$$\alpha_{11}a'_1+(\mu_1\alpha_{11}-\alpha_{12})b_1+\alpha_{12}a'_2+\mu_2\alpha_{12}b_2 \text{ on } T_2 .$$

Suppose that $k_1\sim\varepsilon k_2$ in V_2 for $\varepsilon=1$ or -1. Since $k_1\sim 0$ on T_2, one of the following systems of equations holds.

$$\begin{cases}\mu_1\alpha_{11}=0\,,\\ \alpha_{11}-\mu_2\alpha_{12}=0\,.\end{cases}\qquad\qquad \begin{cases}\mu_1\alpha_{11}-\alpha_{12}=0\,,\\ \mu_2\alpha_{12}=0\,.\end{cases}$$

Using $\mu_1\mu_2\neq 0$, we can show that it is impossible. This completes the proof.

Now we return to the proof for Case 2. Without loss of generality, we may assume that $\varphi_1F_3\cap F_3$ contains c^*. Suppose that there exists a curve of type IV on φ_1F_3 in $\varphi_1F_3\cap F_3$. Let c be a simple closed curve of type IV on φ_1F_3 which bounds a Möbius band B such that $B\cap F_3=c\cup c^*$. Since $B\cap V_2$ is an annulus, it follows from Assertion B that c is of type IV on F_3. Hence c bounds a Möbius band B' on F_3. Let F'_3 denote the surface obtained by deforming $F_3-B'\cup B$ slightly so that it is disjoint from B. Then, as is similar to Case 1 [Fig. 4.1], $\varphi_1F'_3\cap F'_3$ contains fewer curves of type IV on $\varphi_1F'_3$ than $\varphi_1F_3\cap F_3$. Repeating these procedures, we can show that φ_1 is equivalent to φ_2 such that $\varphi_2F_3\cap F_3$ does not contain a curve of type IV on φ_2F_3 and F_3.

Suppose that $\varphi_2F_3\cap F_3$ contains at least two curves of type III on φ_2F_3. Then there exists an annulus A on φ_2F_3 such that $A\cap F_3=\partial A$. Let A' be an annulus on F_3 which bounds ∂A. Deforming $F_3-A'\cup A$ slightly until it is disjoint from A, we obtain A' such that $\varphi_2F'_3\cap F'_3$ has fewer components than $\varphi_2F_3\cap F_3$. Hence we can find an involution φ which is equivalent to φ_2 such that $\varphi F_3\cap F_3$ consists of c^* and at most one curve of type III on φF_3. If $\varphi F_3\cap F_3$ contains a curve c of type III, then c is φ-invariant. Since any two-sided curve in $\varphi F_3\cap F_3$ is not φ-invariant [12], the proof is completed.

Case 3. We will show that this case can not occur except for $\mu_1\mu_2=-2$.

Assertion C. *Suppose that $\mu_1\mu_2\neq -2$. Let l_1 and l_2 be disjoint simple closed curves on T_2 such that πl_1 is of type* II *or* V, *and πl_2 is of type* III *on F_3. Then l_1 is not homologous to εl_2, for $\varepsilon=1$ and -1, in V_2.*

Proof. Let ρ be an autohomeomorphism of F_3 such that $\rho\pi l_1$ coincides with $\partial N(c_1)$ or c_1 and $\rho\pi l_2=b^*$. By $\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}$ we denote a matrix corresponding to ρ^{-1}. Then, by Lemma 3.2, l_1 is homologous to $\alpha_{22}b_1-\alpha_{21}b_2$ and l_2 is homologous to either

$$\varepsilon(-\alpha_{21}a'_1-\mu_1\alpha_{21}b_1-\alpha_{22}a'_2+(\alpha_{21}-\mu_2\alpha_{22})b_2)$$

or

$$\varepsilon(\alpha_{21}a'_1+(\mu_1\alpha_{21}-\alpha_{22})b_1+\alpha_{22}a'_2+\mu_2\alpha_{22}b_2),\ \text{for } \varepsilon = 1 \text{ or } -1, \text{ on } T_2 .$$

and $\rho c = \partial N(c_\mu)$, $\mu = 1$, 2 or 3. An annulus $B \cap V_2$ has the boundaries k_1 and k_2 such that $\pi k_1 = c'$ and $\pi k = c$. If we suppose that $B' \not\supset c'$, then $\rho c = \partial N(c_2)$ or $\partial N(c_3)$. Hence, in order to show that $B' \supset c'$, it suffices to prove that $k_1 \not\sim \tilde{\rho}^{-1} c_2$, $-\tilde{\rho}^{-1} c_3$, $\tilde{\rho}^{-1} c_3$ and $-\tilde{\rho}^{-1} c_3$ in V_2, where $\tilde{\rho}$ denotes the lifting of ρ. Let $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$ be a matrix in $GL(2, Z)$ corresponding to the isotopy class of ρ^{-1}. Then it follows from Lemma 3.2 that

