Commun. math. Phys. 38, 1—10 (1974)

On the Equivalence of the *KMS* Condition and the Variational Principle for Quantum Lattice Systems

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Received May 16, 1974

Abstract. For quantum spin systems on a lattice of an arbitrary dimension, the *KMS* condition and the variational principle are shown to be equivalent at an arbitrary temperature for translationally invariant states.

§ 1. Main Result

The *KMS* condition and the variational principle are known to be equivalent for classical spin lattice systems [8]. The equivalence has been shown also for quantum spin lattice systems when either the dimension of the lattice is one or the temperature is high [7]. We shall prove the equivalence for any spin lattice system at arbitrary non-zero temperature.

We use the same notation as in [7]. The assumption on the interaction potential $\Phi(I)$ is as follows:

(i) Translational covariance: $\Phi(I+a)=\tau(a)\Phi(I)$.

(ii) Finite-body interaction: $\Phi(I)=0$ if $\mathrm{N}(I)\geqq N_0$ for some N_0.

(iii) Relatively short range: $\|\Phi\| = \sum_{I\ni 0} \|\Phi(I)\|/\mathrm{N}(I) < \infty$.

For a state ψ of the C^*-algebra $\mathfrak{A}$ (of quasi-local operators) and a finite subset Λ of the lattice, ψ_Λ denotes the restriction of ψ to $\mathfrak{A}(\Lambda)$ (the local subalgebra) and ϱ_ψ^Λ denotes the density matrix for ψ_Λ:

$$\varrho_\psi^\Lambda \in \mathfrak{A}(\Lambda), \quad \psi(Q)=\mathrm{tr}(\varrho_\psi^\Lambda Q) \quad \text{for all} \quad Q\in\mathfrak{A}(\Lambda). \tag{1.1}$$

The variational principle at the inverse temperature β is satisfied by a translationally invariant state ψ of $\mathfrak{A}$ if

$$\mathrm{s}(\psi)-\beta\psi(A)=P\equiv\lim_{\Lambda\uparrow}\mathrm{N}(\Lambda)^{-1}\log\mathrm{tr}(e^{-\beta \mathrm{U}(\Lambda)}) \tag{1.2}$$

where $s(\psi)$ is the mean entropy of the state ψ:

$$\mathrm{s}(\psi)=-\lim_{\Lambda\uparrow}\mathrm{N}(\Lambda)^{-1}\,\psi(\log\varrho_\psi^\Lambda)\,, \tag{1.3}$$

$\psi(A)$ is the mean energy of the state ψ:

$$A \equiv \sum_{I \ni 0} \mathrm{N}(I)^{-1} \Phi(I) \in \mathfrak{A}, \tag{1.4}$$

$$\psi(A) = \lim_{\Lambda\uparrow} \mathrm{N}(\Lambda)^{-1} \psi(\mathrm{U}(\Lambda)), \tag{1.5}$$

and $\mathrm{U}(\Lambda)$ is the total energy in Λ:

$$\mathrm{U}(\Lambda) = \sum_{I \subset \Lambda} \Phi(I). \tag{1.6}$$

The time translation automorphisms σ_t of $\mathfrak{A}$ are given by

$$\sigma_t Q = \lim_{\Lambda\uparrow} e^{i\mathrm{U}(\Lambda)t} Q e^{-i\mathrm{U}(\Lambda)t}, \quad Q \in \mathfrak{A}. \tag{1.7}$$

A state ψ of $\mathfrak{A}$ satisfies the *KMS* condition at the inverse temperature β if for any given Q_1 and Q_2 in $\mathfrak{A}$ there exists a function $F(z)$ of a complex variable z in the strip $0 \leqq \operatorname{Im} z \leqq \beta$ such that F is continuous and bounded on the strip, holomorphic inside the strip and

$$F(t) = \psi(Q_2 \sigma_t Q_1), \quad F(t + i\beta) = \psi(\{\sigma_t Q_1\} Q_2)$$

for all real t.

