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ON THE ETALE COHOMOLOGY OF TORSORS FOR ABELIAN SCHEMES

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Abstract. We study the Leray spectral sequence for the integral l-adic cohomology of torsors for abelian schemes. For an abelian scheme over a smooth, geometrically irreducible curve over a finite field, we relate the question of whether a torsor is determined by its etale cohomology with the conjectured finiteness of the set of all such torsors.

CONTENTS

Introduction.

(0.1) The theme of this paper is the Leray spectral sequence for the integral l-adic etale cohomology of torsors for abelian schemes, with special regard to the case of varieties over finite fields. As motivating background for our interest in the subject, we recall the following.

(0.2) The Tate conjecture (cf. [12, Conjecture 1]) in codimension 1 can be reduced to the case of surfaces. Over finite fields, this particular case of the conjecture is amenable to an earlier conjecture, asserting the finiteness of the Shafarevich-Tate group of an abelian variety over a global field—here of finite characteristic, say p_0—(cf. [14], [5]). Actually, it would be sufficient to show the finiteness of a single primary component of this group—for a prime different from 2, if $p_0=2$—(cf. [14], [8]). Now, for any prime l different from p_0, this is equivalent to the statement that, given an abelian scheme $\mathscr{A}$ over a smooth geometrically irreducible (but not necessarily projective) curve B over a finite field of characteristic p_0, the set $H^1(B, \mathscr{A})(l)$ of torsors for $\mathscr{A}$ killed by a power of l is finite.

(0.3) Torsors are a geometric incarnation of cohomology. The set of all torsors for $\mathscr{A}$ constitute the etale cohomology group $H^1(B, \mathscr{A})$. A possible way to split up the finiteness question occurs when one has attached an invariant to each torsor in some natural way. One can first try to prove the finiteness of the set of invariants so obtained—

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a necessary condition—and then hope that the invariant be strong enough to determine the torsor it comes from.

Here we are interested in testing a linear algebraic realization of the twist that makes up a torsor, extracted from its cohomology. Something reminiscent, in a sense, of the Tate module of an abelian variety over—say—a finite field, which is a Galois module that determines the isogeny class of the abelian variety it comes from (cf. [13]). It turns out that, sticking to $H^1(B, \mathscr{A})(l)$ with l fixed (outside an a priori determined finite set of primes), this invariant satisfies the first condition—its range is finite—and that the second condition—its strength—is equivalent to the finiteness statement itself. That is, it provides a quite exact measurement of the problem—although no step is made towards its solution.

(0.4) Write k for the finite groundfield and $\bar{k}$ for an algebraic closure of k. For a k-scheme X (resp. a k-morphism f) we shall write $\bar{X}=X\otimes_k\bar{k}$ (resp. $\bar{f}=f\otimes_k\bar{k}$). Fix a prime $l\neq p_0$. For a torsor $\mathscr{X}$ for $\mathscr{A}$, denoting by $\chi:\mathscr{X}\to B$ and $\alpha:\mathscr{A}\to B$ the respective structure maps, the higher direct image sheaves $R^i\bar{\chi}_*\mathbf{Z}_{l,\bar{\mathscr{X}}}$, $i\geq 0$, do not bear any information, for they are canonically isomorphic for all $\mathscr{X}$: $R^i\bar{\chi}_*\mathbf{Z}_{l,\bar{\mathscr{X}}}=R^i\bar{\alpha}_*\mathbf{Z}_{l,\bar{\mathscr{A}}}$. On the other hand, it turns out that for all but at most a finite set of primes l, the Leray spectral sequence

$$H^pR^q\bar{\alpha}_*\mathbf{Z}_{l,\bar{\mathscr{A}}}=H^pR^q\bar{\chi}_*\mathbf{Z}_{l,\bar{\mathscr{X}}}\Rightarrow H^{p+q}\mathbf{Z}_{l,\bar{\mathscr{X}}} \tag{0.5}$$

degenerates for all $\mathscr{A}$-torsors $\mathscr{X}$ (for rational l-adic coefficients this holds without exception, by Lefschetz duality, cf. [3]; cf. [2, p. 153] for the projectivity of $\bar{\chi}$ in the present circumstances). This provides us with short exact sequences of continuous $\mathbf{Z}_l[G]$-modules ($G=\mathrm{Gal}(\bar{k}/k)$)