$$\tilde{\rho}^{-1} \tilde{c}_1 \sim \alpha_{22} b_1 - \alpha_{21} b_2, \ \tilde{\rho}^{-1} \tilde{c}_2 \sim -(\alpha_{22} + \alpha_{21}) b_1 + (\alpha_{21} + \alpha_{11}) b_2$$

and

$$\tilde{\rho}^{-1} \tilde{c}_3 \sim -\alpha_{12} b_1 + \alpha_{11} b_2 \text{ in } T_2 .$$

Since $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = \pm 1$, it can be easily shown that $k_1 = \tilde{\rho}^{-1} \tilde{c}_1 \not\sim \tilde{\rho}^{-1} \tilde{c}_2$, $-\tilde{\rho}^{-1} \tilde{c}_2$, $\tilde{\rho}^{-1} \tilde{c}_3$ and $-\tilde{\rho}^{-1} \tilde{c}_3$ in V_2. Let F_3' be the surface obtained by deforming $F_3 - B' \cup B$ slightly keeping the exterior of $N(B)$ fixed until it intersects B in c' [Fig. 4.1].

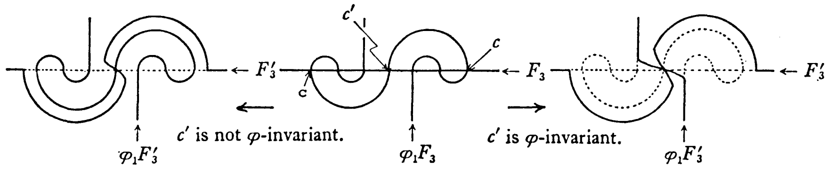


Fig. 4.1

Then $\varphi_1 F_3' \cap F_3'$ has fewer components that $\varphi_1 F_3 \cap F_3$. Since we can deform F_3' onto F_3 by an ambient isotopy, we obtain an involution φ_2 such that $\varphi_2 F_3 \cup F_3$ is isotopic to $\varphi_1 F_3' \cup F_3'$ in $L(2\alpha, \beta)$. Repeating these procedures, we can show that φ_1 is equivalent to φ such that $\varphi F_3 \cap F_3$ consists of three curves of type II on φF_3 and F_3.

Case 2. In this case each curve of $\varphi_1 F_3 \cap F_3$ is of either type I, type III or type IV.

Assertion B. *Let k_1 and k_2 be simple closed curves on T_2 such that πk_1 is of type* I *or* IV, *and πk_2 is of type* III. *Then k_1 is not homologous to εk_2, for $\varepsilon = 1$ and -1, in V_2.*

Proof. Let ρ be an autohomeomorphism of F_3 such that $\rho \pi k_1$ coincides with c^* or $\partial N(c^*)$, and $\rho \pi k_2 = a^*$. By $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$ we denote a matix in $GL(2, Z)$ corresponding to the isotopy class of ρ^{-1}. Then, by using Lemma 3.4, k_2 or $-k_2$ is homologous to either

$$-\alpha_{11} a_1' - \mu_1 \alpha_{11} b_1 - \alpha_{12} a_2' + (\alpha_{11} - \mu_2 \alpha_{12}) b_2$$

or

$$Id\times i\colon\ M\times S(E\oplus F)\to M\times V_m^{\Lambda}(E\oplus F\oplus\Lambda^{m-1})$$

induces a bundle embedding $\tilde{j}\colon B'\to B$. There is a one-to-one correspondence between G-maps from M to $V_m^{\Lambda}(E\oplus F\oplus\Lambda^{m-1})$ and cross sections of B, and there is also a one-to-one correspondence between their homotopies. This shows that the following two lemmas are equivalent:

Lemma 3. *Let*

$$f\colon\ M\to V_m^{\Lambda}(E\oplus F\oplus\Lambda^{m-1})$$

be a G-map, and let

$$P\colon\ \partial M\times[0,1]\to V_m^{\Lambda}(E\oplus F\oplus\Lambda^{m-1})$$

be a G-homotopy with $P_0=f|\partial M$ *and* $P_1(\partial M)\subset i(S(E\oplus F))$. *Then P extends to a G-homotopy*

$$Q\colon\ M\times[0,1]\to V_m^{\Lambda}(E\oplus F\oplus\Lambda^{m-1})$$

with $Q_0=f$ *and* $Q_1(M)\subset j(S(E\oplus F))$.