We shall prove the following:

Theorem 1. *A translationally invariant state ψ satisfies the KMS condition at the inverse temperature β if and only if it satisfies the variational principle at the inverse temperature β.*

The proof that ψ satisfies the *KMS* condition if it satisfies the variational principle has been known for some time. (Theorems 4.2, 3.2, and 3.4 in [9].) We have only to prove the converse.

It has been shown (Theorem 9.1 in [4]) that ψ satisfies the *KMS* condition if and only if it satisfies the following Gibbs condition:

Let $\mathfrak{H}_\psi$, π_ψ, and Ψ be the cyclic Hilbert space, representation and vector associated with a faithful ψ. Let W_Λ be the interaction energy across the boundary of Λ:

$$W_\Lambda = \Sigma \{\Phi(I); \quad I \cap \Lambda \neq \emptyset, \quad I \cap \Lambda^c \neq \emptyset\}. \tag{1.8}$$

We recall the following notation defined in [1]:

$$\begin{aligned} \Psi(k) = \sum_{n=0} \int_0^{1/2} \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n \\ \cdot \Delta_\Psi^{t_n} k \Delta_\Psi^{t_{n-1}-t_n} k \ldots \Delta_\Psi^{t_1 - t_2} k \Psi \\ (= \exp[(1/2)\{\log \Delta_\Psi + k\}]\, \Psi). \end{aligned} \tag{1.8}$$

A state ψ satisfies the Gibbs condition at the inverse temperature β if and only if it is faithful and the vector state given by the vector $\Psi(\beta W_\Lambda)$ is a product of the Gibbs state

$$\varphi_G^\Lambda(Q) = \mathrm{tr}(e^{-\beta U(\Lambda)}Q)/\mathrm{tr}(e^{-\beta U(\Lambda)})$$

on $\mathfrak{A}(\Lambda)$ and a positive linear functional on $\mathfrak{A}(\Lambda^c)$.

We shall show that the Gibbs condition implies the variational equality (1.2) by using an inequality of Umegaki [10] and Lindblad [11].

§ 2. Continuity Properties of Modular Operators

We need some continuity properties of the modular operators and the modular conjugation operators when there is a monotonously increasing net of von Neumann algebras $\mathfrak{M}_\alpha$ with

$$\mathfrak{M} = \Big(\bigcup_\alpha \mathfrak{M}_\alpha\Big)''.$$

Let Ψ be a cyclic and separating vector for the von Neumann algebra $\mathfrak{M}$. Let E_α be the projection onto the subspace $\overline{\mathfrak{M}_\alpha \Psi}$. Let Δ and J be the modular operator and the modular conjugation operator for Ψ relative to $\mathfrak{M}$. Define Δ_α and J_α to be the same for Ψ relative to $\mathfrak{M}_\alpha$ on $\overline{\mathfrak{M}_\alpha \Psi}$. They are defined to be the identity operator and an antiunitary involution on $(\mathfrak{M}_\alpha \Psi)^\perp$, respectively, and are defined additively on the sum $\overline{\mathfrak{M}_\alpha \Psi} + (\mathfrak{M}_\alpha \Psi)^\perp$.

Theorem 2. *Δ_α^{it} and J_α have strong limits which are Δ^{it} and J, respectively, where the convergence is uniform in t over any compact set.*

We shall present the proof as a series of Lemmas. We first recall Sakai's theorem on the linear Radon-Nicodym derivative. (For example, see Lemmas 1 and 2 in [6].) Let ψ and φ be normal positive linear functionals on a von Neumann algebra $\mathfrak{M}$ and assume that ψ is faithful and $\varphi \leqq \psi$ (i.e. $\varphi(Q) \leqq \psi(Q)$ for all positive Q in $\mathfrak{M}$). Then there exists a unique $h \in \mathfrak{M}^+$ (the positive elements of $\mathfrak{M}$) such that $\|h\| \leqq 1$ and

$$\varphi(Q) = \psi(hQ + Qh)/2 \tag{2.1}$$

for all $Q \in \mathfrak{M}$.