$$0\to H^{p+1}R^{q-1}\bar{\alpha}_*\mathbf{Z}_{l,\bar{\mathscr{A}}}\to F^pH^{p+q}\mathbf{Z}_{l,\bar{\mathscr{X}}}/F^{p+2}H^{p+q}\mathbf{Z}_{l,\bar{\mathscr{X}}}\to H^pR^q\bar{\alpha}_*\mathbf{Z}_{l,\bar{\mathscr{A}}}\to 0\,, \tag{0.6}$$

and hence with natural maps, for any prime number l as above:

$$H^1(B,\mathscr{A})\to \mathrm{Ext}^1_{\mathbf{Z}_l[G]}(H^pR^q\bar{\alpha}_*\mathbf{Z}_{l,\bar{\mathscr{A}}},\, H^{p+1}R^{q-1}\bar{\alpha}_*\mathbf{Z}_{l,\bar{\mathscr{A}}})\,. \tag{0.7}$$

This map is actually a group homomorphism, and the first group is a torsion group, while the second one is a $\mathbf{Z}_l$-module of finite type. Hence (0.7) boils down to a group homomorphism into a finite group:

$$H^1(B,\mathscr{A})(l)\to \mathrm{Tors}(\mathrm{Ext}^1_{\mathbf{Z}_l[G]}(H^pR^q\bar{\alpha}_*\mathbf{Z}_{l,\bar{\mathscr{A}}},\, H^{p+1}R^{q-1}\bar{\alpha}_*\mathbf{Z}_{l,\bar{\mathscr{A}}}))\,. \tag{0.8}$$

The injectivity of this map would settle the finiteness question (0.2). Our result in this paper goes in the opposite direction: we show that, conversely, for $(p,q)=(0,2)$, the finiteness of $H^1(B,\mathscr{A})(l)$ implies the injectivity of (0.8) (Here, as before, l is taken outside a certain a priori determined finite set of primes).

For $(p,q)=(0,2)$ the sequence (0.6) reads

$$0\to H^1R^1\bar{\alpha}_*\mathbf{Z}_{l,\bar{\mathscr{A}}}\to H^2\mathbf{Z}_{l,\bar{\mathscr{X}}}/H^2\mathbf{Z}_{l,\bar{B}}\to H^0R^2\bar{\alpha}_*\mathbf{Z}_{l,\bar{\mathscr{A}}}\to 0\,. \tag{0.9}$$

(0.10) This sequence is reminiscent of, and related to the exact sequence defined by the torsor $\mathscr{X}$ (cf. [11, pp. 162, 192]):

$$
(0.11)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{P}ic^0_{\mathscr{X}/B} & \longrightarrow & \mathscr{P}ic_{\mathscr{X}/B} & \longrightarrow & \mathscr{N}\mathscr{S}_{\mathscr{X}/B} & \longrightarrow & 0\,. \\
 & & \| & & & & \| & & \\
 & & \hat{\mathscr{A}} & & & & \mathscr{N}\mathscr{S}_{\mathscr{A}/B} & &
\end{array}
$$

At least with a slight additional hypothesis, the sequence (0.11) determines the torsor $\mathscr{X}$, as follows: Let $\lambda\colon \mathscr{A}\to\hat{\mathscr{A}}$ be a polarization of the abelian scheme $\mathscr{A}$. The image $[\mathscr{X}']\in H^1(B, \hat{\mathscr{A}})$ of $\lambda\in H^0(B, \mathscr{N}\mathscr{S}_{\mathscr{A}/B})$ by the connecting homomorphism δ of (0.11) is given by the pullback diagram

$$
\begin{array}{ccc}
\mathscr{X}' & \longrightarrow & \boldsymbol{Z}_B \\
\big\downarrow & & \big\downarrow \lambda \\
\mathscr{P}ic_{\mathscr{X}/B} & \longrightarrow & \mathscr{N}\mathscr{S}_{\mathscr{A}/B}\,,
\end{array}
$$