Lemma 4. *Let* $N=M/G$. *Let* $s\colon N\to B$ *be a cross section of B, and let P:* $\partial N\times[0,1]\to B|\partial N$ *be a homotopy of cross section of* $B|\partial N$ *with* $P_0=s|\partial N$ *and* $P_1(\partial N)\subset\tilde{j}(B')$. *Then P extends to a homotopy of cross section of B,* $Q\colon N\times[0,1]\to B$, *with* $Q_0=s$ *and* $Q_1(N)\subset\tilde{j}(B')$.

We give a proof of Lemma 4 making use of the obstruction theory. Refer to [4; Part III] for the obstruction theory.

Proof of Lemma 4. Since N is a smooth manifold, we obtain a triangulation of N. Let $n=\dim S(E\oplus F)$. Then $\dim N\leq n$, and $S(E\oplus F)$, which is the fibre of B', is $(n-1)$-connected. So the cross section $\tilde{j}^{-1}P_1$ of $B'|\partial N$ extends to a cross section $s_1; N\to B'$ of B'. We see from Proposition 1 that $V_m^{\Lambda}(E\oplus F\oplus\Lambda^{m-1})$ is also $(n-1)$-connected. Let N^{n-1} denote the $(n-1)$-skeleton of N, which contains ∂N. Then P extends to a homotopy of cross section,

$$R\colon\ N^{n-1}\times[0,1]\to B|N^{n-1},$$

with $R_0=s|N^{n-1}$ and $R_1=\tilde{j}s_1|N^{n-1}$. So, if $\dim N<n$, the lemma is proved.

Now let $\dim N=n$. Let $B(\pi_n)$ and $B'(\pi_n)$ be the bundles of coefficients associated with the bundles B and B' by the n-th homotopy group, respectively. Also let $C^n(N; B(\pi_n))$ and $C^n(N; B'(\pi_n))$ be the groups of n-cochains of N with coefficients in $B(\pi_n)$ and $B'(\pi_n)$, respectively. The bundle embedding $\tilde{j}\colon B'\to B$ induces a group homomorphism

$$\tilde{j}_*\colon\ C^n(N; B'(\pi_n))\to C^n(N; B(\pi_n)).$$

We see from Proposition 1 that $\tilde{j}_*$ is an epimorphism. Let homotopy set.

Our result is

Theorem 2. *Let E, F be representations of a compact Lie group G over Λ. Let*

$$j_*: [S(E), S(E\oplus F)]_G \to [S(E), V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})]_G$$

be the transformation induced from the G-map

$$j: S(E\oplus F) \to V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1}).$$

Then

(a) *j_* is surjective,*

(b) *j_* is bijective in particular in each case of the followings* (i), (ii):

(i) $\Lambda=\boldsymbol{R}$, $\dim_R E^H$ *is odd for any* $H\in\mathfrak{N}(E, F)=\{H\in\mathfrak{M}_r(E)\mid \dim_R F^H=0\}$, *and*

$$r = \prod_{H\in\mathfrak{N}(E,F)} r_H: [S(E), S(E\oplus F)]_G \to \prod_{H\in\mathfrak{N}(E,F)}[S(E^H), S(E^H\oplus F^H)]$$

is injective,

(ii) $\Lambda=\boldsymbol{C}$ *or* $\boldsymbol{Q}$, *and r is injective,*

(c) *if* $\dim_R E^G\geq 2$ *then* $[S(E), V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})]_G$ *has a group structure and j_* is a group homomorphism.*

NOTE. The injectivity of r is studied by several authors, e.g., Hauschild [1; Satz 4.5].

In the subsequent sections we prove Theorem 2. Section 2 is devoted to preliminary lemmas. Section 3 is devoted to proving the surjectivity of j_*, and section 4 is devoted to proving the injectivity of j_*. In section 5 we give a group structure to $[S(E), V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})]_G$ so that j_* is a group homomorphism.

2. Preliminary lemmas

Let E, F be representations of a compact Lie group G over Λ, and let M be a compact, smooth, free G-manifold with $\dim M\leq\dim S(E\oplus F)$. Consider the fibre bundles

$$B = M\times_G V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1}) \to M/G$$

with fibre $V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$, and

$$B' = M\times_G S(E\oplus F) \to M/G$$

with fibre $S(E\oplus F)$. The G-map

$$d = d(s, R, \tilde{j}s_1) \in C^n(N; B(\pi_n))$$

be the deformation n-cochain. (See [4; p 172].) There is $d' \in C^n(N; B'(\pi_n))$ with $\tilde{j}_*(d')=d$. By [4; 33.9] there is a cross section s_2 of B' such that s_2 agrees with s_1 on N^{n-1} and $d(s_1, s_2)=-d'$, where $d(s_1, s_2)$ is the difference n-cochain. (Also see [4; p 172].) We see

