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On the equivariant self homotopy equivalences of spheres

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§ 1. Introduction.

Let G be a compact Lie group, V its orthogonal representation with a G-invariant metric, and $S(V)$ the unit sphere in V. Let $[S(V), S(V)]_G$ be the set of all G-homotopy classes of G-maps of $S(V)$ into itself. If $\dim_{\boldsymbol{R}} V^G \geqq 2$, then this set has a natural ring structure.

R. L. Rubinsztein [**3**] discussed the ring structure of $[S(V), S(V)]_G$. Moreover he gave a classification of G-maps $f: S(V) \to S(V)$. Another classification of G-maps were given by S. J. Willson [**5**]. T. tom. Dieck [**2**] and G. B. Segal [**4**] gave several important results for the Burnside ring and the equivariant stable homotopy group.

We are interested in the multiplicative group of the ring $[S(V), S(V)]_G$, denoted by $\boldsymbol{E}_G[S(V)]$, which consists of all G-homotopy equivalences of $S(V)$ into itself. In this paper we shall prove the following results. (Notations are given in § 2.)

THEOREM I. *Let G be a finite abelian group, and let V be its orthogonal representation such that $\dim_{\boldsymbol{R}} V^G \geqq 2$. Then we have*

$$|\boldsymbol{E}_G[S(V)]| = 2^{N+1},$$

where $N = Car.\{H \mid H \in O(V)$ and $|G/H| = 2\}$.

THEOREM II. *Let D_n be the dihedral group generated by a and b with relation $a^n = b^2 = abab = 1$, and let V be its complex representation such that $\dim_{\boldsymbol{R}} V^G \geqq 2$. We put*

$$N_1 = Car.\{i \mid i \in [n]^*,\ i \text{ is odd},\ i \neq 1, \text{ and } (b, a^i) \in O(V)\}$$

and

$$N_2 = Car.\{i \mid i \in [n]^*,\ i \text{ is even},\ i \neq 2,\ (ba, a^i), (b, a^i) \in O(V)\}.$$

Then we have

$$|\boldsymbol{E}_{D_n}[S(V)]|=2^{N_1+N_2+N_0+1}$$

where $N_0=Car.(\{(a), (b, a^2), (ba, a^2)\} \cap O(V))$.

§ 2. Preparation.

2.1. From now on, let G be a finite group and V its orthogonal representation. Throughout this paper we use the following notations:

(H)	the conjugacy class of a subgroup H of G,
G_X	the isotropy group at $x \in S(V)$,
$O(V)$	the set of orbit types on $S(V)$,
$\langle Y \rangle$	the subgroup generated by a subset Y of G,
$Car. X$	the cardinal number of a set X,
$\|K\|$	the order of a group K,
$A(V)$	the free abelian group generated by the set $O(V)$,
$C(G)$	the set of conjugacy classes of all subgroups of G,
$X_{(H)}$	the set $\{x \mid x \in S(V)$ and $(G_X)=(H)\}$,
V^G	the set $\{x \mid x \in V$ and $G_X=G\}$,
R^*	the multiplicative group of a ring R,
$\boldsymbol{Z}$	the ring of rational integeres,
$[n]$	the set $\{1, \cdots, n\}$ for a natunal number n,
$[\underline{n}]$	the set $\{0, \cdots, n-1\}$,
$[n]^*$	the set $\{i \mid i \in [n]$ and $i \mid n\}$.

Let Γ be the set of isomorphism classes of all finite G-sets. Addition and multiplication in Γ are defined by the disjoint union and the cartesian product, respectively. The Burnside ring $A(G)$ is defined to be the Grothendieck ring of Γ. Any finite G-set can be written as the disjoint union of its orbits under the G-action, each of which is isomorphic to a homogeneous G-space. So that equivalently, $A(G)$ is (additively) the free abelian group generated by the set $\{G/H \mid (H) \in C(G)\}$. We denote by $[X]$ the element of $A(G)$ represented by a finite G-set X. Then we have the formula

$$[X]=\sum_{(H)\in C(G)} \lambda_{(H)}[G/H], \tag{2.1.1}$$

where $\lambda_{(H)}=Car.\{e \mid x \in e \in X/G$ and $(G_x)=(H)\}$. For each element (H) of $O(V)$, we denote by the same letter (H) the corresponding element of $A(V)$ when there arises no confussion.

LEMMA 2.2 (Remark 8.2 [3]). *For any* $x, y \in S(V)$, *there is a point* $z \in S(V)$ *such that*

$$G_x \cap G_y = G_z.$$

2.3. There is a canonical group homomorphism

$$i_V : A(V) \longrightarrow A(G)$$

defined by $i_V((H)) = [G/H]$. We define a partial order on $O(V)$ by $(H) \leqq (K)$ if and only if H is conjugate to a subgroup of K. Suppose $(H_0), \cdots, (H_k)$ are all orbit types on $S(V)$ with

(2.3.1) $$(H_i) \not\geqq (H_j) \quad \text{for} \quad i<j.$$

Let $x=(g_1 H_i, g_2 H_j)$ be an element of the G-set $G/H_i \times G/H_j$, then we have

$$(G_x) \leqq (H_i) \quad \text{and} \quad (G_x) \leqq (H_j).$$

Therefore, from (2.1.1) and Lemma 2.2, we have

(2.3.2) $$[G/H_i][G/H_j] = \sum_{s=j}^{k} \lambda(s, i, j)[G/H_s] \qquad \text{for} \quad i \leqq j,$$

where $\lambda(s, i, j) = Car. \{e \mid x \in e \in (G/H_i \times G/H_j)/G \text{ and } (G_x)=(H_s)\}$. From (2.3.2), $i_V(A(V))$ is a subring of $A(G)$. So we consider $A(V)$ as a ring. If $(G) \in O(V)$, then $A(V)$ is a ring with unit element $1=(G)$.