hence $\delta(\lambda)=[\mathscr{P}ic^{\lambda}_{\mathscr{X}/B}]$. The class of this $\hat{\mathscr{A}}$-torsor equals the sum of $[\mathscr{P}ic^{\lambda}_{\mathscr{A}/B}]$ and the image $\lambda[\mathscr{X}]$ of $[\mathscr{X}]$ by the morphism $\lambda\colon H^1(B, \mathscr{A})\to H^1(B, \hat{\mathscr{A}})$. As $2[\mathscr{P}ic^{\lambda}_{\mathscr{A}/B}]=0$ by [9, p. 121], it follows that $2\lambda[\mathscr{X}]=2\delta(\lambda)$. Let $d=\deg(\lambda)$, and put $d=\mu\lambda$ for a suitable $\mu\colon \hat{\mathscr{A}}\to\mathscr{A}$. Then $2d[\mathscr{X}]=2\mu\delta(\lambda)$. If the period of $[\mathscr{X}]$ is prime to $2d$, then $[\mathscr{X}]$ is determined by $\mathscr{X}]$, and hence by the sequence (0.11).

We shall mimic (0.10), with the sequence (0.11) replaced by (0.9). It turns out that, sticking to torsors $[\mathscr{X}]$ such that $[\mathscr{X}]\in H^1(B, \mathscr{A})(l)$ with l an odd prime as in (0.4) and furthermore l prime to the degree of a previously arbitrarily chosen polarization of $\mathscr{A}$, a torsor is determined by its corresponding sequence (0.9) up to a divisible element of $H^1(B, \mathscr{A})(l)$. Hence, if this group is finite, $\mathscr{X}$ is uniquely determined in this way.

In Sections 1–3 we study the Leray spectral sequence

$$
(0.12)\qquad H^pR^q\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}\Rightarrow H^{p+q}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}}\,,\qquad m\in\boldsymbol{N}\cap\Gamma(B, \mathscr{O}_B)^*\,,
$$

for a torsor $\chi\colon \mathscr{X}\to B$ for an abelian scheme $\alpha\colon \mathscr{A}\to B$ over a general base scheme B and, specifically, the following aspects of it:

(i) Computation of the differentials d_2^{pq}.

(ii) Degeneration of (0.12), if the torsor $\mathscr{X}$ is sufficiently highly m-divisible.

(iii) When B is a scheme over a finite field k, and the torsor $\bar{\mathscr{X}}$, obtained by base change to $\bar{k}$, is sufficiently highly m-divisible, computation of the class of the extension of discrete $(\boldsymbol{Z}/m\boldsymbol{Z})[G]$-modules

$$
(0.13)\quad 0\to H^{p+1}R^{q-1}\bar{\alpha}_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{A}}}\to\frac{F^pH^{p+q}(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{X}}}}{F^{p+2}H^{p+q}(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{X}}}}\to H^pR^q\bar{\alpha}_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{A}}}\to 0\,.
$$

This is applied then, in Section 4, to the purpose explained in this introduction.

Due to expository reasons, (i) and (ii) above are treated in reverse order.

(0.14) NOTE. In Sections 2 and 3 we shall use the Čech cohomology as a working tool. Čech cohomology agrees with derived functor cohomology, for instance for quasicompact schemes such that every finite subset is contained in an affine open subset (cf. [1], or [7, p. 104]). This happens in particular for quasiprojective schemes over affine schemes, hence the condition holds for all schemes in Section 4. In Sections 2 and 3 we shall simply *assume* that this condition is satisfied everywhere. This is sufficient for our purposes. (It would be nice to understand things from the more generic perspective of derived categories, but we have not investigated this).

1. Divisible torsors.

(1.1) Let B be an arbitrary scheme, and let $\alpha: \mathscr{A} \to B$ be an abelian scheme over B. We shall consider torsors (=principal homogeneous spaces) $\chi: \mathscr{X} \to B$ for $\mathscr{A}$. We refer to [2, p. 152] or [7, p. 120] for definitions. The set of all isomorphism classes of such torsors is a subgroup of the etale cohomology group $H^1(B, \mathscr{A})$, written $H^1(B, \mathscr{A})_{\text{rep}}$ (cf. [7, p. 123]). In [11, p. 178] (cf. also [7, loc. cit.]) one finds conditions ensuring that $H^1(B, \mathscr{A})_{\text{rep}} = H^1(B, \mathscr{A})$. In particular, this equality will hold throughout in Section 4 below. We shall write $[\mathscr{X}] \in H^1(B, \mathscr{A})$ for the isomorphism class of the torsor $\mathscr{X}$.