$$d(\tilde{j}s_1, \tilde{j}s_2) = \tilde{j}_*(d(s_1, s_2)) = -d\,.$$

We define a homotopy of cross section of $B|N^{n-1}$,

$$S\colon N^{n-1}\times[0, 1] \to B|N^{n-1}\,,$$

by

$$S(x, t) = \begin{cases} R(x, 2t) & \text{if } 0\leq t\leq 1/2 \\ \tilde{j}s_2(x) & \text{if } 1/2\leq t\leq 1\,. \end{cases}$$

By [4; 33.7],

$$\begin{aligned} d(s, S, \tilde{j}s_2) &= d(s, R, \tilde{j}s_1)+d(\tilde{j}s_1, \tilde{j}s_2) \\ &= d-d \\ &= 0\,. \end{aligned}$$

By this and [4; 33.8], S extends to a homotopy of cross section of B, $T\colon N\times[0, 1]\to B$, with $T_0=s$ and $T_1=\tilde{j}s_2$. Let $\partial N\times[0, 1]\subset N$ be a collar of ∂N in N. Then we define a homotopy $Q\colon N\times[0, 1]\to B$ as, for $x\in N$ and $t\in[0, 1]$,

$$\begin{aligned} &Q(x, t) = T(x, t) \quad \text{if} \quad x\in N-\partial N\times[0, 1)\,, \\ &Q(x, t) = T(x, t/(2-r)) \quad \text{if} \quad x=(y, r)\in\partial N\times[0, 1] \quad \text{and} \quad 2t+r\leq 2\,, \\ &Q(x, t) = T(x, (t+r-1)/r) \quad \text{if} \quad x=(y, r)\in\partial N\times[0, 1]\,, \quad 2t+r\geq 2 \\ &\qquad \text{and} \quad r\neq 0\,. \end{aligned}$$

Q is well-defined, and is an extension of P with the desired property.

Q.E.D.

3. The surjectivity of j_*

Let E, F be representations of a compact Lie group G over Λ, and let

$$j_*\colon [S(E), S(E\oplus F)]_G \to [S(E), V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})]_G$$

be the transformation induced from the G-map

$$j\colon S(E\oplus F) \to V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})\,.$$

The purpose of this section is to prove the surjectivity of j_*. Since j is an embedding, it suffices to prove the following fact:

Lemma 5. *Let*

$$f: S(E) \to V_m^{\Lambda}(E \oplus F \oplus \Lambda^{m-1})$$

be a G-map, and let N be a compact smooth G-submanifold of S(E) with dim $N=$ dim $S(E)$. *Let*

$$T: N \times [0, 1] \to V_m^{\Lambda}(E \oplus F \oplus \Lambda^{m-1})$$

be a G-homotopy with $T_0=f|N$ *and* $T_1(N) \subset j(S(E \oplus F))$. *Then T extends to a G-homotopy*

$$R: S(E) \times [0, 1] \to V_m^{\Lambda}(E \oplus F \oplus \Lambda^{m-1})$$

with $R_0=f$ *and* $R_1(S(E)) \subset i(S(E \oplus F))$.

Proof. $\mathfrak{M}(E)$ is a finite set. Let us number its elements

$$\mathfrak{M}(E) = \{(H_1), (H_2), \cdots, (H_a)\}$$

in such a way that if $i<k$ then $(H_i) \not\subset (H_k)$. Consider the following Assertion:

Assertion. *There are compact smooth G-submanifolds* $M_0, M_1, \cdots, M_a$ *of* $S(E)$ *such that*

$$\dim M_i = \dim S(E) \quad \text{for} \quad i = 0, 1, \cdots, a, \quad M_0 \supset N, \quad \textit{and}$$
$$\operatorname{Int} M_i \supset M_{i-1} \cup S(E)_{(H_i)} \quad \textit{for} \quad i = 1, 2, \cdots, a.$$

Furthermore there are G-homotopies $R^{(0)}, R^{(1)}, \cdots, R^{(a)}$ *such that*

$$R^{(i)}: M_i \times [0, 1] \to V_m^{\Lambda}(E \oplus F \oplus \Lambda^{m-1}) \quad \textit{for} \quad i = 0, 1, \cdots, a,$$
$$R_0^{(i)} = f | M_i \quad \textit{for} \quad i = 0, 1, \cdots, a,$$
$$R_1^{(i)}(M_i) \subset j(S(E \oplus F)) \quad \textit{for} \quad i = 0, 1, \cdots, a,$$
$$R^{(0)} | N \times [0, 1] = T, \quad \textit{and}$$
$$R^{(i)} | M_{i-1} \times [0, 1] = R^{(i-1)} \quad \textit{for} \quad i = 1, 2, \cdots, a.$$

Lemma 5 follows from the Assertion since $M_a=S(E)$. In the following we prove the Assertion.

