225. On the Essential Set of Function Algebras

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Let A be a function algebra on a compact Hausdorff space X, that is, A is a closed subalgebra of $C(X)$ which separates the points of X and contains the constants. In the following we shall present several results relating to the essential set of A, some of which are regarded as generalizations of the results published in several literatures [4], [7], and [8]. Complete proofs of these theorems and other details will be published elsewhere.

Throughout this paper M will indicate the maximal ideal space of A. The Šilov boundary of A will be denoted by ∂A. For a subset F in X, we shall denote by $A|F$ the restricted algebra of A to F. If $A|F$ is closed in $C(F)$, $A|F$ is regarded as a function algebra on F. A closed subset F in X is called an interpolation set for A if $A|F=C(F)$, and is called a closed restriction set if $A|F$ is closed in $C(F)$. Let G be an open set in X. G is called a w-interpolation set for A if any compact subset in G is an interpolation set for A.

Theorem 1. *Let A be a function algebra on X and let $A\neq C(X)$. If G is any w-interpolation set for A, then $G\cap\partial_{A|E}=\phi$, where E is the essential set of A in X.*

Corollary. *Let A be a function algebra on X and suppose $E=\partial_{A|E}$. Then the set $X\sim E$ is the largest w-interpolation set for A.*

The hypothesis of the corollary is necessary. Let X be the set consisting of the unit circle and the origin 0 in the unit disc and let A be the restriction of A_0 to X, where A_0 denotes the function algebra of all continuous functions on the closed unit disc which are analytic on the open unit disc. Then $E=X$. But we here see that $G=\{0\}$ is a w-interpolation set and $G\supsetneqq X\sim E=\phi$.

Bishop [3] and Glicksberg [6] have proved that A is characterized by the disjoint closed partitions of its maximal antisymmetric sets and Tomiyama [11] has shown that among these sets the set P of all maximal antisymmetric sets in X consisting of one point is free from the representing space X and plays a special rôle in determining the struc-

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ture of the essential set E; in fact

$$E=X\sim \text{int}(P),$$

where int (P) is again free from the representing space X.

Now by the above mentioned result by Bishop and Glicksberg one easily sees that int (P) is a w-interpolation set for A, and for each point $x\in$ int (P) one can find a closed neighborhood V of x such that $A|V=C(V)$. Mullins [8] has proved that the converse is true if $X=M_A$ is metrizable. Here we shall show, using Theorem 1, that the result is generally true so far as $X=M_A$.

Theorem 2. *Let A be a function algebra on X with $X=M_A$ and let E be the essential set of A in X. Then, a point x belongs to the set $X\sim E$ ($=$int (P)) if and only if $A|V=C(V)$ for some closed neighborhood V of x.*

Once we could succeed to generalize the result of Mullins [8; Theorem 1], the same idea which derives his Theorem 2 from Theorem 1 would lead us to get the following.

Theorem 3. *Let A be a function algebra on X with $X=M_A$ and let $\{F_j\}_{j=1}^{\infty}$ be a closed cover of X such that $A|F_j$ is closed in $C(F_j)$ for each j. Then the closure of $\cup_{j=1}^{\infty}E_j$ is the essential set of A in X where E_j denotes the essential set of $A|F_j$ in F_j.*

Corollary. *Let A be a function algebra on a compact Hausdorff space X. If X is covered by interpolation sets for A of countable number, then $A=C(X)$.*

The above corollary has also been proved by Gamelin and Wilken [5], Chalice [4], and Mullins [8] (but [4] and [8] suppose more restricted conditions on A).

It is to be noticed that in Theorems 2 and 3 one can not expect the same results for an arbitrary representing space X. The example cited after the corollary of Theorem 1 shows that the origin 0 satisfies the condition in Theorem 2 but belongs to the essential set. And, if we consider the cover $\{F_1, F_2\}$ of this representing space X as $F_1=\{$unit circle$\}$ and $F_2=\{0\}$, then $E_1=\{$unit circle$\}$ and $E_2=\phi$ and $E=E_1\cup\{0\}\supsetneqq E_1=\bar{E}_1$. The reason which we could expect the above theorems is that in case $X=M_A$ the well known Šilov's theorem prevent us from facing the situation described in the above example. However, for some specialized fuction algebra one might expect this kind of generalization. Chalice [4], indeed, announces the results of this type. Including these results we shall present here more general results.

A function algebra A is said to be ε-regular on X for some (fixed positive number) ε if for each point x in X and each closed set F in X not containing x, there is a function f in A with $|1-f(x)|<\varepsilon$ and $|f|<\varepsilon$ on F.

Theorem 4. *Let A be an ε-regular function algebra on X for $0 < \varepsilon \leqq 1/2$. Then, a point x in X belongs to $X \sim E$ $(=int\ (P))$ if and only if $A \mid V = C(V)$ for some closed neighborhood V of x.*

Corollary 1. *If A is ε-normal function algebra on X for $0 < \varepsilon \leqq 1/2$, the same conclusion holds.*

Corollary 2. *If A is approximately regular, still more approximately normal, function algebra on X the consequence in the theorem is also true.*

Glicksberg [7] has proved the following theorem; If any closed subset in X is a closed restriction set for A, then $A = C(X)$. We moreover obtain the following.

Theorem 5. *Let A be a function algebra on X and let F_0 be a closed set in X. If $A \mid F_0$ is dense in $C(F_0)$ and if any compact subset F in $X \sim F_0$ is an interpolation set for A (or a closed restriction set for A), then $A = C(X)$.*

Corollary. *Let A be a function algebra on X. If F_0 is a closed set in X without perfect subsets (in particular, a closed countable set) and if any compact subset F in $X \sim F_0$ is an interpolation set for A (or a closed restriction set for A), then $A = C(X)$.*

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