(1.2) Let $m \in \boldsymbol{N} \cap \Gamma(B, \mathcal{O}_B)^*$. For all $q \geq 0$, the higher direct image scheaves for the etale topology $R^q\chi_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}}$ and $R^q\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}$ are canonically isomorphic: If $\mathscr{U} = (U_i)_{i \in I}$ is a covering of B in the etale topology trivializing the torsor $\mathscr{X}$, choose isomorphisms of $\mathscr{A}_{U_i}$-torsors $\varphi_i: \mathscr{A}_{U_i} \xrightarrow{\sim} \mathscr{X}_{U_i}$. Any two choices differ by a translation by a section of $\mathscr{A}_{U_i}$, and, similarly, the restrictions of φ_i and φ_j above $U_i \times_B U_j$ differ by a translation by a section of $\mathscr{A}_{U_i \times_B U_j}$. Since translations act trivially on the etale cohomology of the geometric fibers of α, it follows that the isomorphisms $\varphi_i^*: (R^q\chi_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}})|U_i \xrightarrow{\sim} (R^q\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}})|U_i$ are independent of the choice of the φ_i and that they coincide on the $U_i \times_B U_j$, fitting together into a canonical isomorphism $R^q\chi_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}} \xrightarrow{\sim} R^q\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}$.

We shall identify these sheaves throughout. Thus the Leray spectral sequence for the sheaf $(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}}$ with respect to the structure map χ reads

$$H^pR^q\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}} \Rightarrow H^{p+q}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}}\,. \tag{1.3}$$

(For notational convenience, we shall often drop the symbol for the etale site on which a sheaf is defined, when referring to its cohomology). The differentials in the r-th term of this spectral sequence, $r \geq 2$, will be denoted $d_r(\mathscr{X}) = \bigoplus d_r^{pq}(\mathscr{X})$. Note that, as $H^pR^q\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}} = 0$ for $q \geq 2\dim_B\mathscr{A} + 1$, one has $d_r(\mathscr{X}) = 0$ for $r \geq 2\dim_B\mathscr{A} + 2$.

We start with a sufficient condition for the degeneracy of the Leray spectral sequence (1.3), in terms of m-divisibility of the torsor $\mathscr{X}$. Given $s \in \boldsymbol{N}$, let us say that the torsor $\mathscr{X}$ is s times m-divisible, if there exists a torsor $\mathscr{X}_0$ such that $[\mathscr{X}] = m^s[\mathscr{X}_0]$.

(1.4) PROPOSITION. *Let B be an arbitrary scheme, and let $\alpha: \mathscr{A} \to B$ be an abelian scheme. Let $m \geq 2$ be an integer relatively prime to the residual characteristics of B and* *such that m contains no prime factors that are less than or equal to* $2\dim_B\mathscr{A}$. *Let* $\chi: \mathscr{X}\to B$ *be a torsor for* $\mathscr{A}$. *If* $\mathscr{X}$ *is s times m-divisible, then* $d_r(\mathscr{X})=0$ *for all* $2\le r\le s+1$. *Thus, if* $\mathscr{X}$ *is* $2\dim_B\mathscr{A}$ *times m-divisible, the Leray spectral sequence for* $(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}}$ *degenerates.*

(1.5) REMARKS. (i) Our reason for imposing the second restriction on the prime factors of m, here and in several other statements below, is technical. We do not know, whether these results continue to hold without this condition.