N and T satisfy the conditions for M_0 and $R^{(0)}$, respectively. Suppose that $M_0, \cdots, M_{i-1}$, and $R^{(0)}, \cdots, R^{(i-1)}$ are constructed. Put

$$M = (S(E) - \operatorname{Int} M_{i-1})^{H_i} = S(E^{H_i}) - \operatorname{Int} M_{i-1}^{H_i}.$$

Then M is a compact smooth manifold with boundary $\partial M = M \cap \partial M_{i-1}$. Moreover M is $N(H_i)$-invariant, and all isotropy subgroups on M are H_i. So M becomes a free $N(H_i)/H_i$-manifold. Regard E^{H_i} and F^{H_i} as representations of $N(H_i)/H_i$. By Lemma 3 there is an $N(H_i)/H_i$-homotopy

$$Q\colon M\times[0, 1]\to V_m^\Lambda(E^{H_i}\oplus F^{H_i}\oplus\Lambda^{m-1})$$

such that

$$Q_0=f|M,$$
$$Q_1(M)\subset i(S(E^{H_i}\oplus F^{H_i}))\,, \qquad \text{and}$$
$$Q|\partial M\times[0, 1]=R^{(i-1)}|\partial M\times[0, 1]\,.$$

Since $G(M)=G\times_{N(H_i)}M$, we may extend Q to a G-homotopy

$$Q'\colon G(M)\times[0, 1]\to G(V_m^\Lambda(E^{H_i}\oplus F^{H_i}\oplus\Lambda^{m-1}))\subset V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$$

such that

$$Q_0'=f|G(M),$$
$$Q_1'(G(M))\subset i(S(E\oplus F))\,, \qquad \text{and}$$
$$Q'|\partial G(M)\times[0, 1]=R^{(i-1)}|\partial G(M)\times[0, 1]\,.$$

Applying [3; Lemma 1.1] to the G-manifold $A=S(E)-\operatorname{Int}M_{i-1}$ and the submanifold $G(M)$ of A, we obtain compact G-submanifolds K, L of A such that

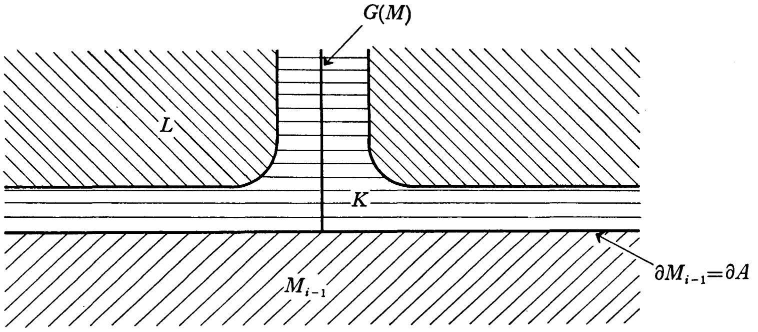
(i) $K\cup L=A$,

(ii) $\partial L=L\cap K$,
$\partial K=\partial L\cup\partial A=\partial L\cup\partial M_{i-1}$,
$\partial M_{i-1}\cap\partial L=\phi$,

(iii) $\partial M_{i-1}\cup G(M)\subset K$, and

(iv) K is a mapping cylinder of some G-map

$$\psi\colon \partial L\to\partial M_{i-1}\cup G(M)\,.$$



Put $M_i=M_{i-1}\cup K$ in $S(E)$. Then M_i is a compact smooth G-submanifold of $S(E)$ with $\dim M_i=\dim S(E)$, and with $\operatorname{Int}M_i\supset M_{i-1}\cup S(E)_{(H_i)}$. According to (iv), let us denote a point of K by the form $[y, s]$, where $y\in\partial L$ and $s\in[0, 1]$. Under this form $[y, 1]=y$ and $[y, 0]=\psi(y)$. We define a G-homotopy

$$R^{(i)}\colon M_i\times[0, 1]\to V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$$

as the following: For $(x, t)\in M_i\times[0, 1]$,

$$R^{(i)}(x, t) = R^{(i-1)}(x, t) \quad \text{if} \quad x\in M_{i-1},$$
$$R^{(i)}(x, t) = f([y, s-2t]) \quad \text{if} \quad x = [y, s]\in K \quad \text{and} \quad 2t\leq s,$$
$$R^{(i)}(x, t) = R^{(i-1)}(\psi(y), (2t-s)/(2-s)) \quad \text{if} \quad x=[y, s]\in K,$$
$$\psi(y)\in\partial M_{i-1} \quad \text{and} \quad s\leq 2t,$$
$$R^{(i)}(x, t) = Q'(\psi(y),\ (2t-s)/(2-s)) \quad \text{if} \quad x = [y, s]\in K,$$
$$\psi(y)\in G(M) \quad \text{and} \quad s\leq 2t.$$

