivity on $\mathcal{K}_{\psi,\theta}$ follows if we fix any $\varphi \in \mathcal{K}_{\psi,\theta}$ and consider a diverging sequence $\{u_i\} \subset \mathcal{K}_{\psi,\theta}$ and if we calculate

$$\frac{\langle \mathcal{F}(u_i) - \mathcal{F}(\varphi) | u_i - \varphi\rangle}{\|u_i - \varphi\|_{W^{1,p}(\Omega)}} \overset{\text{(B3)}}{\geq} \frac{\langle \mathcal{A}(u_i) - \mathcal{A}(\varphi) | u_i - \varphi\rangle}{\|u_i - \varphi\|_{W^{1,p}(\Omega)}}$$

$$\overset{\text{(A1),(A2)}}{\geq} \frac{a_1\left(\|\nabla u_i\|_p^p + \|\nabla\varphi\|_p^p\right) - a_2\left(\|\nabla u_i\|_p^{p-1}\|\nabla\varphi\|_p + \|\nabla u_i\|_p\|\nabla\varphi\|_p^{p-1}\right)}{\|u_i - \varphi\|_{W^{1,p}(\Omega)}}.$$

This last quantity tends to infinity as i goes to infinity, thanks to the Poincaré inequality on Ω:

$$\|u_i - \varphi\|_{L^p(\Omega)} \leq C\,\|\nabla u_i - \nabla\varphi\|_{L^p(\Omega)},$$

which leads to $\|\nabla u_i\|_{L^p(\Omega)} \geq C_1 + C_2\,\|u_i\|_{W^{1,p}(\Omega)}$ for some constants C_1, C_2, where C_1 depends on $\|\varphi\|_{W^{1,p}(\Omega)}$. □

A very important characterization of the solution of the obstacle problem is a corollary to the following comparison, whose proof closely follows that of the comparison proposition, Proposition 3.1.

Proposition 3.12. *If u is a solution to the obstacle problem $\mathcal{K}_{\psi,\theta}$ and if w is a supersolution such that $\min\{u,w\} \in \mathcal{K}_{\psi,\theta}$, then $u \leq w$ a.e.*

Proof. Define $U = \{x|\ u(x) > w(x)\}$. Suppose by contradiction that U has positive measure. Since u solves the obstacle problem, using (3.6) with the function $\varphi = \min\{u,w\} \in \mathcal{K}_{\psi,\theta}$ we get

$$(3.7) \qquad 0 \leq \langle \mathcal{F}(u), \varphi - u\rangle = \int_U \langle A(\nabla u)|\nabla w - \nabla u\rangle + \int_U B(x,u)(w-u).$$

On the other hand, applying the definition of the supersolution w with the test function $0 \leq \phi = u - \min\{u,w\} \in W_0^{1,p}(\Omega)$ we get

$$(3.8) \qquad 0 \leq \langle \mathcal{F}(w), \phi\rangle = \int_U \langle A(\nabla w)|\nabla u - \nabla w\rangle + \int_U B(x,w)(u-w).$$

Adding the two inequalities we get, by (Mo) and (B2),

$$0 \leq \int_U \langle A(\nabla u) - A(\nabla w)|\nabla w - \nabla u\rangle + \int_U \big[B(x,u) - B(x,w)\big](w-u) \leq 0.$$

Since A is strictly monotone, $\nabla u = \nabla w$ a.e. on U so that $\nabla((u-w)_+) = 0$ a.e. on Ω. Consequently, since U has positive measure, $u - w = c$ a.e. on Ω, where c is a positive constant. Since $\min\{u,w\} \in \mathcal{K}_{\psi,\theta}$, we get $c = u - w = u - \min\{u,w\} \in W_0^{1,p}(\Omega)$, a contradiction. □

Corollary 3.13. *The solution u to the obstacle problem in $\mathcal{K}_{\psi,\theta}$ is the smallest supersolution in $\mathcal{K}_{\psi,\theta}$.*

Proposition 3.14. *Let $w_1, w_2 \in W^{1,p}_{\mathrm{loc}}(M)$ be supersolutions for $L_\mathcal{F}$. Then, $w \doteq \min\{w_1, w_2\}$ is a supersolution. Analogously, if $u_1, u_2 \in W^{1,p}_{\mathrm{loc}}(M)$ are subsolutions for $L_\mathcal{F}$, then so is $u \doteq \max\{u_1, u_2\}$.*

Proof. Consider a smooth exhaustion $\{\Omega_j\}$ of M and the obstacle problem $\mathcal{K}_w$ on Ω_j. By Corollary 3.13 its solution is necessarily $w_{|\Omega_j}$, and so w is a supersolution that is locally the solution of an obstacle problem. As for the second part of the statement, define $\widetilde{A}(X) \doteq -A(-X)$ and $\widetilde{B}(x,t) \doteq -B(x,-t)$. Then, $\widetilde{A}, \widetilde{B}$ satisfy the set of assumptions $\mathscr{S}$. Denote by $\widetilde{\mathcal{F}}$ the operator associated to $\widetilde{A}, \widetilde{B}$. Then, it is easy to see that $L_\mathcal{F} u_i \geq 0$ if and only if $L_{\widetilde{\mathcal{F}}}(-u_i) \leq 0$, and to conclude it is enough to apply the first part with the operator $L_{\widetilde{\mathcal{F}}}$. □

The next version of the pasting lemma generalizes the previous proposition to the case when one of the supersolutions is not defined on the whole M. Before stating it, we need a preliminary definition. Given an open subset $\Omega \subset M$, possibly with non-compact closure, we recall that the space $W^{1,p}_{\mathrm{loc}}(\overline{\Omega})$ is the set of all functions u on Ω such that, for every relatively compact open set $V \Subset M$ that intersects Ω, $u \in W^{1,p}(\Omega \cap V)$. A function u in this space is, loosely speaking, well behaved on relatively compact portions of $\partial\Omega$, while no global control on the $W^{1,p}$ norm of u is assumed. Clearly, if Ω is relatively compact, $W^{1,p}_{\mathrm{loc}}(\overline{\Omega}) = W^{1,p}(\Omega)$. We identify the following subset of $W^{1,p}_{\mathrm{loc}}(\overline{\Omega})$, which we call $X^p_0(\Omega)$:

$$X^p_0(\Omega) = \begin{cases} u \in W^{1,p}_{\mathrm{loc}}(\overline{\Omega}) \text{ such that, for every open set } U \Subset M \text{ that} \\ \text{intersects } \Omega, \text{ there exists } \{\phi_n\}_{n=1}^{+\infty} \subset C^0(\overline{\Omega \cap U}) \cap W^{1,p}(\Omega \cap U), \\ \text{with } \phi_n \equiv 0 \text{ in a neighbourhood of } \partial\Omega, \text{ satisfying} \\ \varphi_n \to u \text{ in } W^{1,p}(\Omega \cap U) \text{ as } n \to +\infty. \end{cases} \tag{3.9}$$

If Ω is relatively compact, then $X^p_0(\Omega) = W^{1,p}_0(\Omega)$.

Remark 3.15. Observe that if $u \in C^0(\overline{\Omega}) \cap W^{1,p}_{\mathrm{loc}}(\overline{\Omega})$, then $u \in X^p_0(\Omega)$ if and only if $u = 0$ on $\partial\Omega$. This is the version, for non-compact domains Ω, of a standard result. However, for the convenience of the reader we briefly sketch the proof. Up to working with positive and negative parts separately, we can suppose that $u \geq 0$ on Ω. If $u = 0$ on $\partial\Omega$, then choosing the sequence $\phi_n = \max\{u - 1/n, 0\}$ it is easy to check that $u \in X^p_0(\Omega)$. Vice versa, if $u \in X^p_0(\Omega)$, let $x_0 \in \partial\Omega$ be any point. Choose $U_1 \Subset U_2 \Subset M$ such that $x_0 \in U_1$, and a sequence $\{\phi_n\} \in C^0(\overline{\Omega \cap U_2}) \cap W^{1,p}(\Omega \cap U_2)$ as in the definition of $X^p_0(\Omega)$. If $\psi \in C^\infty_c(U_2)$ is a smooth cut-off function such that $\psi = 1$ on U_1, then $\psi\phi_n \to \psi u$ in $W^{1,p}(\Omega \cap U_2)$. Since $\psi\phi_n$ is compactly supported in $\Omega \cap U_2$, then $\psi u \in W^{1,p}_0(\Omega \cap U_2)$. It is a standard fact that, in this case, $\psi u = 0$ on $\partial(\Omega \cap U_2)$. Since $x_0 \in \partial\Omega \cap U_2 \subset \partial(\Omega \cap U_2)$, $u(x_0) = u\psi(x_0) = 0$. By the arbitrariness of x_0, this shows that $u = 0$ on $\partial\Omega$.