Lemma 1. *Let Ψ be a cyclic and separating vector for $\mathfrak{M}$ such that*

$$\omega_\Psi = \psi \quad (\textit{here } \omega_\Psi(Q) \equiv (\Psi, Q\Psi)). \tag{2.2}$$

Then $h\Psi$ is in the domain of the modular operator Δ_Ψ and

$$\Delta_\Psi h\Psi = 2h'\Psi - h\Psi \tag{2.3}$$

where h' is the unique positive element in $\mathfrak{M}'$ satisfying

$$\varphi(Q) = (h'\Psi, Q\Psi). \tag{2.4}$$

Proof. For all $Q \in \mathfrak{M}$ we have

$$2\varphi(Q) = (2h'\Psi, Q\Psi) = (h\Psi, Q\Psi) + (Q^*\Psi, h\Psi).$$

By properties of Δ_Ψ and J_Ψ, we have

$$(\Delta_\Psi^{1/2} h\Psi, \Delta_\Psi^{1/2} Q\Psi) = (J_\Psi \Delta_\Psi^{1/2} Q\Psi, J_\Psi \Delta_\Psi^{1/2} h\Psi)$$
$$= (Q^*\Psi, h\Psi) = ((2h' - h)\Psi, Q\Psi).$$

Since $\mathfrak{M}\Psi$ is a core of $\Delta_\Psi^{1/2}$, we see that $\Delta_\Psi^{1/2} h\Psi$ is in the domain of $\Delta_\Psi^{1/2}$ and

$$\Delta_\Psi^{1/2}(\Delta_\Psi^{1/2} h\Psi) = (2h' - h)\Psi.$$

This proves Lemma 1.

We now investigate the linear Radon-Nikodym derivatives h_α of the restrictions φ_α and ψ_α of φ and ψ to $\mathfrak{M}_\alpha \subset \mathfrak{M}$. Since $\varphi_\alpha \leqq \psi_\alpha$ follows from $\varphi \leqq \psi$ and ψ_α is faithful, we have the unique existence of $h_\alpha \in \mathfrak{M}_\alpha^+$ with $\|h_\alpha\| \leqq 1$.

Lemma 2. *h_α and $\Delta_\alpha h_\alpha \Psi$ strongly tend to h and $\Delta h\Psi$, respectively.*

Proof. By weak compactness, there exists a weak accumulation point h_∞ of h_α. We then have

$$\varphi(Q) = \psi(h_\infty Q + Q h_\infty)/2, \qquad Q \in \mathfrak{M}_\alpha$$

for an arbitrary α due to (2.1) for φ_γ, $\gamma \geqq \alpha$. Since $\left(\bigcup_\alpha \mathfrak{M}_\alpha\right)'' = \mathfrak{M}$, we have $h_\infty = h$. Hence h_α has a weak limit which is h. From (2.1) for φ_α again, we obtain

$$\|h\Psi\|^2 = \varphi(h) = \lim_\alpha \varphi(h_\alpha) = \lim_\alpha \psi(h_\alpha^2) = \lim_\alpha \|h_\alpha \Psi\|^2.$$

This implies that $h_\alpha Q'\Psi$ tends strongly to $hQ'\Psi$ for $Q' = 1$ and hence for any $Q' \in \mathfrak{M}' \subset \mathfrak{M}'_\alpha$. Therefore h_α tends strongly to h.

Since $h' \in \mathfrak{M}'$ in Lemma 1 satisfies $h' \in \mathfrak{M}'_\alpha$ $(\supset \mathfrak{M}')$ and $\varphi_\alpha(Q) = \varphi(Q) = (h'\Psi, Q\Psi)$ for $Q \in \mathfrak{M}_\alpha$, we obtain

$$\Delta_\alpha h_\alpha \Psi = 2h'\Psi - h_\alpha \Psi.$$

Hence $\Delta_\alpha h_\alpha \Psi$ tends strongly to

$$\Delta h\Psi = 2h'\Psi - h\Psi.$$

This proves Lemma 2.