(ii) The proposition applies in particular to the trivial torsor $\mathscr{X}=\mathscr{A}$. As a matter of fact, the arguments in the proof dwell on the original idea of Lieberman for this special case, in which actually more can be said (cf. [3, p. 116]):

Suppose that the integer m in the statement of (1.4) has no prime factors which are less than or equal to $2\dim_B\mathscr{A}+1$ (instead of $2\dim_B\mathscr{A}$). Then we have a canonical direct sum decomposition, for all $n\ge 0$: $H^n(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}=\bigoplus_{p+q=n}H^pR^q\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}$. Although this decomposition is well known, even at the more basic level of (relative) motives, we give a proof here, since we are dealing with torsion coefficients. For each fixed $p=0,\ldots,n-1$ write $n=p+q$ and set $F^p=F^pH^n(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}$ and $R^j=R^j\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}$. There exists $d\in\boldsymbol{Z}_{\ge 1}$ such that, for all $j=0,\ldots,q-1$: d^j-d^q is a unit in $\boldsymbol{Z}/m\boldsymbol{Z}$. From the diagrams for $i=p,p+1,\ldots,n$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{i+1} & \longrightarrow & F^i & \longrightarrow & H^iR^{n-i} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle d^*-d^q} & & \uparrow{\scriptstyle d^*-d^q} & & \uparrow{\scriptstyle d^{n-i}-d^q} & & \\ 0 & \longrightarrow & F^{i+1} & \longrightarrow & F^i & \longrightarrow & H^iR^{n-i} & \longrightarrow & 0 \end{array}$$

(d^* being the maps deduced from multiplication by d on the abelian scheme $\mathscr{A}$), we deduce by descending induction that d^*-d^q are isomorphisms on F^i, $i\ge p+1$ and, since this is 0 on H^iR^{n-i}, that $F^p=F^{p+1}\oplus M_q$, where M_q is the kernel of the map d^*-d^q on F^p, and $M_q \xrightarrow{\sim} H^pR^q$. It follows that, for all $r\in\boldsymbol{Z}_{\ge 1}$, $r^*=r^q$ on M_q. Hence finally a decomposition $F^pH^{p+q}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}=M_q\oplus M_{q-1}\oplus\cdots\oplus M_0$, where the piece M_i is characterized by the property that $r^*=r^i$ for all $r\in\boldsymbol{Z}_{\ge 1}$.

The following two remarks will be used in the proof of (1.4) and also at other places, below.

(1.6) Given a morphism $\phi: \mathscr{A}_1\to\mathscr{A}$ of abelian schemes over B, and an equivariant morphism $f: \mathscr{X}_1\to\mathscr{X}$ between respective torsors, there is attached functorially a morphism of spectral sequences, which at the E_2-level equals $\phi^*: H^pR^q\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}\to H^pR^q\alpha_{1*}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}_1}$, and which converges to $f^*: H^{p+q}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}}\to H^{p+q}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}_1}$.

We shall apply this to the following morphisms of abelian schemes and respective torsors over B, deduced from $\mathscr{A}$ and $\mathscr{X}$:

$$\begin{array}{ccccccc} \mathscr{A}\times_B\mathscr{A} & & & \mathscr{X}\times_B\mathscr{A} & & \\ \| & & & \| & & \\ \tilde{\mathscr{A}} & \xrightarrow{\Sigma} & \mathscr{A} & \tilde{\mathscr{X}} & \xrightarrow{\Sigma} & \mathscr{X} \end{array}$$

where Σ is the summation map, resp. the action of $\mathscr{A}$ on $\mathscr{X}$. For simplicity, let us write here $R^i=R^i\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}$ and $\tilde{R}^i=R^i\tilde{\alpha}_*(\mathbf{Z}/m\mathbf{Z})_{\tilde{\mathscr{A}}}$, where $\tilde{\alpha}: \tilde{\mathscr{A}}\to B$ is the structure map. The Künneth formula reads $\tilde{R}^q=\bigoplus_{i+j=q}R^i\otimes R^j$.

(1.7) Given a morphism $\phi: \mathscr{A}_1\to\mathscr{A}$ of abelian schemes over B, and respective torsors $\mathscr{X}_1$ and $\mathscr{X}$ for them, there exists a morphism of B-schemes $f: \mathscr{X}_1\to\mathscr{X}$ which is ϕ-equivariant if and only if the morphism $\phi: H^1(B,\mathscr{A}_1)\to H^1(B,\mathscr{A})$ satisfies $[\mathscr{X}]=\phi[\mathscr{X}_1]$. Moreover, the map f is then uniquely determined up to translations of $\mathscr{X}$ by global sections of $\mathscr{A}$.