Lemma 3.16. *Let $w_1 \in W^{1,p}_{\mathrm{loc}}(M)$ be a supersolution for $L_\mathcal{F}$, and let $w_2 \in W^{1,p}_{\mathrm{loc}}(\overline{\Omega})$ be a supersolution on some open set Ω with $\overline{\Omega} \subset M$, $\overline{\Omega}$ being possibly non-compact. Suppose that $\min\{w_2 - w_1, 0\} \in X^p_0(\Omega)$. Then, the function*

$$m \doteq \begin{cases} \min\{w_1, w_2\} & \text{on } \Omega, \\ w_1 & \text{on } M \setminus \Omega \end{cases}$$

is a supersolution for $L_{\mathcal{F}}$ on M. In particular, if further $w_1 \in C^0(M)$ and $w_2 \in C^0(\overline{\Omega})$, then m is a supersolution on M whenever $w_1 = w_2$ on $\partial\Omega$. A similar statement is valid for subsolutions, replacing min *with* max.

Proof. We first need to check that $m \in W^{1,p}_{\mathrm{loc}}(M)$. Let $U \Subset M$ be an open set. By assumption, there exists a sequence of functions $\{\phi_n\} \in C^0(\overline{\Omega \cap U}) \cap W^{1,p}(\Omega \cap U)$, each ϕ_n being zero in some neighbourhood of $\partial\Omega$, which converges in the $W^{1,p}$ norm to $\min\{w_2 - w_1, 0\}$. We can thus continuously extend ϕ_n on the whole U by setting $\phi_n = 0$ on $U \backslash \Omega$, and the resulting extension is in $W^{1,p}(U)$. Define $u = \min\{w_2 - w_1, 0\}\chi_\Omega$, where χ_Ω is the indicatrix function of Ω. Then, $\phi_n \to u$ in $W^{1,p}(U)$ so that $u \in W^{1,p}(U)$. It follows that $w_1 + \phi_n \in W^{1,p}(U)$ converges to $m = w_1 + u$, which shows that $m \in W^{1,p}(U)$. To prove that $L_{\mathcal{F}} m \le 0$ we use a technique similar to Proposition 3.12. Let $U \Subset M$ be a fixed relatively compact open set, and let s be the solution to the obstacle problem $\mathcal{K}_m$ on U. Then by Corollary 3.13 we have $s \le w_1$ a.e. on U, and so $s = w_1 = m$ on $U \backslash \Omega$. Since s solves the obstacle problem, using $\varphi = m$ in equation (3.6) we have

$$(3.10) \quad 0 \le \langle \mathcal{F}(s), m - s\rangle = \int_{\Omega \cap U} \langle A(\nabla s) | \nabla m - \nabla s\rangle + \int_{\Omega\cap U} B(x,s)(m-s).$$

On the other hand m is a supersolution in $\Omega \cap U$, being the minimum of two supersolutions, by Proposition 3.14. To apply the weak definition of $L_{\mathcal{F}} m \le 0$ on $\Omega \cap U$ to the test function $s - m$, we first claim that $s - m \in W^{1,p}_0(\Omega \cap U)$. Since we know that $s \le w_1$ on U, then on $\Omega \cap U$

$$0 \le s - m \le w_1 - \min\{w_2, w_1\} = -\min\{w_2 - w_1, 0\} \in X^p_0(\Omega).$$

The claim now follows by a standard result (see for example [9], Lemma 1.25), but for the sake of completeness we sketch the proof. Since $0 \le s - m \in W^{1,p}_0(U)$ by the definition of the obstacle problem, there exists a sequence of non-negative functions $\psi_n \in C^\infty_c(U)$ converging to $s - m$. We further consider the sequence $\{\phi_n\}$ of continuous functions, converging to $\min\{w_2 - w_1, 0\}$, defined at the beginning of this proof. Then, on $\Omega \cap U$, $0 \le s - m \le \lim_n \min\{-\phi_n, \psi_n\}$, where the limit is taken in $W^{1,p}(\Omega \cap U)$. Now, $\min\{-\phi_n, \psi_n\}$ has compact support in $\Omega \cap U$, and this proves the claim. Applying the definition of $L_{\mathcal{F}} m \le 0$ to the test function $s - m$ we get

$$(3.11) \quad 0 \le \langle \mathcal{F}(m), s - m\rangle = \int_{\Omega \cap U} \langle A(\nabla m) | \nabla s - \nabla m\rangle + \int_{\Omega\cap U} B(x,m)(s-m).$$

Summing inequalities (3.10) and (3.11), we conclude as in Proposition 3.12 that $\nabla(s - m) = 0$ in $\Omega \cap U$ with $s - m \in W^{1,p}_0(\Omega \cap U)$, and so the two functions there are equal. Since $s = w = m$ on $U \backslash \Omega$, then $m = s$ is a supersolution on U. The thesis follows by the arbitrariness of U. If further $w_1 \in C^0(M)$ and $w_2 \in C^0(\overline{\Omega})$, then the conclusion follows by Remark 3.15. The proof of the statement for subsolutions is obtained via the same trick as in Proposition 3.14. ☐

As for the regularity of solutions of the obstacle problem, we have

Theorem 3.17 ([18], Theorem 5.4 and Corollary 5.6)**.** *If the obstacle ψ is continuous in Ω, then the solution u to $\mathcal{K}_{\psi,\theta}$ has a continuous representative in the Sobolev sense. Furthermore, if $\psi \in C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1)$, then there exist $C, \beta > 0$ depending only on $p, \alpha, \Omega, \|u\|_{L^\infty(\Omega)}$ and on the parameters in $\mathscr{S}$ such that*

$$\|u\|_{C^{0,\beta}(\Omega)} \le C(1 + \|\psi\|_{C^{0,\alpha}(\Omega)}).$$

Remark 3.18. The interested reader should be advised that, in the notation of [18], b_0 and $\mathbf{a}$ are both zero with our assumptions. Stronger results, for instance $C^{1,\alpha}$ regularity, can be obtained from stronger requirements on ψ, A and B which are stated for instance in [18], Theorem 5.14.

In the proof of our main theorem, and to get some boundary regularity results, it will be important to see what happens on the set where the solution of the obstacle problem is strictly above the obstacle.

Proposition 3.19. *Let u be the solution of the obstacle problem $\mathcal{K}_{\psi,\theta}$ with continuous obstacle ψ. If $u > \psi$ on an open set D, then u is a solution of $L_{\mathcal{F}}u = 0$ on D.*

Proof. Consider any test function $\phi \in C^\infty_c(D)$. Since $u > \psi$ on D and since ϕ is bounded, by continuity there exists $\delta > 0$ such that $u \pm \delta\phi \in \mathcal{K}_{\psi,\theta}$. From the definition of the solution to the obstacle problem we have that

$$\pm\langle \mathcal{F}(u), \phi\rangle \; = \frac{1}{\delta}\langle \mathcal{F}(u), \pm\delta\phi\rangle \; = \frac{1}{\delta}\langle F(u), (u \pm \delta\phi) - u\rangle \; \geq 0,$$

hence $\langle \mathcal{F}(u), \phi\rangle \; = 0$ for every $\phi \in C^\infty_c(D)$, as required. □

As for boundary regularity, to the best of our knowledge there is no result for solutions to the kind of obstacle problems we are studying. However, if we restrict ourselves to Dirichlet problems (i.e. obstacle problems with $\psi = -\infty$), some results are available. We briefly recall that a point $x_0 \in \partial\Omega$ is called "regular" if for every function $\theta \in W^{1,p}(\Omega)$ continuous in a neighbourhood of x_0, the unique solution to the relative Dirichlet problem is continuous in x_0, and that a necessary and sufficient condition for x_0 to be regular is the famous Wiener criterion (which has a local nature). For our purposes, it is enough to use some simpler sufficient conditions for regularity, so we just cite the following corollary of the Wiener criterion:

Theorem 3.20 ([6], Theorem 2.5)**.** *Let Ω be a domain, and suppose that $x_0 \in \partial\Omega$ has a neighbourhood where $\partial\Omega$ is Lipschitz. Then x_0 is regular for the Dirichlet problem.*

For a more specific discussion of the subject, we refer the reader to [6]. We mention that Dirichlet and obstacle problems have also been studied in metric space setting, and boundary regularity theorems with the Wiener criterion have been obtained for example in [2], Theorem 7.2.

Remark 3.21. Note that [6] deals only with the case $1 < p \leq m$, but the other cases follow from standard Sobolev embeddings.

Using the comparison principle and Proposition 3.19, it is possible to obtain a corollary to this theorem which deals with boundary regularity of some particular obstacle problems.