THEOREM 2.4 (Theorem 7.2 and Theorem 8.4 [3]). *There is a bijection* Φ ; $[S(V), S(V)]_G \rightarrow A(V)$ *such that the diagram*

$$\begin{array}{ccc} [S(V), S(V)]_G & \xrightarrow{\ \Phi\ } & A(V) \\ \downarrow r_H & & \downarrow \chi_H \\ [S(V)^H, S(V)^H] & \xrightarrow{\ \deg\ } & \boldsymbol{Z} \end{array}$$

commutes for all subgroup H *of* G, *where* r_H *is the restriction transformation, and* χ_H *is the homomorphism defined by*

$$\chi_H((K)) = Car.(G/K)^H$$

for each generator (K) *of* $A(V)$. *If* $\dim_R V^G \geqq 2$ *and* $X_{(H)}/G$ *is connected for each orbit types* (H) *on* $S(V)$, *then* Φ *is a ring isomorphism and two G-maps*

(2.4.1) $$f_1, f_2 : S(V) \longrightarrow S(V) \quad \textit{are G-homotopic}$$

if and only if

$$\deg(f_1^H) = \deg(f_2^H) \quad \textit{for all} \quad (H) \in O(V).$$

THEOREM 2.5 (Proposition 8.1 [3]). *For any* $(H)\in O(V)$, $X_{(H)}/G$ *is connected, provided one of the following two conditions is satisfied:*

(2.5.1) $$G \text{ is a finite abelian group and } \dim_{\boldsymbol{R}} V^G \geqq 2$$

and

(2.5.2) $$V \text{ is a complex representation of } G.$$

From Theorem 2.4 and Theorem 2.5, we have

COROLLARY 2.6. *If* $\dim_{\boldsymbol{R}} V^G \geqq 2$ *and one of the conditions* (2.5.1) *and* (2.5.2) *is satisfied, then*

$$\Phi|\boldsymbol{E}_G[S(V)] : \boldsymbol{E}_G[S(V)] \longrightarrow A(V)^*$$

is a group isomorphism.

LEMMA 2.7. *We have*

$$\Delta^2=1 \qquad \text{for any} \quad \Delta\in A(G)^*.$$

PROOF. It is shown in Bredon [1] that there is an orthogonal representation $V(H)$ of G and a point $x\in S(V(H))$ such that $G_x=H$ for each subgroup H of G. Now we consider the representation

$$V_0=\bigoplus_{H\subset G} 2V(H)\oplus \boldsymbol{R}^2, \quad (G \text{ acts trivially on } \boldsymbol{R}^2)$$

then we have $O(V_0)=C(G)$. From Theorem 2.5, $X_{(H)}/G$ is connected for each subgroup H of G. Each element of $\boldsymbol{E}_G[S(V_0)]$ is of order 2 by (2.4.1). So the desired result follows from Corollary 2.6. Q.E.D.

§ 3. Proof of Theorem I.

In this section we assume that G is a finite abelian group, $\dim_{\boldsymbol{R}} V^G \geqq 2$, $O(V)=\{(H_0)=G, \cdots, (H_k)\}$ satisfies (2.3.1), $|G/H_i|=2$ for $1\leqq i\leqq N\leqq k$, and $|G/H_j|>2$ for $j>N$.

LEMMA 3.1. *We have*

(3.1.1) $$Car.((G/H_i\times G/H_j)/G)=|G/H_i\cdot H_j|,$$

(3.1.2) $$(H_i)(H_j)=|G/H_i\cdot H_j|(H_i\cap H_j) \quad \text{in} \quad A(V)$$

and

(3.1.3) $$s\geqq i, j \quad \text{if} \quad H_i\cap H_j=H_s,$$

$$s>i, j \quad \text{if} \quad H_i\cap H_j=H_s \quad \text{and} \quad H_i\cap H_j\neq H_i, H_j.$$

PROOF. (3.1.1) is trivial. (3.1.2) and (3.1.3) follows from (2.3.2) and (2.3.1), respectively. Q.E.D.

LEMMA 3.2. *For each subset* $I=\{i_1, \cdots, i_s\}$ *of* $[N]$, *we define an element* Δ_I *of* $A(V)$ *by*

$$\Delta_I=\prod_{t=1}^{s}(1-(H_{i_t})),$$

then $\Delta_I^2=1$.