M_i and $R^{(i)}$ constructed above satisfy the conditions in the Assertion. Thus this completes the proof. Q.E.D.

Now let X, Y be G-spaces, and let $x_0\in X^G$, $y_0\in Y^G$. Denote by

$$[(X, x_0), (Y, y_0)]_G$$

the set of G-homotopy classes rel. x_0 of G-maps $f\colon X\to Y$ with $f(x_0)=y_0$.
The following Proposition is required in section 5.

Proposition 6. *Let E, F be representations of G, and let $x_0\in S(E^G)$, $y_0\in S(E^G\oplus F^G)$. Then*

$$j_*\colon\ [(S(E), x_0), (S(E\oplus F), y_0)]_G \to [(S(E), x_0), (V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1}), j(y_0))]_G$$

is surjective.

Proof. Let

$$f\colon\ S(E) \to V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$$

be a G-map with $f(x_0)=j(y_0)$. Let D be a G-invariant, top-dimensional, small disc in $S(E)$ with x_0 as its center. We may deform f to a G-map f' such that $f'(D)=j(y_0)$ and $f'\simeq f$ rel. x_0. By Lemma 5 there is a G-homotopy

$$R\colon\ S(E)\times[0, 1] \to V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$$

such that

$$R_0 = f',$$
$$R_1(S(E))\subset j(S(E\oplus F)), \qquad \text{and}$$
$$R(D\times[0, 1]) = j(y_0).$$

Then f' is G-homotopic to R_1 rel. x_0. This proves the Proposition. Q.E.D.

4. The injectivity of j_*

Let E, F be representations of a compact Lie group G over Λ, and let

$$j_*\colon [S(E),\ S(E\oplus F)]_G \to [S(E),\ V_m^\Lambda(E\oplus F\oplus \Lambda^{m-1})]_G$$

be the transformation induced from the G-map

$$j\colon S(E\oplus F) \to V_m^\Lambda(E\oplus F\oplus \Lambda^{m-1}).$$

The purpose of this section is to prove the injectivity of j_* under the assumption (i) or (ii) in Theorem 2.

For any closed subgroup H of G, let

$$j^H = j\,|\,S(E^H\oplus F^H)\colon\ S(E^H\oplus F^H) \to V_m^\Lambda(E^H\oplus F^H\oplus \Lambda^{m-1})\,.$$

The following diagram is commutative:

$$\begin{array}{ccc}
[S(E),\ S(E\oplus F)]_G & \xrightarrow{\ j_*\ } & [S(E),\ V_m^\Lambda(E\oplus F\oplus \Lambda^{m-1})]_G \\
\downarrow r_H & & \downarrow r'_H \\
[S(E^H),\ S(E^H\oplus F^H)] & \xrightarrow[\ j_*^H\ ]{} & [S(E^H),\ V_m^\Lambda(E^H\oplus F^H\oplus \Lambda^{m-1})]
\end{array}$$

where r_H and r'_H are the transformations restricting to the fixed point set by H.

Now suppose

$$j_*(\alpha) = j_*(\beta)$$

for $\alpha,\ \beta\in [S(E),\ S(E\oplus F)]_G$. Then, by the commutativity of the above diagram,

$$j_*^H r_H(\alpha) = j_*^H r_H(\beta)$$

for any closed subgroup H of G. Proposition 1 implies that j_*^H is an isomorphism under the assumption (i) or (ii) in Theorem 2. Thus

$$r_H(\alpha) = r_H(\beta)$$

for any H. Hence $r(\alpha)=r(\beta)$. By the assumption r is injective, hence $\alpha=\beta$. Thus j_* is injective.

5. The group structure

Let E, F be representations of a compact Lie group G over Λ. Suppose $\dim_R E^G\geq 2$. Then, according to [3; Section 6], $[S(E),\ S(E\oplus F)]_G$ has a group structure. In the similar way we may give a group structure to

$$[S(E),\ V_m^\Lambda(E\oplus F\oplus \Lambda^{m-1})]_G$$

so that

$$j_*\colon [S(E),\ S(E\oplus F)]_G \to [S(E),\ V_m^\Lambda(E\oplus F\oplus \Lambda^{m-1})]_G$$

is a group homomorphism. To show this is the purpose of this section.