Corollary 3.22. *Consider the obstacle problem $\mathcal{K}_{\psi,\theta}$ on Ω, and suppose that Ω has Lipschitz boundary and that both θ and ψ are continuous up to the boundary. Then the solution w to $\mathcal{K}_{\psi,\theta}$ is continuous up to the boundary (for convenience we denote w as the continuous representative of the solution).*

Proof. If we want $\mathcal{K}_{\psi,\theta}$ to be non-empty, it is necessary to assume $\psi(x_0) \leq \theta(x_0)$ for all $x_0 \in \partial\Omega$.

Let $\tilde{\theta}$ be the unique solution to the Dirichlet problem relative to θ on Ω. Then Theorem 3.20 guarantees that $\tilde{\theta} \in C^0(\overline{\Omega})$ and the comparison principle allows us to conclude that $w(x) \geq \tilde{\theta}(x)$ everywhere in Ω.

Suppose first that $\psi(x_0) < \theta(x_0)$. Then in a neighbourhood U of x_0 $(U \subset \Omega)$. $w(x) \geq \tilde{\theta}(x) > \psi(x)$. By Proposition 3.19, $L_{\mathcal{F}} w = 0$ on U, and so by Theorem 3.20 w is continuous in x_0.

If $\psi(x_0) = \theta(x_0)$, consider w_ϵ to be the solutions to the obstacle problem $\mathcal{K}_{\tilde{\theta}+\epsilon,\psi}$. By the same argument as above we have that w_ϵ are all continuous at x_0, and by the comparison principle $w(x) \leq w_\epsilon(x)$ for every $x \in \Omega$ (recall that both functions are continuous in Ω). So we have on one hand

$$\liminf_{x \to x_0} w(x) \geq \liminf_{x \to x_0} \psi(x) = \psi(x_0) = \theta(x_0)$$

and on the other

$$\limsup_{x \to x_0} w(x) \leq \limsup_{x \to x_0} w_\epsilon(x) = \theta(x_0) + \epsilon.$$

This proves that w is continuous in x_0 with value $\theta(x_0)$. □

Finally, we present some results on the convergence of supersolutions and their approximation with regular ones.

Proposition 3.23. *Let w_j be a sequence of supersolutions on some open set Ω. Suppose that either $w_j \uparrow w$ or $w_j \downarrow w$ pointwise monotonically, for some locally bounded w. Then, w is a supersolution and there exists a subsequence of $\{w_j\}$ that converges locally strongly in $W^{1,p}$ to w on each compact subset of Ω. Furthermore, if $\{u_j\}$ is a sequence of solutions of $L_{\mathcal{F}} u_j = 0$ which are locally uniformly bounded in L^∞ and pointwise convergent to some u, then u solves $L_{\mathcal{F}} u = 0$ and, up to choosing a subsequence, $\{u_j\}$ converges to u locally strongly on each compact subset of Ω.*

Proof. Suppose that $w_j \uparrow w$. Up to changing the representative in the Sobolev class, by Theorem 3.7 we can assume that w_j is lower semicontinuous. Hence, w_j has minimum on compact subsets of Ω. Since w is locally bounded and the convergence is monotone up to a set of zero measure, the sequence $\{w_j\}$ turns out to be locally bounded in the L^∞-norm. The elliptic estimate in Proposition 3.9 ensures that $\{w_j\}$ is locally bounded in $W^{1,p}(\Omega)$. Fix a smooth exhaustion $\{\Omega_n\}$ of Ω. For each j, up to passing to a subsequence, $w_j \rightharpoonup z_n$ weakly in $W^{1,p}(\Omega_n)$ and strongly in $L^p(\Omega_n)$. By the Riesz theorem, $z_j = w$ for every j; hence $w \in W^{1,p}_{\mathrm{loc}}(\Omega)$. With a Cantor argument, we can select a sequence, still called w_j, such that w_j converges to w both weakly in $W^{1,p}(\Omega_n)$ and strongly in $L^p(\Omega_n)$ for every fixed n. To prove that w is a supersolution, fix $0 \leq \eta \in C^\infty_c(\Omega)$ and choose a smooth relatively compact open set $\Omega_0 \Subset \Omega$ that contains the support of η. Define $M \doteq \max_j \|w_j\|_{W^{1,p}(\Omega_0)} < +\infty$. Since w_j is a supersolution and $w \geq w_j$ for every j,

$$\langle \mathcal{F}(w_j), \eta(w - w_j) \rangle \ \geq 0.$$

Using (A1) we can rewrite the above inequality as follows:

$$\int \langle A(\nabla w_j) | \eta(\nabla w - \nabla w_j) \rangle \geq - \int \Big[B(x, w_j) + \langle A(\nabla w_j) | \nabla \eta \rangle \Big] (w - w_j). \tag{3.12}$$

Using (B1), (A2) and suitable Hölder inequalities, the RHS can be bounded from below with the following quantity:

$$
\begin{aligned}
&-b_1 \|\eta\|_{L^\infty(\Omega)} \int_{\Omega_0} (w - w_j) - b_2 \|\eta\|_{L^\infty(\Omega)} \int_{\Omega_0} |w_j|^{p-1}|w - w_j| \\
&-a_2 \|\nabla\eta\|_{L^\infty(\Omega)} \int_{\Omega_0} |\nabla w_j|^{p-1}|w - w_j| \\
&\geq - \|\eta\|_{C^1(\Omega)} \left[b_1 |\Omega_0|^{\frac{p-1}{p}} - b_2 M^{p-1} - a_2 M^{p-1} \right] \|w - w_j\|_{L^p(\Omega_0)} \to 0
\end{aligned}
\tag{3.13}
$$

as $j \to +\infty$. Combining this with (3.12) and the fact that $w_j \rightharpoonup w$ weakly on $W^{1,p}(\Omega_0)$, by assumption (Mo) the following inequality holds true:

$$
0 \leq \int \eta \,\langle A(\nabla w) - A(\nabla w_j) | \nabla w - \nabla w_j \rangle \leq o(1) \qquad \text{as } \; j \to +\infty. \tag{3.14}
$$

By a lemma due to F. Browder (see [3], p. 13, Lemma 3), the combination of assumptions $w_j \rightharpoonup w$ both locally weakly in $W^{1,p}$ and locally strongly in L^p, and (3.14) for every $0 \leq \eta \in C^\infty_c(\Omega)$, implies that $w_j \to w$ locally strongly in $W^{1,p}$. Since the operator $\mathcal{F}$ is weakly continuous, as shown in the proof of Theorem 3.11, this implies that

$$
0 \;\leq\; \langle \mathcal{F}(w_j), \eta\rangle \;\longrightarrow\; \langle \mathcal{F}(w), \eta\rangle;
$$

hence $L_{\mathcal{F}} w \leq 0$, as required.

The case $w_j \downarrow w$ is simpler. By the elliptic estimate, $w \in W^{1,p}_{\mathrm{loc}}(\Omega)$, being locally bounded by assumption. Let $\{\Omega_n\}$ be a smooth exhaustion of Ω, and let u_n be a solution of the obstacle problem relative to Ω_n with obstacle and boundary value w. Then, by (3.13) $w \leq u_n \leq w_j|_{\Omega_n}$, and letting $j \to +\infty$ we deduce that $w = u_n$ is a supersolution on Ω_n, being a solution of an obstacle problem.

The proof of the last part of the proposition exactly follows the same lines as the case $w_j \uparrow w$ done before. Indeed, by the uniform local boundedness, the elliptic estimate gives $\{u_j\} \subset W^{1,p}_{\mathrm{loc}}(\Omega)$. Furthermore, in the definition $\langle \mathcal{F}(u_j), \phi\rangle \;= 0$ we can still use as a test function $\phi = \eta(u - u_j)$, since no sign of ϕ is required. ☐

A few corollaries follow from this theorem. It is in fact easy to see that we can relax the assumption of local boundedness on w if we assume a priori $w \in W^{1,p}_{\mathrm{loc}}(\Omega)$, and moreover with a simple trick we can also prove that local uniform convergence preserves the supersolution property, as in [9], Theorem 3.78.

Corollary 3.24. *Let w_j be a sequence of supersolutions locally uniformly converging to w. Then w is a supersolution.*

Proof. The trick is to transform local uniform convergence into monotone convergence. Fix any relatively compact $\Omega_0 \Subset \Omega$ and a subsequence of w_j (denoted for convenience by the same symbol) with $\|w_j - w\|_{L^\infty(\Omega_0)} \leq 2^{-j}$. The modified sequence of supersolutions $\tilde{w}_j \doteq w_j + \frac{3}{2}\sum_{k=j}^\infty 2^{-k} = w_j + 3\times 2^{-j}$ is easily seen to be a monotonically decreasing sequence on Ω_0, and thus its limit, still w by construction, is a supersolution on any Ω_0 by the previous proposition. The conclusion follows from the arbitrariness of Ω_0. ☐

Now we prove that with continuous supersolutions we can approximate every supersolution.