Lemma 3. *The set of vectors $(\Delta_\Psi + 1) h\Psi$, when φ runs over normal linear functionals on $\mathfrak{M}$ satisfying $\varphi \leqq \psi$, is total.*

Proof. Let $Q \in \mathfrak{M}^+$, $\|Q\| \leqq 1$. Consider

$$h = (1/2)\{1 + \lambda_f \int \sigma_t^{\psi}(Q) f(t)\, \mathrm{d}t\} \tag{2.5}$$

where σ_t^{ψ} denotes the modular automorphisms and the Fourier transform of f is an arbitrary C^{∞}-function with a compact support. Then $\sigma_t^{\psi}(h)$ is an entire function of t and $h\Psi$ is an analytic vector of Δ_{Ψ} (because $h\Psi$ has compact support relative to the spectral measure of Δ_{Ψ}). We choose sufficiently small real positive λ_f satisfying

$$\lambda_f \int |f(t \pm (i/2))|\, \mathrm{d}t < 1\,. \tag{2.6}$$

Then

$$t' \equiv (1/2)\, j_{\Psi}(\sigma_{-i/2}^{\psi}(h) + \sigma_{i/2}^{\psi}(h)) \tag{2.7}$$

is obviously a selfadjoint element of $\mathfrak{M}'$ and satisfies $1 > t' > 0$ due to (2.6). Hence

$$\varphi(Q) \equiv (t'\Psi, Q\Psi)\,, \qquad Q \in \mathfrak{M}$$

defines a normal positive linear functional of $\mathfrak{M}$ satisfying $\varphi < \psi$. Furthermore

$$\begin{aligned} 2\varphi(Q) &= (J_{\Psi}\Delta_{\Psi}^{1/2} h\Psi, Q\Psi) + (\Psi, Q j_{\Psi}(\sigma_{i/2}^{\psi}(h))^* \Psi) \\ &= (h\Psi, Q\Psi) + (\Psi, Qh\Psi) = \psi(hQ + Qh)\,. \end{aligned}$$

The linear span of $h\Psi$ with h given by (2.5) contains Ψ (for $\lambda_f = 0$) and $\int \sigma_t^{\psi}(Q) f(t)\, \mathrm{d}t\Psi$. Hence it is a dense set of analytic vectors of Δ_{Ψ} and is a core of the selfadjoint positive operator Δ_{Ψ}. Hence $(\Delta_{\Psi} + 1)\, h\Psi$ is total.

Lemma 4. *Δ_{α}^{it} tends strongly to Δ^{it} uniformly in t over any compact set.*

Proof. By Lemma 2, we have

$$\lim_{\alpha} \|(\Delta_{\alpha} + 1)\, h_{\alpha}\Psi - (\Delta + 1)\, h\Psi\| = 0\,.$$

Since $\|(\Delta_{\alpha} + 1)^{-1}\| \leqq 1$, we have

$$\lim_{\alpha} \|h_{\alpha}\Psi - (\Delta_{\alpha} + 1)^{-1}(\Delta + 1)\, h\Psi\| = 0\,.$$

Hence we have

$$\lim_{\alpha} \{(\Delta_{\alpha} + 1)^{-1} - (\Delta + 1)^{-1}\}\, x = 0$$

for $x = (\Delta + 1)\, h\Psi$. Since $\|(\Delta_{\alpha} + 1)^{-1}\| \leqq 1$ and since x is total by Lemma 3, we have

$$\lim_{\alpha} (\Delta_{\alpha} + 1)^{-1} = (\Delta + 1)^{-1}\,.$$

This implies the conclusion of Lemma 4.