Lemma 7. *Suppose* $\dim_R E^G=2$ *and* $x_0 \in S(E^G)$. *Let*

$$\omega\colon [0,1] \to S(E^G) \subset S(E)$$

be a path with $\omega(0)=\omega(1)=x_0$. *Then there is a G-homotopy*

$$H\colon S(E)\times[0,1] \to S(E)$$

such that

$$H_0 = H_1 = Id, \quad \textit{and}$$
$$H(x_0, t) = \omega(t) \quad \textit{for any} \quad t\in[0,1].$$

Proof. Choose a homotopy

$$J\colon S(E^G)\times[0,1] \to S(E^G)\subset S(E)$$

such that

$$J_0 = J_1 = \textit{the inclusion}, \quad \text{and}$$
$$J(x_0, t) = \omega(t) \quad \text{for any} \quad t\in[0,1].$$

Denote by $(E^G)^\perp$ the orthogonal complement of E^G in E, and denote a point of E by the form $x+y$ where $x\in E^G$ and $y\in(E^G)^\perp$. Define

$$H\colon S(E)\times[0,1] \to S(E)$$

as

$$H(x+y, t) = ||x||J(x/||x||, t)+y \quad \text{if} \quad x\neq 0, \qquad \text{and}$$
$$H(x+y, t) = y \quad \text{if} \quad x=0.$$

Then H is a G-homotopy with the desired property. Q.E.D.

Lemma 8. *Suppose* $\dim_R E^G\geq 2$, $x_0\in S(E^G)$ *and* $y_0\in S(E^G\oplus F^G)$. *Then the natural transformations*

$$\psi_1\colon [(S(E), x_0), (S(E\oplus F), y_0)]_G \to [S(E), S(E\oplus F)]_G$$

and

$$\psi_2\colon [(S(E), x_0), (V_m^\wedge(E\oplus F\oplus\Lambda^{m-1}), j(y_0))]_G \to [S(E), V_m^\wedge(E\oplus F\oplus\Lambda^{m-1})]_G$$

are bijective.

Proof. Consider the commutative diagram:

$$\begin{array}{ccc}
[(S(E), x_0), (S(E\oplus F), y_0)]_G & \xrightarrow{\psi_1} & [S(E), S(E\oplus F)]_G \\
{\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle j_*} \\
[(S(E), x_0), (V_m^\wedge(E\oplus F\oplus\Lambda^{m-1}), j(y_0))]_G & \xrightarrow[\psi_2]{} & [S(E), V_m^\wedge(E\oplus F\oplus\Lambda^{m-1})]_G
\end{array}$$

In [3; Section 6], ψ_1 is already seen to be bijective. The two j_* are surjective by the arguments in section 3. So it follows that ψ_2 is surjective.

It only remains to show that ψ_2 is injective. Suppose

$$\psi_2(\alpha) = \psi_2(\beta)$$

for $\alpha, \beta \in [(S(E), x_0), (V_m^\Lambda(E \oplus F \oplus \Lambda^{m-1}), j(y_0))]_G$. Since j_* is surjective, there are G-maps

$$f, g: S(E) \to S(E \oplus F)$$

such that $f(x_0) = y_0$, $g(x_0) = y_0$, and jf, jg are representatives of α, β, respectively. There also is a G-homotopy

$$K: S(E) \times [0, 1] \to V_m^\Lambda(E \oplus F \oplus \Lambda^{m-1})$$

with $K_0 = jf$ and $K_1 = jg$. Define a path

$$\omega: [0, 1] \to V_m^\Lambda(E^G \oplus F^G \oplus \Lambda^{m-1})$$

by $\omega(t) = K(x_0, t)$ for $t \in [0, 1]$. Then

$$\omega(0) = \omega(1) = j(y_0).$$

By Proposition 1 there is a path

$$\omega': [0, 1] \to S(E^G \oplus F^G)$$

such that

$$\omega'(0) = \omega'(1) = y_0, \qquad \text{and}$$
$$\omega \simeq j\omega' \text{ rel. } \{0, 1\}.$$

Let D be a G-invariant, top-dimensional, small disc in $S(E)$ with x_0 as its center, and let $D' = \frac{1}{2}D$. By radius contraction we may deform K to a G-homotopy

$$K': S(E) \times [0, 1] \to V_m^\Lambda(E \oplus F \oplus \Lambda^{m-1})$$

such that $K'(x, t) = j\omega'(t)$ for $x \in D'$ and $t \in [0, 1]$. Moreover, if we put $f' = K'_0$ and $g' = K'_1$, then

$$f'(S(E)) \subset i(S(E \oplus F)),$$
$$g'(S(E)) \subset i(S(E \oplus F)),$$

and f', g' are G-homotopic to jf, jg rel. x_0, respectively.