Proposition 3.25. *For every supersolution $w \in W^{1,p}_{\rm loc}(\Omega)$, there exists a sequence w_n of continuous supersolutions which converge monotonically from below and in $W^{1,p}_{\rm loc}(\Omega)$ to w. The same statement is true for subsolutions with monotone convergence from above.*

Proof. Since every w has a lower-semicontinuous representative, it can be assumed to be locally bounded from below, and since $w^{(m)} = \min\{w, m\}$ is a supersolution (for $m \geq 0$) and converges monotonically to w as m goes to infinity, we can assume without loss of generality that w is also bounded from above.

Let Ω_n be a locally finite relatively compact open covering on Ω. Since w is lower semicontinuous it is possible to find a sequence ϕ_m of smooth function converging monotonically from below to w (see [9], Section 3.71, p. 75). Let $w^{(n)}_m$ be the solution to the obstacle problem $\mathcal{K}_{w,\phi_m}$ on Ω_n and define $\bar{w}_m \doteq \min_n\{w^{(n)}_m\}$. Thanks to the local finiteness of the covering Ω_n, $\bar{w}_m$ is a continuous supersolution, being locally the minimum of a finite family of continuous functions.

Monotonicity of the convergence is an easy consequence of the comparison principle for obstacle problems, i.e. Proposition 3.12. To prove convergence in the local $W^{1,p}$ sense, the steps are pretty much the same as for Proposition 3.23, and the statement for subsolutions follows from the usual trick. □

Remark 3.26. With similar arguments and up to some minor technical difficulties, one could strengthen the previous proposition and prove that every supersolution can be approximated by locally Hölder continuous supersolutions.

4. Proof of Theorem 2.8

Theorem 4.1. *Let M be a Riemannian manifold, and let A, B satisfy the set of assumptions $\mathscr{S}$. Define $\mathcal{A}, \mathcal{B}, \mathcal{F}$ as in (2.1), and $L_\mathcal{A}, L_\mathcal{F}$ accordingly. Then, the following properties are equivalent:*

1. *(L) for* $\text{Höl}_{\rm loc}$ *functions,*
2. *(L) for L^∞ functions,*
3. *(K).*

Proof. (2) ⇒ (1) is obvious. To prove that (1) ⇒ (2), we follow the arguments in [25], Lemma 1.5. Assume by contradiction that there exists $0 \leq u \in L^\infty(M) \cap W^{1,p}_{\rm loc}(M)$, $u \not\equiv 0$ such that $L_\mathcal{F} u \geq 0$. We distinguish two cases.

- Suppose first that $B(x,u)u$ is not identically zero in the Sobolev sense. Let $u_2 > u^\star$ be a constant. By (B3), $L_\mathcal{F} u_2 \leq 0$. By the subsolution-supersolution method and the regularity Theorem 3.7, there exists $w \in \text{Höl}_{\rm loc}(M)$ such that $u \leq w \leq u_2$ and $L_\mathcal{F} w = 0$. Since, by (B2), (B3) and $u \leq w$, $B(x,w)w$ is not identically zero, then w is non-constant, contradicting property (1).
- Suppose that $B(x,u)u = 0$ a.e. on M. Since u is non-constant, we can choose a positive constant c such that both $\{u - c > 0\}$ and $\{u - c < 0\}$ have positive measure. By (B2), $L_\mathcal{F}(u-c) \geq 0$; hence by Proposition 3.14 the function $v = (u-c)_+ = \max\{u-c, 0\}$ is a non-zero subsolution. Denoting by $\chi_{\{u<c\}}$ the indicatrix of $\{u < c\}$, we can say that $L_\mathcal{F} v \geq 0 = \chi_{\{u<c\}} v^{p-1}$. Choose any constant $u_2 > v^\star$. Then, clearly $L_\mathcal{F} u_2 \leq \chi_{\{u<c\}} u_2^{p-1}$. Since the potential
$$\widetilde{B}(x,t) \doteq B(x,t) + \chi_{\{u<c\}}(x)|t|^{p-2}t$$

is still a Carathéodory function satisfying the assumptions in $\mathscr{S}$, by Theorem 3.2 there exists a function w such that $v \le w \le u_2$ and $L_{\mathcal{F}} w = \chi_{\{u<c\}} w^{p-1}$. By Theorem 3.7 (ii) w is locally Hölder continuous and, since $\{u < c\}$ has positive measure, w is non-constant, contradicting (1).

To prove the implication (3) $\Rightarrow$ (1), we follow a standard argument in potential theory; see for example [25], Proposition 1.6. Let $u \in \mathrm{Höl}_{\mathrm{loc}}(M) \cap W^{1,p}_{\mathrm{loc}}(M)$ be a non-constant, non-negative, bounded solution of $L_{\mathcal{F}} u \ge 0$. We claim that, by the strong maximum principle, $u < u^\star$ on M. Indeed, let $\widetilde{\mathcal{A}}$ be the operator associated with the choice $\widetilde{A}(X) \doteq -A(-X)$. Then, since $\widetilde{A}$ satisfies all the assumptions in $\mathscr{S}$, it is easy to show that $L_{\widetilde{\mathcal{A}}}(u^\star - u) \le 0$ on M. Hence, by the Harnack inequality $u^\star - u > 0$ on M, as desired.

Let $K \Subset M$ be a compact set. Consider η such that $0 < \eta < u^\star$ and define the open set $\Omega_\eta \doteq u^{-1}\{(\eta, +\infty)\}$. From $u < u^\star$ on M, we can choose η close enough to $u^\star$ so that $K \cap \Omega_\eta = \emptyset$. Let x_0 be a point such that $u(x_0) > \frac{u^\star + \eta}{2}$. Let Ω be such that $x_0 \in \Omega$, and choose a Khas'minskii potential relative to the triple $(K, \Omega, (u^\star - \eta)/2)$. Now, consider the open set V defined as the connected component containing x_0 of the open set

$$\tilde{V} \doteq \{x \in \Omega_\eta \mid u(x) > \eta + w(x)\}.$$

Since u is bounded and w is an exhaustion, V is relatively compact in M and $u(x) = \eta + w(x)$ on ∂V. Since, by (B2), $L_{\mathcal{F}}(\eta + w) \le 0$ and $L_{\mathcal{F}} u \ge 0$; this contradicts the comparison in Proposition 3.1.

We are left to the implication (2) $\Rightarrow$ (3). Fix a triple (K, Ω, ε) and a smooth exhaustion $\{\Omega_j\}$ of M with $\Omega \Subset \Omega_1$. By the existence theorem, Theorem 3.11, with the obstacle $\psi = -\infty$, there exists a unique solution h_j of

$$\begin{cases} L_{\mathcal{F}} h_j = 0 \qquad \text{on } \Omega_j \backslash K, \\ h_j = 0 \quad \text{on } \partial K, \quad h_j = 1 \quad \text{on } \partial \Omega_j, \end{cases}$$

and $0 \le h_j \le 1$ by the comparison theorem, Theorem 3.1, with h_j continuous up to $\partial(\Omega_j \setminus K)$ thanks to Theorem 3.20. Extend h_j by setting $h_j = 0$ on K with $h_j = 1$ on $M \backslash \Omega_j$. Again by comparison, $\{h_j\}$ is a decreasing sequence which, by Proposition 3.23, converges pointwise on M to a solution

$$h \in \cap W^{1,p}_{\mathrm{loc}}(M) \quad \text{of} \quad L_{\mathcal{F}} h = 0 \text{ on } M \backslash K.$$

Since $0 \le h \le h_j$ for every j, and since $h_j = 0$ on ∂K, using Corollary 3.22 with $\psi = -\infty$ we deduce that $h \in C^0(M)$ and $h = 0$ on K. We claim that $h = 0$. Indeed, by Lemma 3.16 $u = \max\{h, 0\}$ is a non-negative, bounded solution of $L_{\mathcal{F}} u \ge 0$ on M. By (1), u has to be constant; hence the only possibility is $h = 0$.

Now we are going to build by induction an increasing sequence of continuous functions $\{w_n\}$, $w_0 = 0$, such that:

(a) $w_n|_K = 0$, w_n are continuous on M and $L_{\mathcal{F}} w_n \le 0$ on $M \backslash K$,
(b) for every n, $w_n \le n$ on all of M and $w_n = n$ in a large enough neighbourhood of infinity denoted by $M \backslash C_n$,
(c) $\|w_n\|_{L^\infty(\Omega_n)} \le \|w_{n-1}\|_{L^\infty(\Omega_n)} + \frac{\varepsilon}{2^n}$.

Once this is done, by (c) the increasing sequence $\{w_n\}$ is locally uniformly convergent to a continuous exhaustion which, by Proposition 3.23, solves $L_{\mathcal{F}} w \le 0$ on

$M \setminus K$. Furthermore,

$$\|w\|_{L^\infty(\Omega)} \leq \sum_{n=1}^{+\infty} \frac{\varepsilon}{2^n} \leq \varepsilon.$$

Note that $w \in W^{1,p}_{\mathrm{loc}}(M \setminus K) \cap C^0(M)$ with $w = 0$ on K, so we can immediately conclude that $w \in W^{1,p}_{\mathrm{loc}}(M)$, and hence w is the desired Khas'minskii potential relative to (K, Ω, ε).