PROOF. For each $i\in[N]$, $(1-(H_i))^2=1-2(H_i)+|G/H_i|(H_i)=1-2(H_i)+2(H_i)=1$ by the assumption and (3.1.2). Since $A(V)$ is a commutative ring, the desired result follows at once. Q.E.D.

LEMMA 3.3. *If* $\left(\sum_{i=0}^{k} x_i(H_i)\right)^2=1$, *where* $x_i\in\boldsymbol{Z}$, *then we have*

(3.3.1) $x_0=\pm 1$ *and* $x_i=0$ *or* $-x_0$ *for all* $i\in[N]$,

(3.3.2) $x_j=0$ *for all* $j>N$ *if* $x_i=0$ *for all* $i\in[N]$.

PROOF. Let $\Delta=\sum_{i=0}^{k} x_i(H_i)$ and let c_i be the coefficient of (H_i) in Δ^2. Since $H_i\cap H_j\neq H_i, H_j$ for $0<i\neq j\leqq N$, we have $c_0^2=x_0^2=1$ and $c_i=2x_0x_i+|G/H_i|x_i^2=2x_i(x_0+x_i)=0$ for all $i\in[N]$. So we have (3.3.1). If $x_i=0$ for all $i\in[N]$, then we have $c_{N+1}=2x_0x_{N+1}+|G/H_{N+1}|x_{N+1}^2=0$ by Lemma 3.1. Since $|G/H_{N+1}|>2$, we have $x_{N+1}=0$. Then (3.3.2) follows from the induction on $j>N$. Q.E.D.

LEMMA 3.4. *Let* $\beta=\sum_{i=N+1}^{k} x_i(H_i)$. *If* $(\Delta_I+\beta)\in A(V)^*$, *then* $\beta=0$.

PROOF. By the assumption and Lemma 3.1, we can write

$$\Delta_I\beta=\sum_{i=N+1}^{k} y_i(H_i),$$

where $y_i\in\boldsymbol{Z}$. So $\Delta_I\beta=0$ by (3.3.2). There are G-maps h and G-homotopy equivalence f such that

$$\Phi([h])=\beta,\quad \Phi([f])=\Delta_I \quad \text{and} \quad [f][h]=0,$$

by Theorem 2.4 and Corollary 2.6. Since f induces an (additive) isomorphism f_*; $[S(V), S(V)]_G\to[S(V), S(V)]_G$, we have $[h]=0$ and $\beta=0$. Q.E.D.

LEMMA 3.5. *Each element of* $A(V)^*$ *is of the form* Δ_I *for some* $I\subset[N]$.

PROOF. Let $\Delta=\sum_{i=0}^{k} x_i(H_i)$ and $I=\{i \mid x_i\neq 0$ and $i\in[N]\}$. If $\Delta\in A(V)^*$, then we have

$$x_i=\begin{cases}\text{the coefficient of } (H_i) \text{ in } \Delta_I=-1 & \text{if } x_0=1\\ \text{the coefficient of } (H_i) \text{ in } -\Delta_I=1 & \text{if } x_0=-1,\end{cases}$$

for all $i\in I$ by Lemma 3.3 and the definition of Δ_I. So the desired result follows from Lemma 3.4. Q.E.D.

PROOF OF THEOREM I.

Since $Car.\ \{I \mid I\subset[N]\}=2^N$, we have

$$|\boldsymbol{E}_G[S(V)]|=|A(V)^*|=2^{N+1}$$

by Lemma 3.5 and Corollary 2.6. Q.E.D.

COROLLARY 3.6. *If $|G|$ is odd, then we have*

$$|\boldsymbol{E}_G[S(V)]|=2.$$

COROLLARY 3.7. *If G acts semi-free on V, then we have*

$$|\boldsymbol{E}_G[S(V)]|=\begin{cases}2 & if \quad |G|\neq 2\\ 4 & if \quad |G|=2.\end{cases}$$

§ 4. Proof of Theorem II.

LEMMA 4.1. *Let D_n be the dihedral group generated by a and b with relation $a^n=b^2=abab=1$. We have*

(4.1.1) $a^i b=ab^{-i}$ *and* $(a^i b)^2=1$ *for any* $i\in\boldsymbol{Z}$,

(4.1.2) *each element of D_n is of the form ba^i or a^i for some $i\in\boldsymbol{Z}$,*

(4.1.3) $(ba^i, a^j)=(ba^{2p-i}, a^j)$ *for any* $i, j, p\in\boldsymbol{Z}$,

(4.1.4) $(ba^i, a^j)=(ba^{i-2p}, a^j)$ *for any* $i, j, p\in\boldsymbol{Z}$,

(4.1.5) $(ba^{2i}, a^j)=(b, a^j)$ *and* $(ba^{2i+1}, a^j)=(ba, a^j)$ *for any* $i, j\in\boldsymbol{Z}$,

(4.1.6) $(ba, a^j)=(b, a^j)$ *if either j is odd or n is odd,*

and

(4.1.7) $(ba, a^j)\neq(b, a^j)$ *if j is even and n is even.*

PROOF. (4. 1. 1) and (4. 1. 2) are trivial. Now we have

(1) $ba^p\langle ba^i, a^j\rangle ba^p=\langle ba^{2p-i}, a^j\rangle$,

(2) $a^p\langle ba^i, a^j\rangle a^{-p}=\langle ba^{i-2p}, a^j\rangle$,

(3) $a^{i+1}\langle ba, a^{2i+1}\rangle a^{-(i+1)}=\langle b, a^{2i+1}\rangle$

and

(4) $\langle ba, a^j\rangle=\langle ba^{1+n}, a^j\rangle$.