So, to show $\alpha = \beta$ we must show that f' is G-homotopic to g' rel. x_0.

(i) Suppose $\dim_{\boldsymbol{R}} E^G \oplus F^G > 2$. By Proposition 1, $j\omega'$ is homotopic to the constant path at $j(y_0)$ rel. $\{0, 1\}$. From this we may deform K' to a G-homotopy

$$K'': S(E) \times [0, 1] \to V_m^\Lambda(E \oplus F \oplus \Lambda^{m-1})$$

such that

$$K''_0=f', \quad K''_1=g', \qquad \text{and}$$
$$K''(x_0, t)=j(y_0) \qquad \text{for any} \quad t\in[0, 1].$$

Therefore f' is G-homotopic to g' rel. x_0.

(ii) Suppose $\dim_R E^G\oplus F^G=2$. Define a path

$$\omega'': [0, 1] \to S(E^G\oplus F^G)$$

by $\omega''=(\omega')^{-1}$, i.e., $\omega''(t)=\omega'(1-t)$. Applying Lemma 7 to the path ω'', there is a G-homotopy

$$H: S(E\oplus F)\times[0, 1] \to S(E\oplus F)$$

such that

$$H_0=H_1=Id, \qquad \text{and}$$
$$H(y_0, t)=\omega''(t) \qquad \text{for any} \quad t\in[0, 1].$$

Define a G-homotopy

$$L: S(E)\times[0, 1] \to V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$$

as, foı $x\in S(E)$ and $t\in[0, 1]$,

$$L(x, t)=K'(x, 2t) \quad \text{if} \quad 0\le t\le 1/2, \qquad \text{and}$$
$$L(x, t)=jH(j^{-1}g'(x), 2t-1) \quad \text{if} \quad 1/2\le t\le 1.$$

Then

$$L_0=f', \quad L_1=g', \qquad \text{and}$$
$$L(x, t)=j\omega'*j\omega''(t) \quad \text{for} \quad x\in D' \quad \text{and} \quad t\in[0, 1]\cdot$$

$j\omega'*j\omega''$ is homotopic to the constant path at $j(y_0)$ rel. $\{0, 1\}$. So we may deform L to a G-homotopy

$$L': S(E)\times[0, 1] \to V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$$

such that

$$L'_0=f', \quad L'_1=g', \qquad \text{and}$$
$$L'(x_0, t)=j(y_0) \quad \text{for any} \quad t\in[0, 1].$$

Therefore f' is G-homotopic to g' rel. x_0. Q.E.D.

Now suppose $\dim_R E^G\ge 2$, $x_0\in S(E^G)$ and $y_0\in S(E^G\oplus F^G)$. Let λ be the real one-dimensional subspace of E spanned by x_0, and let $\lambda^\perp$ be the orthogonal complement of λ in E. We may identify $S(E)$ with a nonreduced suspension

$$\Sigma S(\lambda^\perp)=[0, 1]\times S(\lambda^\perp)/\sim.$$

Under this identification $x_0=[0, x]$ and $-x_0=[1, x]$ for $x\in S(\lambda^{\perp})$. Let Y be one of $S(E\oplus F)$ and $V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$. Put $z_0=y_0$ if Y is the former, and $z_0=j(y_0)$ if Y is the latter. Then we may give a group structure to $[S(E), Y]_G$ as follows. Let $[f]$, $[g]\in[S(E), Y]_G$. By Lemma 8 we may choose f and g in such a way that $f(-x_0)=z_0$ and $g(x_0)=z_0$. Define $h\colon S(E)\to Y$ as, for $[t, x]\in\Sigma S(\lambda^{\perp})=S(E)$,

$$h([t, x])=f([2t, x]) \quad \text{if} \quad 0\leq t\leq 1/2\,, \qquad \text{and}$$
$$h([t, x)]=g([2t-1, x]) \quad \text{if} \quad 1/2\leq t\leq 1\,.$$

Define $[f]+[g]=[h]$. This gives a group structure to $[S(E), Y]_G$, and the transformation

$$j_*\colon [S(E), S(E\oplus F)]_G\to[S(E), V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})]_G$$

becomes a group homomorphism. We note that this group structure does not depend on the choice of $x_0\in S(E^G)$ and $y_0\in S(E^G\oplus F^G)$.

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