We start the induction by setting $w_1 \doteq h_j$, for j large enough in order for property (c) to hold. Define C_1 in order to fix property (b). Suppose now that we have constructed w_n. For notational convenience, write $\bar{w} = w_n$. Consider the sequence of obstacle problems $\mathcal{K}_{\bar{w}+h_j}$ defined on $\Omega_{j+1} \setminus K$ and let s_j be their solution. By Theorem 3.17 and Corollary 3.22 we know that s_j is continuous up to the boundary of its domain. Take for convenience j large enough such that $C_1 \subset \Omega_j$. Note that $s_j|_{\partial K} = 0$, and since the constant function $n+1$ is a supersolution, by comparison $s_j \leq n+1$ and $s_j|_{\Omega_{j+1} \setminus \Omega_j} = n+1$. So we can extend s_j to a function defined on all of M by setting it equal to 0 on K and equal to $n+1$ on $M \setminus \Omega_{j+1}$, and in this fashion, by Lemma 3.16 $L_{\mathcal{F}} s_j \leq 0$ on $M \setminus K$. By Corollary 3.13, $\{s_j\}$ is decreasing, and so it has a pointwise limit $\bar{s}$ which is still a supersolution on $M \setminus K$ by Proposition 3.23. Observe that $\bar{s}(x)$ is continuous. Indeed, since $\bar{s}$ is upper-semicontinuous, $\bar{s} \geq \bar{w}$ and $\bar{w}$ is continuous, $\bar{s}$ is continuous at each point x for which $\bar{s}(x) = \bar{w}(x)$. On the other hand, at points x satisfying $\bar{s}(x) > \bar{w}(x)$, $\bar{s}(x) > \bar{w}(x) + h_j(x)$ definitively in j. Thus, $L_{\mathcal{F}} s_j = 0$ in a neighbourhood of x and so, by the uniform Holder estimates given in Theorem 3.7, (ii), $\bar{s}$ is continuous at x. We are going to prove that $\bar{s} = \bar{w}$. First, we show that $\bar{s} \leq n$ everywhere. Suppose by contradiction that this is false. Then, since h_j converges locally uniformly to zero, on the open set $A \doteq \bar{s}^{-1}\{(n, \infty)\}$ the inequality $s_j > \bar{w} + h_j$ is locally eventually true, so that s_j is locally eventually a solution of $L_{\mathcal{F}} s_j = 0$ by Proposition 3.19, and so $L_{\mathcal{F}} \bar{s} = 0$ on A by Proposition 3.23. By Lemma 3.16 and assumptions $\mathscr{S}$, the function

$$f \doteq \max\{\bar{s} - n, 0\}$$

is a continuous non-negative, bounded solution of $L_{\mathcal{F}} f \geq 0$. By (2), f is constant, hence zero; therefore $\bar{s} \leq n$. This proves that $\bar{s} = \bar{w} = n$ on $M \setminus C_n$. As for the remaining set, an argument similar to the one just used shows that $\bar{s}$ is a solution of $L_{\mathcal{F}} \bar{s} = 0$ on the open, relatively compact set $V \doteq \{\bar{s} > \bar{w}\}$ and that $\bar{s} - w \in W^{1,p}_0(V)$. The comparison principle guarantees that $\bar{s} \leq \bar{w}$ everywhere, which is what we needed to prove. Now, since $s_j \downarrow w$, by Dini's theorem the convergence is locally uniform, and so we can choose $\bar{j}$ large enough in such a way that $s_{\bar{j}} - \bar{w} < \frac{\varepsilon}{2^n}$ on Ω_{n+1}. Define $w_{n+1} \doteq s_{\bar{j}}$, and C_{n+1} in order for (b) to hold, and the construction is completed. $\square$

Remark 4.2. As anticipated in Section 2, the results of our main theorem are the same if we substitute condition (B1) with condition (B1+):

$$|B(x,t)| \leq b(t) \quad \text{instead of} \quad |B(x,t)| \leq b_1 + b_2 |t|^{p-1} \quad \text{for } t \in \mathbb{R}.$$

Although it is not even possible to define the operator $\mathcal{B}$ if we take $W^{1,p}(\Omega)$ as its domain, this difficulty is easily overcome if we restrict the domain to (essentially) bounded functions, i.e. if we define

$$\mathcal{B} : W^{1,p}(\Omega) \cap L^\infty(\Omega) \to W^{1,p}(\Omega)^\star.$$

Now consider that each function used in the proof of the main theorem is either bounded or essentially bounded, so it is quite immediate to see that all the existence and comparison theorems proved in Section 3, along with all the reasoning and tools used in the proof, still work. Consider for example an obstacle problem $\mathcal{K}_{\theta,\psi}$ such that $|\theta| \le C \ge |\psi|$, and define the operator $\tilde{\mathcal{B}}$ relative to the function:

$$\tilde{B}(x,t) = \begin{cases} B(x,t) & \text{for } |t| \le C+1, \\ b_1(x, C+1) & \text{for } t \ge C+1, \\ b_1(x, -(C+1)) & \text{for } t \le -(C+1). \end{cases}$$

$\tilde{\mathcal{B}}$ evidently satisfies condition (B1), so it admits a solution to the obstacle problem, which by comparison Theorem 3.12 is bounded in modulus by C, and now it is evident that this function also solves the obstacle problem relative to the original bad-behaved $\mathcal{B}$.

5. On the links with the weak maximum principle and parabolicity: Proof of Theorem 2.12

As already explained in the introduction, throughout this section we will restrict ourselves to potentials $B(x,t)$ of the form $B(x,t) = b(x)f(t)$, where

$$\begin{aligned} &b, b^{-1} \in L^\infty_{\mathrm{loc}}(M), \quad b > 0 \text{ a.e. on } M, \\ &f \in C^0(\mathbb{R}), \quad f(0) = 0, \quad f \text{ is non-decreasing on } \mathbb{R}, \end{aligned} \tag{5.1}$$

while we require (A1), (A2) on A.

Remark 5.1. As in Remark 3.5, in the case of the operator L_φ in Example 2.2 with h being the metric tensor, (A1) and (A2) can be weakened to (3.1) and (3.2).

We begin with the following lemma characterizing (W), whose proof follows the lines of [24].

Lemma 5.2. *Property (W) for $b^{-1}L_{\mathcal{A}}$ is equivalent to the following property, which we call (P):*

> *For every $g \in C^0(\mathbb{R})$ and for every $u \in C^0(M) \cap W^{1,p}_{\mathrm{loc}}(M)$ bounded above and satisfying $L_{\mathcal{A}}u \ge b(x)g(u)$ on M, $g(u^\star) \le 0$ holds.*

Proof. $(W) \Rightarrow (P)$. From (W) and $L_{\mathcal{A}}u \ge b(x)g(u)$, for every $\eta < u^\star$ and $\varepsilon > 0$ we can find $0 \le \phi \in C^\infty_c(\Omega_\eta)$ such that

$$\varepsilon \int b\phi > -\langle \mathcal{A}(u), \phi\rangle \ \ge \int g(u)b\phi \ge \inf_{\Omega_\eta} g(u) \int b\phi.$$

Since $b > 0$ a.e. on M, we can simplify the integral term to obtain $\inf_{\Omega_\eta} g(u) \le \varepsilon$. Letting $\varepsilon \to 0$ and then $\eta \to u^\star$, and using the continuity of u, g, we get $g(u^\star) \le 0$, as required. To prove that $(P) \Rightarrow (W)$, suppose by contradiction that there exists a bounded above function $u \in C^0 \cap W^{1,p}_{\mathrm{loc}}$, a value $\eta < u^\star$ and $\varepsilon > 0$ such that $\inf_{\Omega_\eta} b^{-1}L_{\mathcal{A}}u \ge \varepsilon$. Let $g_\varepsilon(t)$ be a continuous function on $\mathbb{R}$ such that $g_\varepsilon(t) = \varepsilon$ if $t \ge u^\star - \eta$ and $g_\varepsilon(t) = 0$ for $t \le 0$. Then, by the Pasting Lemma 3.16, $w = \max\{u - \eta, 0\}$ satisfies $L_{\mathcal{A}}w \ge b(x)g_\varepsilon(w)$. Furthermore, $g_\varepsilon(w^\star) = g_\varepsilon(u^\star - \eta) = \varepsilon$, contradicting (P). □

Theorem 2.12 is an immediate corollary of the main Theorem 2.8 and of the following two propositions.