Then (4. 1. 3)$\sim$(4. 1. 6) follows from (1)$\sim$(4). If $(ba, a^{2j})=(b, a^{2j})$ and n is even, then we have

(i) $a^t\langle b, a^{2j}\rangle a^{-t}=\langle ba^{-2t}, a^{2j}\rangle=\langle ba, a^{2j}\rangle$

or

(ii) $ba^t\langle b, a^{2j}\rangle ba^t=\langle ba^{2t}, a^{2j}\rangle=\langle ba, a^{2j}\rangle$,

for some $t\in \boldsymbol{Z}$. If (i) holds, then we have $ba^{1+2sj}=ba^{-2t}$ for some $s\in \boldsymbol{Z}$, so $1+2sj+2t\equiv 0 \pmod n$. If (ii) holds, then we have $ba^{1+2sj}=ba^{2t}$ for some $s\in \boldsymbol{Z}$, so $1+2sj-2t\equiv 0 \pmod n$. Therefore n must be odd. This contradiction establishes (4. 1. 7). Q. E. D.

COROLLARY 4. 2. *We have*

$$C(D_n)=\begin{cases} \{(a^i), (b, a^i) \mid i\in[n]^*\} & \text{if } n \text{ is odd} \\ \{(a^i), (b, a^i), (ba, a^j) \mid i, j\in[n]^* \text{ and } j \text{ is even}\} & \text{if } n \text{ is even.} \end{cases}$$

PROOF. This follows from Lemma 4. 1. Q. E. D.

In $A(D_n)$, let

$$\alpha_i=[D_n/\langle a^i\rangle],\quad \beta_i=[D_n/\langle b, a^i\rangle] \qquad \text{for each } i\in[n]^*$$

and

$$\gamma_i=[D_n/\langle ba, a^i\rangle] \qquad \text{for each even } i\in[n]^*.$$

For $i, j\in[n]^*$, we write $m(i, j)$ (resp. $M(i, j)$) for the greatest common diviser (resp. the least common multiple) of i and j. Throughout this section let us abbreviate $m(i, j)=m$ and $M(i, j)=M$ when there arises no confusion. There exists integers k_1, k_2, q_1 and q_2 such that $i=mk_1$, $j=mk_2$ and $k_1q_1+k_2q_2=1$.

LEMMA 4. 3. *In* $A(D_n)^*$, *we have*

$$\alpha_i\alpha_j=2m\alpha_M.$$

PROOF. Let X_1 denotes the D_n-set $D_n/\langle a^i\rangle\times D_n/\langle a^j\rangle$. For $s_1, s_2\in[\underline{m}]$, we have

(4. 3. 1) $[\langle a^i\rangle, a^{s_1}\langle a^j\rangle]=[\langle a^i\rangle, a^{s_2}\langle a^j\rangle]$ if and only if $s_1=s_2$

and

$$[\langle a^i\rangle, ba^{s_1}\langle a^j\rangle]=[\langle a^i\rangle, ba^{s_2}\langle a^j\rangle] \quad \text{if and only if} \quad s_1=s_2$$

in X_1/D_n. Let $t=hm+s$ ($s\in[\underline{m}]$). Then

$$\begin{aligned}[\langle a^i\rangle, a^t\langle a^j\rangle]&=[a^{-hm}\langle a^i\rangle, a^s\langle a^j\rangle]\\ &=[a^{-hm(k_1q_1+k_2q_2)}\langle a^i\rangle, a^s\langle a^j\rangle]\\ &=[a^{-hmk_2q_2}\langle a^i\rangle, a^s\langle a^j\rangle]=[\langle a^i\rangle, a^{s+mhk_2q_2}\langle a^j\rangle]\\ &=[\langle a^i\rangle, a^s\langle a^j\rangle].\end{aligned}$$

It is trivial that $[\langle a^i\rangle, a^{s_1}\langle a^j\rangle]\neq[\langle a^i\rangle, ba^{s_2}\langle a^j\rangle]$ for any $s_1, s_2\in \boldsymbol{Z}$. Since $\langle a^i\rangle\cap a^s\langle a^j\rangle a^{-s}=\langle a^i\rangle\cap ba^s\langle a^j\rangle ba^s=\langle a^M\rangle$ and $Car.(X_1/D_n)=2m$, the desired result follows from (2.3.2). Q.E.D.