Lemma 5. *J_{α} tends strongly to J.*

Proof. Let

$$x_\alpha(z) \equiv e^{z^2}(\Delta_\alpha^z h_\alpha \Psi - \Delta^z h \Psi).$$

By Lemma 2 and Lemma 4, we have

$$\lim_\alpha \sup_t \|x_\alpha(s+it)\| = 0$$

for $s=0$ and $s=1$. For example

$$x_\alpha(1+it) = \Delta_\alpha^{it} e^{(1+it)^2}(\Delta_\alpha h_\alpha \Psi - \Delta h \Psi) + e^{(1+it)^2}(\Delta_\alpha^{it} - \Delta^{it})\,\Delta h \Psi.$$

By the three lines theorem, we have

$$\lim_\alpha \sup_{\|x\| \leqq 1} \sup_t |(x, x_\alpha(s+it))| = 0$$

for $0 \leqq s \leqq 1$. Hence we have

$$\lim_\alpha \|\Delta_\alpha^z h_\alpha \Psi - \Delta^z h \Psi\| = 0.$$

By setting $z=\frac{1}{2}$, we obtain

$$\lim_\alpha \|J_\alpha h_\alpha \Psi - J h \Psi\| = 0.$$

Hence

$$\lim_\alpha \|(J_\alpha - J)\, h \Psi\| = 0.$$

By the proof of Lemma 3, the set of $h\Psi$ is total and we have $\lim J_\alpha = J$.
Lemmas 4 and 5 prove Theorem 2.

Corollary. *Assume that* $Q_\alpha \in \mathfrak{M}_\alpha$, $Q \in \mathfrak{M}$, $\lim_\alpha Q_\alpha = Q$ *and* $\lim_\alpha Q_\alpha^* = Q^*$ *(strongly). For any z with* $(\mathrm{Re}\, z) \in [0, \frac{1}{2}]$,

$$\lim_\alpha \Delta_\alpha^z Q_\alpha \Psi = \Delta^z Q \Psi, \tag{2.8}$$

where the convergence is uniform in z over any compact subset of the strip $0 \leqq \mathrm{Re}\, z \leqq 1/2$.

Proof. We have

$$\Delta_\alpha^{1/2} Q_\alpha \Psi - \Delta^{1/2} Q \Psi = J_\alpha Q_\alpha^* \Psi - J Q^* \Psi = J_\alpha(Q_\alpha^* \Psi - Q^* \Psi) + (J_\alpha - J)\, Q^* \Psi.$$

By Theorem 2, we have (2.8) for $(\mathrm{Re}\, z) = \frac{1}{2}$ and $(\mathrm{Re}\, z) = 0$ uniformly on any compact set of values of $\mathrm{Im}\, z$. By the three lines theorem, (with e^{z^2} multiplied), we obtain (2.8) for $(\mathrm{Re}\, z) \in [0, \frac{1}{2}]$, with the stated uniformity.

Lemma 6. *If* $k_\alpha \in \mathfrak{M}_\alpha$, $k_\alpha^* = k_\alpha$, $\sup_\alpha \|k_\alpha\| < \infty$ *and* $\lim_\alpha k_\alpha = k$ *(strongly), then*

$$\lim_\alpha \Psi(k_\alpha) = \Psi(k) \tag{2.9}$$

where $\Psi(k_\alpha)$ *is defined in terms of* Δ_α.

Proof. By the preceding Corollary, we have

$$\lim_{\alpha} L_j^{\alpha} = 0\,,$$
$$L_j^{\alpha} \equiv \sup_{-\infty < t < \infty} e^{-t^2} \left\| k_{\alpha}^j \Delta_{\alpha}^{(1/2)+it} k_{\alpha}^{n-j} \Psi - k^j \Delta^{(1/2)+it} k^{n-j} \Psi \right\|$$
$$= \sup \{ e^{-t^2} |(x, k_{\alpha}^j \Delta_{\alpha}^{(1/2)+it} k_{\alpha}^{n-j} \Psi - k^j \Delta^{(1/2)+it} k^{n-j} \Psi)|$$
$$; -\infty < t < \infty,\ \|x\| \leqq 1\}\,.$$