Proposition 5.3. *If $b^{-1}L_{\mathcal{A}}$ satisfies (W), then (L) holds for every operator $L_{\mathcal{F}}$ of type 1. Conversely, if (L) holds for some operator $\mathcal{F}$ of type 1, then $b^{-1}L_{\mathcal{A}}$ satisfies (W).*

Proof. Suppose that (W) is met, and let $u \in \text{Höl}_{\text{loc}} \cap W^{1,p}_{\text{loc}}$ be a bounded, non-negative solution of $L_{\mathcal{F}}u \geq 0$. By Lemma 5.2, $f(u^\star) \leq 0$. Since $\mathcal{F}$ is of type 1, $u^\star \leq 0$, that is, $u = 0$, as desired. Conversely, let $\mathcal{F}$ be an operator of type 1 for which the Liouville property holds. Suppose by contradiction that (W) is not satisfied, so there exists $u \in C^0 \cap W^{1,p}_{\text{loc}}$ such that $b^{-1}L_{\mathcal{A}}u \geq \varepsilon$ on some Ω_{η_0}. Clearly, u is non-constant. Since $f(0) = 0$, we can choose $\eta \in (\eta_0, u^\star)$ in such a way that $f(u^\star - \eta) < \varepsilon$. Hence, by the monotonicity of f, the function $u - \eta$ solves

$$L_{\mathcal{A}}(u-\eta) \geq b(x)\varepsilon \geq b(x)f(u-\eta) \qquad \text{on } \ \Omega_\eta.$$

Thanks to the Pasting Lemma 3.16, $w = \max\{u - \eta, 0\}$ is a non-constant, non-negative bounded solution of $L_{\mathcal{A}}w \geq b(x)f(w)$, that is, $L_{\mathcal{F}}w \geq 0$, contradicting the Liouville property. $\square$

Proposition 5.4. *If $b^{-1}L_{\mathcal{A}}$ is parabolic, then (L) holds for every operator $L_{\mathcal{F}}$ of type 2. Conversely, if (L) holds for some operator $\mathcal{F}$ of type 2, then $b^{-1}L_{\mathcal{A}}$ satisfies (W_{pa}).*

Proof. Suppose that (W_{pa}) is met. Since each bounded, non-negative $u \in \text{Höl}_{\text{loc}} \cap W^{1,p}_{\text{loc}}$ solving $L_{\mathcal{F}}u \geq 0$ automatically solves $L_{\mathcal{A}}u \geq 0$, then u is constant by (W_{pa}), which proves (L). Conversely, let $\mathcal{F}$ be an operator of type 2 for which the Liouville property holds, and let $[0, T]$ be the maximal interval in $\mathbb{R}^+_0$ where $f = 0$. Suppose by contradiction that (W_{pa}) is not satisfied, so there exists a non-constant $u \in C^0 \cap W^{1,p}_{\text{loc}}$ with $b^{-1}L_{\mathcal{A}}u \geq 0$ on M. For η close enough to $u^\star$, $u - \eta \leq T$ on M. Hence $w = \max\{u - \eta, 0\}$ is a non-negative, bounded non-constant solution of $L_{\mathcal{A}}w \geq 0 = b(x)f(w)$ on M, contradicting the Liouville property for $\mathcal{F}$. $\square$

6. The Evans property

We conclude this paper with some comments on the existence of Evans potentials on model manifolds. It turns out that the function-theoretic properties of these potentials can be used to study the underlying manifold. By a way of example, we quote the papers [34] and [31]. In the first one, the authors extend the Kelvin-Nevanlinna-Royden condition and find a Stokes' type theorem for vector fields with integrability condition related to the Evans potential, while in the second article Evans potentials are exploited in order to understand the spaces of harmonic functions with polynomial growth. As a matter of fact, these spaces give much information on the structure at infinity of the manifold. We recall that, only for the standard Laplace-Beltrami operator, it is known that any parabolic Riemannian manifold admits an Evans potential, as proved in [20] or in [30], but the technique involved in this proof heavily relies on the linearity of the operator and cannot be easily generalized, even for the p-Laplacian. In this respect, see [12].

From the technical point of view, we remark that, for the main Theorems 2.8 and 2.12 to hold, no growth control on $B(x, t)$ in the variable t is required. As we will see, for the Evans property to hold for $L_{\mathcal{F}}$ we shall necessarily assume a precise maximal growth of B; otherwise there is no hope to find any Evans potential. This growth is described by the so-called Keller-Osserman condition.

To begin with, we recall that a model manifold M_g is $\mathbb{R}^m$ endowed with a metric $\mathrm{d}s^2$ which, in polar coordinates centered at some origin o, has the expression $\mathrm{d}s^2 = \mathrm{d}r^2 + g(r)^2\mathrm{d}\theta^2$, where $\mathrm{d}\theta^2$ is the standard metric on the unit sphere $\mathbb{S}^{m-1}$ and $g(r)$ satisfies the following assumptions:

$$g \in C^\infty(\mathbb{R}^+_0), \quad g > 0 \text{ on } \mathbb{R}^+, \quad g'(0) = 1, \quad g^{(2k)}(0) = 0$$

for every $k = 0, 1, 2, \ldots$, where $g^{(2k)}$ means the $(2k)$-derivative of g. The last condition ensures that the metric is smooth at the origin o. Note that

$$\Delta r(x) = (m-1)\frac{g'(r(x))}{g(r(x))}, \quad \mathrm{vol}(\partial B_r) = g(r)^{m-1}, \quad \mathrm{vol}(B_r) = \int_0^r g(t)^{m-1}\mathrm{d}t.$$

Consider the operator L_φ of Example 2.2 with h being the metric tensor. If $u(x) = z(r(x))$ is a radial function, a straightforward computation gives

$$L_\varphi u = g^{1-m}\left[g^{m-1}\varphi(|z'|)\mathrm{sgn}(z')\right]'. \tag{6.1}$$

Note that (2.3) implies $\varphi(t) \to +\infty$ as $t \to +\infty$. Let $B(x,t) = B(t)$ be such that

$$B \in C^0(\mathbb{R}^+_0), \ B \geq 0 \text{ on } \mathbb{R}^+, \ B(0) = 0, \ B \text{ is non-decreasing on } \mathbb{R},$$

and set $B = 0$ on $\mathbb{R}^-$. For $c > 0$, define the functions

$$\begin{aligned} &V_{\mathrm{pa}}(r) = \varphi^{-1}\Big(cg(r)^{1-m}\Big), && V_{\mathrm{st}}(r) = \varphi^{-1}\Big(cg(r)^{1-m}\textstyle\int_R^r g(t)^{m-1}\mathrm{d}t\Big), \\ &z_{\mathrm{pa}}(r) = \textstyle\int_R^r V_{\mathrm{pa}}(t)\mathrm{d}t, && z_{\mathrm{st}}(r) = \textstyle\int_R^r V_{\mathrm{st}}(t)\mathrm{d}t. \end{aligned} \tag{6.2}$$

Note that both z_{pa} and z_{st} are increasing on $[R, +\infty)$. By (6.1), the functions $u_{\mathrm{pa}} = z_{\mathrm{pa}} \circ r$, $u_{\mathrm{st}} = z_{\mathrm{st}} \circ r$ are solutions of

$$L_\varphi u_{\mathrm{pa}} = 0, \qquad L_\varphi u_{\mathrm{st}} = c.$$

Therefore, the following property can be easily verified:

Proposition 6.1. *For the operator $L_{\mathcal{F}}$ defined by $L_{\mathcal{F}}u = L_\varphi u - B(u)$, properties (K) and (L) are equivalent to either*

$$V_{\mathrm{st}} \notin L^1(+\infty) \quad \textit{for every } c > 0 \textit{ small enough, if } B > 0 \textit{ on } \mathbb{R}^+ \tag{6.3}$$

or

$$V_{\mathrm{pa}} \notin L^1(+\infty) \quad \textit{for every } c > 0 \textit{ small enough, otherwise.} \tag{6.4}$$

Proof. We sketch the proof when $B > 0$ on $\mathbb{R}^+$, the other case being analogous. If $V_{\mathrm{st}} \in L^1(+\infty)$, then u_{st} is a bounded, non-negative solution of $L_\varphi u \geq c$ on $M \backslash B_R$. Choose $\eta \in (0, u^\star)$ in such a way that $B(u^\star - \eta) \leq c$, and proceed as in the second part of the proof of Proposition 5.3 to contradict the Liouville property of $L_{\mathcal{F}}$. Conversely, if $V_{\mathrm{st}} \notin L^1(+\infty)$, then u_{st} is an exhaustion. For every $\delta > 0$, choose $c > 0$ small enough that $c \leq B(\delta)$. Since $\varphi(0) = 0$, for every $\rho > R$ and $\varepsilon > 0$ we can reduce c in such a way that $w_{\varepsilon,\rho} = \delta + u_{\mathrm{st}}$ satisfies

$$w_{\varepsilon,\rho} = \delta \ \text{ on } \partial B_R, \quad w_{\varepsilon,\rho} \leq \delta + \varepsilon \ \text{ on } B_\rho \backslash B_R, \quad L_\varphi w_{\varepsilon,\rho} = c \leq B(\delta) \leq B(w_{\varepsilon,\rho}).$$

As the reader can check by slightly modifying the argument in the proof of (3) $\Rightarrow$ (1) of Theorem 2.8, the existence of these modified Khas'minskii potentials for every choice of $\delta, \varepsilon, \rho$ is enough to conclude the validity of (L), hence of (K). $\square$

Remark 6.2. In the case $\varphi(t) = t^{p-1}$ of the p-Laplacian, making the conditions on V_{st} and V_{pa} more explicit and using Theorem 2.12 we deduce that, on model manifolds, Δ_p satisfies (W) if and only if

$$\left(\frac{\mathrm{vol}(B_r)}{\mathrm{vol}(\partial B_r)}\right)^{\frac{1}{p-1}} \not\in L^1(+\infty),$$

and Δ_p is parabolic if and only if

$$\left(\frac{1}{\mathrm{vol}(\partial B_r)}\right)^{\frac{1}{p-1}} \not\in L^1(+\infty).$$

This has been observed, for instance, in [25]. See also the end of [26] and the references therein for a thorough discussion of Δ_p on model manifolds.