LEMMA 4.4. *In $A(D_n)^*$, we have*

$$\beta_i\beta_j=\begin{cases}\beta_M+(m-1)/2\alpha_M & \text{if } m \text{ is odd,}\\ 2\beta_M+(m/2-1)\alpha_M & \text{if } m \text{ is even.}\end{cases}$$

PROOF. Let X_2 denotes the D_n-set $D_n/\langle b, a^i\rangle\times D_n/\langle b, a^j\rangle$. For $s, s_1, s_2\in[\underline{m}]$, we have

(4.4.1) $\quad [\langle b, a^i\rangle, a^{s_1}\langle b, a^j\rangle]=[\langle b, a^i\rangle, a^{s_2}\langle b, a^j\rangle]$ in X_2/D_n if and only if $s_1+s_2=m$ or $s_1=s_2$,

and

(4.4.2) $$(\langle b, a^i\rangle\cap a^s\langle b, a^j\rangle a^{-s})=\begin{cases}(b, a^M) & \text{if either } s=m/2 \text{ or } s=0,\\ (a^M) & \text{otherwise.}\end{cases}$$

: Proof of (4.4.1) : If $[\langle b, a^i\rangle, a^{s_1}\langle b, a^j\rangle]=[\langle b, a^i\rangle, a^{s_2}\langle b, a^j\rangle]$, then we can separate two cases:

(1) $\quad a^{(it_1+s_1-s_2)}=a^{jt_2}$ for some $t_1, t_2\in\boldsymbol{Z}$,

(2) $\quad a^{(it_1+s_1+s_2)}=a^{jt_2}$ for some $t_1, t_2\in\boldsymbol{Z}$.

In the case (2), $it_1+s_1+s_2\equiv jt_2 \pmod n$. Since $m\mid i$, $m\mid j$ and $s_1, s_2\in[\underline{m}]$, we have $s_1+s_2=m$. Conversely, if $s_1+s_2=m$, then we have

$$\begin{aligned}[\langle b, a^i\rangle, a^{s_1}\langle b, a^j\rangle]&=[\langle b, a^i\rangle, ba^{(-iq_1+s_1)}\langle b, a^j\rangle]\\ &=[\langle b, a^i\rangle, ba^{(s_1-iq_1-jq_1)}\langle b, a^j\rangle]\\ &=[\langle b, a^i\rangle, ba^{(s_1-m)}\langle b, a^j\rangle]=[\langle b, a^i\rangle, a^{(m-s_1)}\langle b, a^j\rangle]\\ &=[\langle b, a^i\rangle, a^{s_2}\langle b, a^j\rangle].\end{aligned}$$

Therefore we have (4.4.1).

: Proof of (4.4.2) : Let $H=\langle b, a^i\rangle \cap a^s\langle b, a^j\rangle a^{-s}$. It is trivial that H contains $\langle a^M\rangle$. If $\langle a^M\rangle$ is a proper subgroup of H, then we have $ba^{it_1}=ba^{(-2s+jt_2)}$ for some $t_1, t_2\in \boldsymbol{Z}$. So $it_1+2s-jt_2\equiv 0 \pmod n$ and $m|2s$. Since $s\in[\underline{m}]$, we have

$$s=\begin{cases} 0 & \text{if } m \text{ is odd,} \\ 0 \text{ or } m/2 & \text{if } m \text{ is even.} \end{cases}$$

Conversely, if $s=m/2$, then

$$a^s\langle b, a^j\rangle a^{-s}=\langle ba^{-m}, a^j\rangle=\langle ba^{-m(k_1q_1+k_2q_2)}, a^j\rangle$$
$$=\langle ba^{-mk_1q_1}, a^j\rangle=\langle ba^{-iq_1}, a^j\rangle$$

and

$$a^{(-iq_1/2)}(\langle b, a^i\rangle \cap \langle ba^{-iq_1}, a^j\rangle)a^{iq_1/2}$$
$$=\langle ba^{iq_1}, a^i\rangle \cap \langle b, a^j\rangle=\langle b, a^i\rangle \cap \langle b, a^j\rangle=\langle b, a^M\rangle.$$

Therefore we have (4.4.2).

Each element of X_2/D_n is of the form $[\langle b, a^i\rangle, a^s\langle b, a^j\rangle]$ for some $s\in[\underline{m}]$. So the desired result follows from (4.4.1), (4.4.2) and (2.3.2). Q.E.D.

LEMMA 4.5. *In $A(D_n)^*$, we have*

$$\alpha_i\beta_j=m\alpha_M.$$

PROOF. Let X_3 denotes the D_n-set $D_n/\langle a^i\rangle\times D_n/\langle b, a^j\rangle$. Each element of X_3/D_n is of the form $[\langle a^i\rangle, a^s\langle b, a^j\rangle]$ for some $s\in[\underline{m}]$. Since $\langle a^i\rangle\cap a^s\langle b, a^j\rangle a^{-s}=\langle a^M\rangle$ for any $s\in\boldsymbol{Z}$, the desired result follows from (2.3.2).

Q.E.D.

LEMMA 4.6. *In $A(D_n)$, we have*

$$\alpha_i\gamma_j=m\alpha_M.$$

PROOF. This will be proved by the same way as in Lemma 4.5.

Q.E.D.