For the vector

$$\Psi(z_1, \ldots, z_n) \equiv e^{\Sigma z_j^2} \{ \Delta_{\alpha}^{z_1} k_{\alpha} \ldots \Delta_{\alpha}^{z_n} k_{\alpha} \Psi - \Delta^{z_1} k \ldots \Delta^{z_n} k \Psi \}$$

with $\mathrm{Re}(z_1 + \cdots + z_n) \leqq \frac{1}{2}$ and $\mathrm{Re}\, z_j \geqq 0$, we have the following estimate by Corollary 2.2 of [1]:

$$\|\Psi(z_1, \ldots, z_n)\| = \sup\{|x, \Psi(z_1, \ldots, z_n))|\,;\ \|x\| \leqq 1\}$$
$$\leqq e^{1/4} \sup\{L_j^{\alpha};\, 0 \leqq j \leqq n\}\,.$$

Hence we have

$$\lim_{\alpha} \int_0^{1/2} \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n \Delta_{\alpha}^{t_n} k_{\alpha} \Delta_{\alpha}^{t_{n-1}-t_n} k_{\alpha} \ldots \Delta_{\alpha}^{t_1-t_2} k_{\alpha} \Psi$$
$$= \int_0^{1/2} \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n \Delta^{t_n} k \Delta^{t_{n-1}-t_n} k \ldots \Delta^{t_1-t_2} k \Psi\,.$$

Since

$$\sum_{n=0}^{\infty} \left\| \int_0^{1/2} \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n \Delta_{\alpha}^{t_n} k_{\alpha} \Delta_{\alpha}^{t_{n-1}-t_n} k_{\alpha} \ldots \Delta_{\alpha}^{t_1-t_2} k_{\alpha} \Psi \right\|$$
$$\leqq \sum_{n=0}^{\infty} (n!)^{-1} \|k_{\alpha}\|^n \|\Psi\| \leqq \|\Psi\| \exp\left\{\sup_{\alpha} \|k_{\alpha}\|\right\} < \infty\,,$$

we obtain (2.9).

§ 3. An Inequality

The main tool for our proof of Theorem 1 is the following:

Theorem 3. *Let* $\mathfrak{N}$ *be a finite Type* I *subfactor of a hyperfinite von Neumann algebra* $\mathfrak{M}$, Ψ *be a cyclic and separating unit vector for* $\mathfrak{M}$, $k = k^* \in \mathfrak{M}$, $\varrho^{\mathfrak{N}}(\Psi)$ *and* $\varrho^{\mathfrak{N}}(\Psi(k))$ *be the density matrices for the restrictions of vector states* ω_{Ψ} *and* $\omega_{\Psi(k)}$ *to* $\mathfrak{N}$, *i.e. the unique positive elements in* $\mathfrak{N}$ *satisfying*

$$(\Psi, Q\Psi) = \mathrm{tr}(\varrho^{\mathfrak{N}}(\Psi)\, Q)\,, \qquad (\Psi(k), Q\Psi(k)) = \mathrm{tr}(\varrho^{\mathfrak{N}}(\Psi(k))\, Q)$$

for all $Q \in \mathfrak{N}$. *Then*

$$(\Psi, k\Psi) \leqq (\Psi, \{\log \varrho^{\mathfrak{N}}(\Psi(k)) - \log \varrho^{\mathfrak{N}}(\Psi)\}\, \Psi) \leqq \log\{\|\Psi(k)\|^2\}\,. \quad (3.1)$$

First we prove the finite matrix case:

Lemma 7. *If* $\mathfrak{M}$ *is a finite Type* I *factor, then* (3.1) *holds.*

Proof. As is well known, there exists a unitary map u from the underlying Hilbert space to $\mathfrak{M}$ [considered as the Hilbert space with inner product $(Q_1, Q_2) = \mathrm{tr}(Q_1^* Q_2)$] such that $u(Qx) = Q(ux)$ for all $Q \in \mathfrak{M}$ and $(u\Psi) > 0$. From the characterization of J_Ψ and Δ_Ψ in [3], it is easy to see that $u(J_\Psi x) = (ux)^*$ and $u(\Delta_\Psi^\alpha x) = \varrho(\Psi)^\alpha x \varrho(\Psi)^{-\alpha}$ where $\varrho(\Psi) = (u\Psi)^2$ is the density matrix for ω_Ψ. Hence

$$u\Psi(k) = \sum_{n=0}^{\infty} \int_0^{1/2} \mathrm{d}t_1 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n \varrho(\Psi)^{t_n} k \varrho(\Psi)^{t_{n-1}-t_n} k \ldots$$

$$\ldots \varrho(\Psi)^{t_1 - t_2} k \varrho(\Psi)^{(1/2)-t_1} .$$

By the formula (5.4) in [2], with $A = k/2$ and $B = (\log \varrho(\Psi))/2$, we obtain

$$u\Psi(k) = e^{\{k + \log \varrho(\Psi)\}/2} .$$

Hence

$$\log \varrho(\Psi(k)) - \log \varrho(\Psi) = k . \tag{3.2}$$

We now recall an inequality derived by Lindblad. Let A and B be strictly positive elements of $\mathfrak{M}$ which we assume to be a finite Type I factor. Let $\mathfrak{N}$ be a subfactor of $\mathfrak{M}$ and π be the conditional expectation from $\mathfrak{M}$ onto $\mathfrak{N}$. Namely, for each $C \in \mathfrak{M}$, $\pi(C)$ is defined as the element of $\mathfrak{N}$ satisfying $\varphi_0(\pi(C)Q) = \varphi_0(CQ)$ for all $Q \in \mathfrak{N}$ where φ_0 denotes the tracial state on $\mathfrak{M}$. If $\mathrm{tr} A = \mathrm{tr} B$, Umegaki defines the information between A and B by

$$I(A, B) = \mathrm{tr}(A \log A - A \log B)$$

which is always positive. (Umegaki's definition is for any semifinite $\mathfrak{M}$ and operators A and B affiliated with $\mathfrak{M}$ satisfying $A \geqq 0$, $B \geqq 0$, $s(A) \geqq s(B)$ and $\varphi_0(A) = \varphi_0(B) < \infty$ where $s(C)$ denotes the support projection of C.) Lindblad obtains the following inequality in Theorem 1 of [11] (also see Theorem 4 of [10]).

$$0 \leqq I(\pi(A), \pi(B)) \leqq I(A, B) . \tag{3.3}$$

We set $A = \varrho(\Psi)$ and $B = \varrho(\Psi(k))/\|\Psi(k)\|^2$. We then have $\pi(A) = \varrho^{\mathfrak{N}}(\Psi)$, $\pi(B) = \varrho^{\mathfrak{N}}(\Psi(k))/\|\Psi(k)\|^2$. Substituting these into (3.3) and using (3.2), $\mathrm{tr} A = \mathrm{tr} B = \mathrm{tr}\,\pi(A) = \mathrm{tr}\,\pi(B) = 1$ ($\|\Psi\| = 1$), $\mathrm{tr} AQ = (\Psi, Q\Psi)$ for $Q \in \mathfrak{M}$ and $\mathrm{tr}\,\pi(A) Q = (\Psi, Q\Psi)$ for $Q \in \mathfrak{N}$, we obtain (3.1).