We now study the existence of an Evans potential on M_g. First, we need to produce radial solutions of $L_\varphi u = B(u)$ which are zero on some fixed sphere ∂B_R. To do so, the first step is to solve locally the related Cauchy problem. The next result is a modification of Proposition A.1 of [4].

Lemma 6.3. *In our assumptions, for every fixed $R > 0$ and $c \in (0, 1]$ the problem*

$$\left\{ \begin{array}{ll} \left[g^{m-1}\varphi(c|z'|)\mathrm{sgn}(z')\right]' = g^{m-1}B(cz) & \text{on } [R, +\infty), \\ z(R) = \vartheta \geq 0, \quad z'(R) = \mu > 0 & \end{array} \right. \tag{6.5}$$

has a positive, increasing C^1 solution z_c defined on a maximal interval $[R, \rho)$, where ρ may depend on c. Moreover, if $\rho < +\infty$, then $z_c(\rho^-) = +\infty$.

Proof. We sketch the main steps. First, we prove local existence. For every chosen $r \in (R, R+1)$, denote with A_ε the ε-ball centered at the constant function ϑ in $C^0([R, r], \|\cdot\|_{L^\infty})$. We look for a fixed point of the Volterra operator T_c defined by

$$T_c(u)(t) = \vartheta + \frac{1}{c}\int_R^t \varphi^{-1}\left(\frac{g^{m-1}(R)\varphi(c\mu)}{g^{m-1}(s)} + \int_R^s \frac{g^{m-1}(\tau)}{g^{m-1}(s)}B(cu(\tau))\mathrm{d}\tau\right)\mathrm{d}s. \tag{6.6}$$

It is a simple matter to check the following properties:

(i) If $|r - R|$ is sufficiently small, $T_c(A_\varepsilon) \subset A_\varepsilon$.

(ii) There exists a constant $C > 0$, independent of $r \in (R, R+1)$, such that $|T_c u(t) - T_c u(s)| \leq C|t - s|$ for every $u \in A_\varepsilon$. By the Ascoli-Arzelà theorem, T_c is a compact operator.

(iii) T_c is continuous. To prove this, let $\{u_j\} \subset A_\varepsilon$ be such that $\|u_j - u\|_{L^\infty} \to 0$, and use the Lebesgue convergence theorem in the definition of T_c to show that $T_c u_j \to T_c u$ pointwise. The convergence is indeed uniform by (ii).

By the Schauder theorem ([7], Theorem 11.1), T_c has a fixed point z_c. Differentiating $z_c = T_c z_c$ we deduce that $z_c' > 0$ on $[R, r]$; hence z_c is positive and increasing. Therefore, z_c is also a solution of (6.5). This solution can be extended up to a maximal interval $[R, \rho)$. If by contradiction the (increasing) solution z_c satisfies $z_c(\rho^-) = z_c^\star < +\infty$, then by differentiating $z_c = T_c z_c$ we would argue that $z_c'(\rho^-)$ exists and is finite. Hence, by local existence z_c could be extended past ρ, a contradiction. □

We are going to prove that if $B(t)$ does not grow too fast and under a reasonable structure condition on M_g, the solution z_c of (6.5) is defined on $[R, +\infty)$. To do this, we first need some definitions. We consider the initial condition $\vartheta = 0$. For convenience, we further require the following assumptions:

$$\varphi \in C^1(\mathbb{R}^+), \qquad a_2^{-1} t^{p-1} \le t\varphi'(t) \le a_1 + a_2 t^{p-1} \quad \text{on } \mathbb{R}^+, \tag{6.7}$$

for some positive constants a_1, a_2. Define

$$K_\mu(t) = \int_\mu^t s\varphi'(s)\mathrm{d}s, \qquad \beta(t) = \int_0^t B(s)\mathrm{d}s.$$

Note that $\beta(t)$ is non-decreasing on $\mathbb{R}^+$ and that, for every $\mu \ge 0$, K_μ is strictly increasing. By (6.7), $K_\mu(+\infty) = +\infty$. We focus our attention on the condition

$$\frac{1}{K_\mu^{-1}(\beta(s))} \notin L^1(+\infty). \tag{$\neg KO$}$$

This (or, better, its opposite) is called the Keller-Osserman condition. Originating, in the quasilinear setting, from works of J. B. Keller [10] and R. Osserman [21], it has been the subject of increasing interest in the last few years. The interested reader can consult, for instance, [5], [17], [19]. Note that the validity of ($\neg KO$) is independent of the choice of $\mu \in [0,1)$, and we can thus refer ($\neg KO$) to $K_0 = K$. This follows since, by (6.7), $K_\mu(t) \asymp t^p$ as $t \to +\infty$, where the constant is independent of μ, and thus $K_\mu^{-1}(s) \asymp s^{1/p}$ as $s \to +\infty$, for some constants which are uniform when $\mu \in [0,1)$. Therefore, ($\neg KO$) is also equivalent to

$$\frac{1}{\beta(s)^{1/p}} \notin L^1(+\infty). \tag{6.8}$$

Lemma 6.4. *Under the assumptions of the previous proposition and subsequent discussion, suppose that $g' \ge 0$ on $\mathbb{R}^+$. If*

$$\frac{1}{K^{-1}(\beta(s))} \notin L^1(+\infty), \tag{$\neg KO$}$$

then, for every choice of $c \in (0,1]$, the solution z_c of (6.5) *is defined on $[R, +\infty)$.*

Proof. From $[g^{m-1}\varphi(cz')]' = g^{m-1}B(cz)$ and $g' \ge 0$ we deduce that

$$\varphi'(cz')cz'' \le B(cz) \quad \text{so that} \quad cz'\varphi'(cz')cz'' \le B(cz)cz' = (\beta(cz))'.$$

Hence integrating and changing variables we obtain

$$K_\mu(cz') = \int_\mu^{cz'} s\varphi'(s)\mathrm{d}s \le \int_0^{cz} B(s)\mathrm{d}s = \beta(cz).$$

Applying K_μ^{-1}, $cz' = K_\mu^{-1}(\beta(cz))$. Since $z' > 0$, we can divide the last equality by $K_\mu^{-1}(\beta(cz))$ and integrate on $[R, t)$ to get, after changing variables,

$$\int_0^{cz(t)} \frac{\mathrm{d}s}{K_\mu^{-1}(\beta(s))} \le t - R.$$

By ($\neg KO$), we deduce that ρ cannot be finite for any fixed choice of c. □

For every $R > 0$, we have produced a radial function $u_c = (cz_c) \circ r$ which solves $L_\varphi u_c = B(u_c)$ on $M \backslash B_R$ and $u_c = 0$ on B_R. The next step is to guarantee that, up to choosing μ, c appropriately, u_c can be arbitrarily small on some bigger ball B_{R_1}. The basic step is a uniform control of the norm of z_c on $[R, R_1]$ with respect to the variable c, up to choosing $\mu = \mu(c)$ appropriately small. This requires a further control on $B(t)$, this time on the whole $\mathbb{R}^+$ and not only in a neighbourhood of $+\infty$.