LEMMA 4.7. *In $A(D_n)$, we have*

$$\beta_i\gamma_j=\begin{cases} m/2\alpha_M & \textit{if } m \textit{ is even,} \\ ((m+1)/2-1)\alpha_M+\gamma_M & \textit{if } m \textit{ is odd.} \end{cases}$$

PROOF. Let X_4 denotes the D_n-set $D_n/\langle b, a^i\rangle\times D_n/\langle ba, a^j\rangle$. For $s, s_1, s_2 \in[\underline{m}]$, we have

(4.7.1) $[\langle b, a^i\rangle, a^{s_1}\langle ba, a^j\rangle]=[\langle b, a^i\rangle, a^{s_2}\langle ba, a^j\rangle]$ in X_4/D_n

if and only if $s_1+s_2=m+1$ or $s_1=s_2$ or $s_1=0$ and $s_2=1$,

and

$$(4.7.2)\qquad (\langle b,\ a^i\rangle \cap a^s\langle ba,\ a^j\rangle a^{-s}) = \begin{cases} (ba,\ a^M) & \text{if either } s=0,\ m=1 \\ & \text{or } s=(m+1)/2,\ m\neq 1, \\ (a^M) & \text{otherwise.} \end{cases}$$

(4.7.1) and (4.7.2) will be proved by the same way as in (4.4.1) and (4.4.2). Since each element of X_4/D_n is of the form $[\langle b,\ a^i\rangle,\ a^s\langle ba,\ a^j\rangle]$ for some $s\in[\underline{m}]$, the desired result follows from (4.7.1), (4.7.2) and (2.3.2).

Q. E. D.

LEMMA 4.8. *In $A(D_n)$, we have*

$$\gamma_i\gamma_j=2\gamma_M+(m/2-1)\alpha_M.$$

PROOF. This will be proved by the same way as in Lemma 4.4.

Q. E. D.

From the above Lemmas 4.3-4.8, we have

LEMMA 4.9. *We put*

$$\Delta_i=1+\alpha_i-2\beta_i \quad \textit{for each odd} \quad i\in([n]^*-\{1\}),$$

$$\nabla_i=1+\alpha_{2i}-\beta_{2i}-\gamma_{2i} \quad \textit{for each} \quad i\in([n/2]^*-\{1\}),$$

$$\Delta_1=1-\alpha_1,$$

$$\nabla_{(1,0)}=1-\beta_2,$$

and

$$\nabla_{(1,1)}=1-\gamma_2,$$

then those elements are in $A(D_n)^*$.

4.10. Let M_s $(s=1, 2, 3, 4)$ be the submodule of $A(D_n)$ defined as follows:

M_1=the submodule generated by the set $\{1, \alpha_1\}$,

M_2=the submodule generated by the set
$\{\alpha_i, \beta_i \mid i\in([n]^*-\{1\})$ and i is odd$\}$,

M_3=the submodule generated by the set $\{\alpha_2, \beta_2, \gamma_2\}$,

and

M_4=the submodule generated by the set
$\{\alpha_{2i}, \beta_{2i}, \gamma_{2i} \mid i\in([n/2]^*-\{1\})\}$.

Then it is trivial that $A(D_n)$ and $(M_1\oplus M_2\oplus M_3\oplus M_4)$ are isomorphic as additive groups. Let $\Delta=X+Y$, where $X\in(M_1\oplus M_2)$ and $Y\in(M_3\oplus M_4)$. Since M(odd, odd)=odd, M(odd, even)=even and M(even, even)=even, we have

(4.10.1) $$(2XY+Y^2)\in(M_3\oplus M_4) \text{ and } X^2\in(M_1\oplus M_2),$$

by Lemmas 4.3-4.8. If $\Delta\in A(D_n)^*$, then we have

(4.10.2) $$2XY+Y^2=0,\ X^2=1 \text{ and } X=\Delta(1+XY).$$

So we have

(4.10.3) $$A(D_n)^*=(M_1\oplus M_2)^*((1+M_3\oplus M_4)\cap A(D_n)^*).$$

LEMMA 4.11. *Let $\Delta=1+x\alpha_1+X$, where $x\in\boldsymbol{Z}$ and $X\in M_2$. If $\Delta\in A(D_n)^*$, then $x=0$ or $x=-1$.*

PROOF. Since $\Delta^2=1$ and $\alpha_1^2=2\alpha_1$, we have

$$\Delta^2=1+(2x^2+2x)\alpha_1+X^2+2x\alpha_1X+2X=1,$$

and

$$(X^2+2x\alpha_1+2X)\in M_2.$$

So $(2x^2+2x)=0$. Q.E.D.

LEMMA 4.12. *Let $\Delta=1+\sum_{(2i+1)|n}(x_i\alpha_{2i+1}+y_i\beta_{2i+1})$ and $l=\mathrm{Min}\{i\mid\{x_i, y_i\}\neq\{0\}\}$. If $l>0$ and $\Delta^2=1$, then we have $x_l=1$ and $y_l=-2$.*

PROOF. Let a_1 and b_1 be the coefficients of α_{2l+1} and β_{2l+1}, respectively, in Δ^2. Then we have

$$a_1=2(2l+1)x_l^2+ly_l^2+2x_l+2x_ly_l(2l+1)=0,$$

and

$$b_1=y_l^2+2y_l=0 \quad (y_l=0 \text{ or } -2),$$

by Lemmas 4.3-4.5. If $y_l=0$, then

$$a_1=2x_l((2l+1)+1)=0.$$

If $y_l=-2$, then

$$a_1=2(2l+1)(x_l-1)(x_l-2l/(2l+1))=0.$$

Since $l>0$ and $\{x_l, y_l\}\neq\{0\}$, we have $x_l=1$ and $y_l=-2$. Q.E.D.