Proof of Theorem 3. There exists an increasing sequence of finite Type I factors $\mathfrak{M}_n$ with $\mathfrak{M}_n \supset \mathfrak{N}$ and $\mathfrak{M} = \left(\bigcup_n \mathfrak{M}_n\right)''$ since $\mathfrak{N}' \cap \mathfrak{M}$ is hyperfinite. Let $k_n \in \mathfrak{M}_n$ be such that $\|k_n\| \leqq \|k\|$, $k_n^* = k_n$ and $\lim_n k_n = k$. By

Lemma 7, we have

$$(\Psi, k_n \Psi) \leqq (\Psi, \{\log \varrho^{\mathfrak{N}}(\Psi(k_n)) - \log \varrho^{\mathfrak{N}}(\Psi)\} \Psi) \leqq \log\{\|\Psi(k_n)\|^2\}. \quad (3.4)$$

By Lemma 6, we have $\lim_{\alpha} \Psi(k_n) = \Psi(k)$. Then the vector state $\omega^{\mathfrak{N}}_{\Psi(k_n)}$ of $\mathfrak{N}$ tends to $\omega^{\mathfrak{N}}_{\Psi(k)}$ in norm. Hence $\lim \varrho^{\mathfrak{N}}(\Psi(k_n)) = \varrho^{\mathfrak{N}}(\Psi(k))$. Since $\Psi(k)$ is separating by Corollary 4.4 of [1], $\varrho^{\mathfrak{N}}(\Psi(k))$ is a strictly positive matrix. Hence $\lim_n \log \varrho^{\mathfrak{N}}(\Psi(k_n)) = \log \varrho^{\mathfrak{N}}(\Psi(k))$. We then obtain (3.1) as the limit of (3.4).

§ 4. Proof of Theorem 1

By Theorem 1 of [5], we have

$$\log\{\|\Psi(k)\|^2\} \leqq \log(\Psi, e^k \Psi).$$

Hence we have the estimate

$$2\|k\| \geqq \varepsilon(k) \geqq 0,$$

$$\varepsilon(k) \equiv \log\{\|\Psi(k)\|^2\} - (\Psi, \{\log \varrho^{\mathfrak{N}}(\Psi(k)) - \log \varrho^{\mathfrak{N}}(\Psi)\}\Psi).$$

For $k = \beta W_\Lambda$, we have $\lim_{\Lambda\uparrow} \|k\|/\mathrm{N}(\Lambda) = 0$ by Lemma 4 of [7]. Therefore

$$\lim_{\Lambda\uparrow} \{\mathrm{N}(\Lambda)^{-1} \varepsilon(k)\} = 0. \quad (4.1)$$

By the Gibbs condition as formulated in Section 1 (see [4]), the restriction $\omega^{\mathfrak{N}}_{\Psi(\beta W_\Lambda)}$ of the vector state $\omega_{\Psi(\beta W_\Lambda)}$ to $\mathfrak{N} = \mathfrak{A}(\Lambda)$ is the Gibbs state $\varphi^{\Lambda}_{\mathrm{G}}$ up to a proportionality constant, which is $\omega_{\Psi(\beta W_\Lambda)}(1) = \|\Psi(\beta W_\Lambda)\|^2$. Since $\varrho^{\mathfrak{N}}(\varphi^{\Lambda}_{\mathrm{G}}) = e^{-\beta \mathrm{U}(\Lambda)}/\mathrm{tr}(e^{-\beta \mathrm{U}(\Lambda)})$, we obtain

$$\begin{aligned} &-\mathrm{N}(\Lambda)^{-1} \psi(\log \varrho^{\mathfrak{N}}(\Psi)) - \beta \mathrm{N}(\Lambda)^{-1} \psi(\mathrm{U}(\Lambda)) \\ &\quad = -\mathrm{N}(\Lambda)^{-1} \varepsilon(\beta W_\Lambda) + \mathrm{N}(\Lambda)^{-1} \log \mathrm{tr}(e^{-\beta \mathrm{U}(\Lambda)}). \end{aligned} \quad (4.2)$$

By taking the limit of large Λ and using (4.1), (1.3), (1.5) and the definition of P in (1.2), we obtain the variational equality:

$$P = \mathrm{s}(\psi) - \beta \psi(A).$$

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Communicated by G. Gallavotti

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