Lemma 6.5. *Under the assumptions of the previous proposition, suppose further that*

$$B(t) \le b_1 t^{p-1} \qquad on\ \mathbb{R}^+. \tag{6.9}$$

Then, for every $R_1 > R$ and every $c \in (0,1]$, there exists $\mu > 0$ depending on c such that the solution z_c of (6.5) *with $\vartheta = 0$ satisfies*

$$\|z_c\|_{L^\infty([R,R_1])} \le K, \tag{6.10}$$

for some $K > 0$ depending on R, R_1, on a_2 in (6.7) *and on b_1 in* (6.9) *but not on c.*

Proof. Note that, by (6.9), ($\neg KO$) (equivalently, (6.8)) is satisfied. Hence, z_c is defined on $[R, +\infty)$ for every choice of μ, c. Fix $R_1 > R$. Setting $\vartheta = 0$ in the expression (6.6) of the operator T_c, and using the monotonicity of g and z_c, we deduce that

$$\begin{aligned} u_c(t) &\le \frac{1}{c}\int_R^t \varphi^{-1}\left(\varphi(c\mu) + \int_R^s B(cu(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ &\le \frac{1}{c}\int_R^t \varphi^{-1}\Big(\varphi(c\mu) + (R_1 - R)B(cu_c(s))\Big)\mathrm{d}s. \end{aligned}$$

Differentiating, this gives

$$\varphi(cu_c'(t)) \le \varphi(c\mu) + (R_1 - R)B(cu_c(t)).$$

Now, from (6.7) and (6.9) we get

$$c^{p-1}(u_c')^{p-1} \le a_2\varphi(c\mu) + a_2(R_1 - R)b_2 c^{p-1}u_c^{p-1}. \tag{6.11}$$

Choose μ in such a way that

$$\varphi(c\mu) \le c^{p-1}; \qquad \text{that is,} \qquad \mu \le \frac{1}{c}\varphi^{-1}(c^{p-1}).$$

Then, dividing (6.11) by c^{p-1} and applying the elementary inequality $(x+y)^a \le 2^a(x^a + y^a)$, we obtain the existence of a constant $K = K(R_1, R, a_2, b_2)$ such that

$$u_c'(t) \le K(1 + u_c(t)).$$

Estimate (6.10) follows by applying Gronwall's inequality. $\square$

Corollary 6.6. *Let the assumptions of the last proposition be satisfied. Then, for each triple $(B_R, B_{R_1}, \varepsilon)$, there exists a positive, radially increasing solution of $L_\varphi u = B(u)$ on $M_g \backslash B_R$ such that $u = 0$ on ∂B_R and $u < \varepsilon$ on $B_{R_1}\backslash B_R$.*

Proof. By the previous lemma, for every $c \in (0,1]$ we can choose $\mu = \mu(c) > 0$ such that the resulting solution z_c of (6.5) is uniformly bounded on $[R, R_1]$ by some K independent of c. Since, by (6.1), $u_c = (cz_c) \circ r$ solves $L_\varphi u_c = B(u_c)$, it is enough to choose $c < \varepsilon/K$ to obtain a desired $u = u_c$ for the triple $(B_R, B_{R_1}, \varepsilon)$. $\square$

To conclude, we shall show that Evans potentials exist for any triple (K, Ω, ε), not necessarily given by concentric balls centered at the origin. In order to do so, we use a comparison argument with suitable radial Evans potentials. Consequently, we need to ensure that, for careful choices of c, μ, the radial Evans potentials do not overlap.

Lemma 6.7. *Under the assumptions of Lemma* 6.4, *let* $0 < R$ *be chosen, and let* w *be a positive, increasing* C^1 *solution of*

$$\begin{cases} \left[g^{m-1}\varphi(w')\right]' = g^{m-1}B(w) & \text{on } [R,+\infty), \\ w(R) = 0, \quad w'(R) = w'_R > 0. \end{cases} \tag{6.12}$$

Fix $\hat{R} > R$. *Then, for every* $c > 0$, *there exists* $\mu = \mu(c, R, \hat{R})$ *small enough that the solution* z_c *of* (6.5), *with* R *replaced by* $\hat{R}$, *satisfies* $cz_c < w$ *on* $[\hat{R}, +\infty)$.

Proof. Let μ satisfy $g^{m-1}(R)\varphi(w'_R) > g^{m-1}(\hat{R})\varphi(c\mu)$. Suppose by contradiction that $\{cz_c \geq w\}$ is a closed, non-empty set. Let $r > \hat{R}$ be the first point where $cz_c = w$. Then, $cz_c \leq w$ on $[\hat{R}, r]$, and thus $cz'_c(r) \geq w'(r)$. However, from the chain of inequalities

$$\begin{aligned} \varphi(w'(r)) &= \frac{g^{m-1}(R)\varphi(w'_R)}{g^{m-1}(r)} + \int_R^r B(w(\tau))\mathrm{d}\tau \\ &> \frac{g^{m-1}(\hat{R})\varphi(c\mu)}{g^{m-1}(r)} + \int_{\hat{R}}^r B(cz_c(\tau))\mathrm{d}\tau = \varphi(cz'_c(r)) \end{aligned}$$

and from the strict monotonicity of φ we deduce $w'(r) > cz'_c(r)$, a contradiction. □

Corollary 6.8. *For each* u *constructed in Corollary* 6.6 *and for every* $R_2 > R$, *there exists a positive, radially increasing solution* w *of* $L_{\mathcal{F}}w = 0$ *on* $M_g \backslash B_{R_2}$ *such that* $w = 0$ *on* ∂B_{R_2} *and* $w \leq u$ *on* $M \backslash B_{R_2}$.

Proof. It is a straightforward application of the last lemma. □

We are now ready to state the main result of this section.

Theorem 6.9. *Let* M_g *be a model with origin* o *and non-decreasing defining function* g. *Let* φ *satisfy* (6.7) *with* $a_1 = 0$, *and suppose that* $B(t)$ *satisfies* (6.9). *Define* $L_{\mathcal{F}}$ *according to* $L_{\mathcal{F}}u = L_\varphi u - B(u)$. *Then, properties* (K), (L) *(for* Höl$_{\mathrm{loc}}$ *or* L^∞*) and* (E) *restricted to triples* (K, Ω, ε) *with* $o \in K$ *are equivalent and also equivalent to either*

$$\left(\frac{\mathrm{vol}(B_r)}{\mathrm{vol}(\partial B_r)}\right)^{\frac{1}{p-1}} \notin L^1(+\infty) \quad \text{if } B > 0 \text{ on } \mathbb{R}^+ \tag{6.13}$$

or

$$\left(\frac{1}{\mathrm{vol}(\partial B_r)}\right)^{\frac{1}{p-1}} \notin L^1(+\infty) \quad \text{otherwise.} \tag{6.14}$$

Proof. From (6.7), assumptions (6.13) and (6.14) are equivalent, respectively, to (6.3) and (6.4). Therefore, by Proposition 6.1 and Theorem 2.8, the result will be proved once we show that (L) implies (E) restricted to the triples (K, Ω, ε) such that $o \in K$. Fix such a triple (K, Ω, ε). Since $o \in K$ and K is open, let $R < \rho$ be such that $B_R \Subset K \Subset \Omega \Subset B_\rho$. By making use of Corollary 6.6 we can construct a radially increasing solution w_2 of $L_{\mathcal{F}}w_2 = 0$ associated to the triple $(B_R, B_\rho, \varepsilon)$. By (L), u must tend to $+\infty$ as x diverges. Otherwise by the Pasting Lemma, Lemma 3.16, the function s obtained by extending w_2 with zero on B_R would be a bounded, non-negative, non-constant solution of $L_{\mathcal{F}}s \geq 0$, a contradiction. From Corollary 6.8 and the same reasoning, we can produce another exhaustion w_1 solving $L_{\mathcal{F}}w_1 = 0$ on $M \backslash B_\rho$, $w_1 = 0$ on ∂B_ρ and $w_1 \leq w_2$ on $M \backslash B_\rho$. Setting w_1

equal to zero on B_ρ, by the Pasting Lemma w_2 is a global subsolution on M below w_2. By the subsolution-supersolution method on $M\backslash K$, there exists a solution w such that $w_1 \le w \le w_2$. By construction, w is an exhaustion and $w \le \varepsilon$ on $\Omega\backslash K$. Note that, by Remark 3.6, from (6.7) with $a_1 = 0$ we deduce that $w \in C^1(M\backslash K)$. We claim that $w > 0$ on $M\backslash K$. To prove the claim we can avail ourselves of the strong maximum principle in the form given in [27], Theorem 1.2. Indeed, again from (6.7) with $a_1 = 0$ we have (in their notation)

$$pa_2^{-1}s^p \le K(s) \le pa_2 s^p \ \text{ on } \mathbb{R}^+, \qquad 0 \le F(s) \le \frac{b_1}{p} s^p \ \text{ on } \mathbb{R}^+,$$

and hence

$$\frac{1}{K^{-1}(F(s))} \notin L^1(0^+).$$

The last expression is a necessary and sufficient condition for the validity strong maximum principle for C^1 solutions u of $L_{\mathcal{F}} u \le 0$. Therefore, $w > 0$ on $M\backslash K$ follows since w is not identically zero by construction. In conclusion, w is an Evans potential relative to (K, Ω, ε), as desired. □

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