LEMMA 4.13. *Let $\nabla=1+\sum_{2i|n}(x_i\alpha_{2i}+y_i\beta_{2i}+z_i\gamma_{2i})$ and $l=\mathrm{Min}\{i\mid\{x_i, y_i, z_i\}\neq\{0\}\}$. If $l>1$ and $\nabla^2=1$, then we have $x_l=1$ and $y_l=z_l=-1$.*

PROOF. Let a_1, b_1 and c_1 be the coefficients of α_{2l}, β_{2l} and γ_{2l}, respectively, in ∇^2. Then we have

(4.13.1) $$a_1=4lx_l^2+(l-1)(y_l^2+z_l^2)+4lx_l(y_l+z_l)+2x_l+2ly_lz_l=0,$$

$$b_1=2y_l^2+2y_l=0 \quad (y_l=0 \text{ or } -1),$$

and

$$c_1 = 2z_l^2 + 2z_l = 0 \quad (z_l = 0 \text{ or } -1).$$

If $y_l = z_l = 0$, then

$$a_1 = 4l(x_l + 1/2l)x_l = 0.$$

If either $y_l = 0$ and $z_l = -1$ or $y_l = -1$ and $z_l = 0$, then

$$a_1 = 4l(x_l - 1/2)(x_l - (l-1)2l) = 0.$$

If $y_l = z_l = -1$, then

$$a_1 = 4l(x_l - 1)(x_l - (2l-1)/2l) = 0.$$

Since $l > 1$, $\{x_l, y_l, z_l\} \subset \boldsymbol{Z}$ and $\{x_l, y_l, z_l\} \neq \{0\}$, we have $x_l = 1$ and $y_l = z_l = -1$.

Q. E. D.

Let S_k $(k=1, 2)$ be the subgroups of $A(D_n)^*$ defined as follows:

S_1 = the subgroup generated by the set $\{1, \Delta_1, \Delta_i\}$,

and

S_2 = the subgroup generated by the set

$\{1, \nabla_{(1,0)}, \nabla_{(1,1)}, \nabla_i\}$ (cf. Lemma 4.9).

LEMMA 4.14. *Let* $\Delta = 1 + x\alpha_1 + X$, *where* $x \in \boldsymbol{Z}$ *and* $X \in M_2$. *If* $\Delta \in A(D_n)^*$, *then* $\Delta \in S_1$.

PROOF. From Lemma 4.11, $x = 0$ or -1.

: In the case $x = 0$: From Lemma 4.12, we can write

$$\Delta = 1 + \alpha_{l_1} - 2\beta_{l_1} + X_1$$

for some $l_1 \in ([n]^* - 1)$ and $X_1 \in M_2$ such that l_1 is odd and the coefficients of α_i and β_i in X_1 are equal to zero if $i \leqq l_1$ and $X \neq 0$. So we have

$$1 + X_1 = \Delta_{l_1} \Delta \in (1 + M_2) \cap A(D_n)^*.$$

Therefore we have

$$1 = \Delta_{l_k} \cdots \Delta_{l_2} \Delta_{l_1} \Delta$$

for some $l_t \in [n]^*$ $(t = i, \cdots, k)$ by the induction. So $\Delta \in S_1$.

: In the case $x = -1$: Since $\Delta_1 \Delta = 1 + \Delta_1 X$ and $\Delta_1 X \in M_2$, so the desired result follows from : In the case $x = 0$: .

Q. E. D.

LEMMA 4.15. *Let* $\nabla = 1 + x\alpha_2 + y\beta_2 + z\gamma_2 + X$, *where* $\{x, y, z\} \subset \boldsymbol{Z}$ *and* $X \in M_4$. *If* $\nabla \in A(D_n)^*$, *then* $\nabla \in S_2$.

PROOF. Since (4. 13. 1) is true for case $l=1$, we can separate four cases:

(1) $x=y=z=0$,

(2) $x=y=0$ and $z=-1$,

(3) $x=z=0$ and $y=-1$,

and

(4) $x=y=z=-1$.

: In the case (1): From Lemma 4. 13, we can write

$$\nabla=1+\alpha_{2l_1}-\beta_{2l_1}-\gamma_{2l_1}+X_1$$

for some $l_1\in[n/2]^*$ and $X_1\in M_4$ such that the coefficients of α_{2i}, β_{2i} and γ_{2i} in X_1 are equal to zero if $i\leqq l_1$ and $X\neq 0$. Therefore we have

$$1=\nabla_{l_k}\cdots\nabla_{l_2}\cdot\nabla_{l_1}\cdot\nabla$$

for some $l_t\in[n/2]^*$ $(t=1, \cdots, k)$ by the induction. So $\nabla\in S_2$.

: In the cases (2)~(4): This will be proved by the same way as in Lemma 4. 14 by the use of the elements $\nabla_{(1,0)}$, $\nabla_{(1,1)}$ and $\nabla_{(1,0)}\cdot\nabla_{(1,1)}$.

Q. E. D.

THEOREM 4. 16. *We put*

$$\bar{N}_1=Car. \{i \mid i \text{ is odd and } i\in([n]^*-\{1\})\}$$

and

$$\bar{N}_2=Car. \{i \mid i \text{ is even and } i\in([n]^*-\{2\})\}.$$

Then we have

$$A(D_n)^*=S_1\cdot S_2\cup -S_1\cdot S_2 \quad \text{and} \quad |A(D_n)^*|=2\cdot 2^{(\bar{N}_1+\bar{N}_2+3)}.$$

PROOF. For any $\Theta\in A(D_n)^*$, we can write $\Theta=\varDelta\cdot\nabla$, where $\varDelta\in(M_1\oplus M_2)^*$ and $\nabla\in(1+M_3\oplus M_4)\cap A(D_n)^*$, by (4. 10. 3). If the coefficient of β_1 ($\beta_1=1$ in $A(D_n)$) is equal to 1, then $\varDelta\in S_1$ and $\nabla\in S_2$ by Lemmas 4. 14 and 4. 15. From Lemmas 4. 3-4. 8, $\varDelta_{i_1}\cdot\varDelta_{i_2}\cdots\varDelta_{i_p}\neq\varDelta_{j_1}\cdot\varDelta_{j_2}\cdots\varDelta_{j_q}$ and $\nabla_{i_1}\cdot\nabla_{i_2}\cdots\nabla_{i_p}\neq\nabla_{j_1}\cdot\nabla_{j_2}\cdots\nabla_{j_q}$ if there exists some i_t such that $i_t\neq j_s$ for any s $(1\leqq s\leqq q)$. Therefore the desired result follows at once. Q. E. D.

PROOF OF THEOREM II. From (4. 4. 2), we have

$$\langle b, a^i\rangle\cap a\langle b, a^i\rangle a^{-1}=\langle a^i\rangle \qquad \text{if} \quad i\neq 1, 2.$$

Therefore if (b, a^i) $(i\neq 1, 2)$ is in $O(V)$, then (a^i) is also in $O(V)$ by Lemma 2. 2. From Lemma 4. 8, we have

$$\langle ba, a^i\rangle\cap a^s\langle ba, a^i\rangle a^{-s}\neq\langle b, a^i\rangle \qquad \text{if} \quad i\neq 1.$$

So the desired result follows from Theorem 4. 16. Q. E. D.

§ 5. Example.

Let $n=p^N$ (p is an odd prime), then we define a homomorphism (complex representation) $\varphi: D_n \to U(2N+2)$ as follows:

$$\varphi(a)=\begin{pmatrix} A_0 & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_N \end{pmatrix} \qquad A_i=\begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$$
$$\theta_i=2\pi/p^i \ (i=0, \cdots, N),$$

and

$$\varphi(b)=\begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}.$$

We define D_n-equivariant maps f_i $(i=0, \cdots, N)$ and $h: S^{4N+3} \to S^{4N+3}$ as follows:

$$f_i(z)=(z_0, w_0, \cdots, \bar{z}_i, \bar{w}_i, \cdots, z_N, w_N)$$

and

$$h(z)=(\bar{z}_0, w_0, \cdots, z_i, w_i, \cdots, z_N, w_N),$$

where $z=(z_0, w_0, \cdots, z_i, w_i, \cdots, z_N, w_N) \in S^{4N+3}$ and $\bar{z}_i$ is the conjugation of z_i. Since $D_n \cdot (z_0, 0, \cdots, 0)=(z_0, 0, \cdots, 0)$, we can use the Theorem II.

Now we have the following tables:

	Δ_{p^i}
$\langle a^{p^j} \rangle$	1
$\langle b, a^{p^j} \rangle$	$\begin{cases} 1 & \text{if } j<i \\ -1 & \text{if } j \geqq i \end{cases}$

$$\left(\begin{array}{c|c} & \Delta \in A(D_n) \\ \hline H & \chi_H(\Delta) \end{array} \right),$$

and

	$[f_i]$
$\langle a^{p^j} \rangle$	1
$\langle b, a^{p^j} \rangle$	$\begin{cases} 1 & \text{if } j<i \\ -1 & \text{if } j \geqq i \end{cases}$

$$\left(\begin{array}{c|c} & [f] \in \boldsymbol{E}_{D_n}[S^{4N+3}] \\ \hline H & \deg(f^H) \end{array} \right).$$

Therefore we have

$$\Phi([f_i])=\Delta_{pi} \quad \text{and} \quad \Phi([h])=-1,$$

by Theorem 2.4. Therefore $E_{D_n}[S^{4N+3}]$ is the group of order 2^{N+2} and the group generated by the set

$$\{[identity\ map],\ [h],\ [f_i] \mid i=0, \cdots, N\}.$$

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