(1.8) Proof of (1.4): Case $s=1$. Let $\mathscr{X}_0$ be a torsor such that $[\mathscr{X}]=m[\mathscr{X}_0]$. Let $f: \mathscr{X}_0\to\mathscr{X}$ be a morphism of B-schemes, equivariant for $m: \mathscr{A}\to\mathscr{A}$. Following (1.6), we put $\tilde{\mathscr{A}}=\mathscr{A}\times_B\mathscr{A}$, and $\tilde{\mathscr{X}}=\mathscr{X}\times_B\mathscr{A}$, $\tilde{\mathscr{X}}_0=\mathscr{X}_0\times_B\mathscr{A}$ are torsors for $\tilde{\mathscr{A}}$. The map $f\times_B m: \tilde{\mathscr{X}}_0\to\tilde{\mathscr{X}}$ is equivariant for $m=m\times_B m: \tilde{\mathscr{A}}\to\tilde{\mathscr{A}}$, hence $[\tilde{\mathscr{X}}]=m[\tilde{\mathscr{X}}_0]$ in $H^1(B,\tilde{\mathscr{A}})$ and, in particular, $\tilde{\mathscr{X}}$ is m-divisible.

(1.9) We claim that the differential $d_2^{pq}(\tilde{\mathscr{X}}): H^p\tilde{R}^q\to H^{p+2}\tilde{R}^{q-1}$ can be written as $d_2^{pq}(\tilde{\mathscr{X}})=\bigoplus\lambda_i$ for suitable morphisms $\lambda_i: H^p(R^i\otimes R^{q-i})\to H^{p+2}(R^{i-1}\otimes R^{q-i})$. To see this, for arbitrary $d\in\mathbf{Z}_{\geq 1}$, the equivariant map $1\times_B d: \tilde{\mathscr{X}}\to\tilde{\mathscr{X}}$ for $1\times_B d: \tilde{\mathscr{A}}\to\tilde{\mathscr{A}}$ yields a commutative diagram (cf. (1.6))

$$\begin{array}{ccc} H^p\tilde{R}^q & \xleftarrow{(1\times_B d)^*} & H^p\tilde{R}^q \\ {\scriptstyle d_2^{pq}(\tilde{\mathscr{X}})}\downarrow & & \downarrow{\scriptstyle d_2^{pq}(\tilde{\mathscr{X}})} \\ H^{p+2}\tilde{R}^{q-1} & \xleftarrow[(1\times_B d)^*]{} & H^{p+2}\tilde{R}^{q-1}. \end{array}$$

Since $(1\times_B d)^*$ is the direct sum of $d^{q-i}: H^p(R^i\otimes R^{q-i})\to H^p(R^i\otimes R^{q-i})$, it follows that the entry $\lambda_{ii'}: H^p(R^i\otimes R^{q-i})\to H^{p+2}(R^{i'}\otimes R^{q-1-i'})$ of $d_2^{pq}(\tilde{\mathscr{X}})$ satisfies: $(d^{q-1-i'}-d^{q-i})\lambda_{ii'}=0$. Hence, by our assumption on m, $\lambda_{ii'}=0$ if $i'\neq i-1$.

(1.10) On the other hand, from the equivariant map $f\times_B 1: \tilde{\mathscr{X}}_0\to\tilde{\mathscr{X}}$ for $m\times_B 1: \tilde{\mathscr{A}}\to\tilde{\mathscr{A}}$ one deduces a commutative diagram

$$\begin{array}{ccc} H^p(R^i\otimes R^{q-i}) & \xleftarrow{m^i} & H^p(R^i\otimes R^{q-i}) \\ {\scriptstyle \lambda_{i,i-1}(\tilde{\mathscr{X}}_0)}\downarrow & & \downarrow{\scriptstyle \lambda_i} \\ H^{p+2}(R^{i-1}\otimes R^{q-i}) & \xleftarrow[m^{i-1}]{} & H^{p+2}(R^{i-1}\otimes R^{q-i}), \end{array}$$

which, for $i=1$, yields $\lambda_1=0$.

(1.11) Finally, the equivariant map $\Sigma: \tilde{\mathscr{X}} \to \mathscr{X}$ for $\Sigma: \tilde{\mathscr{A}} \to \mathscr{A}$ yields the right hand side square of the following commutative diagram

$$\begin{array}{ccccc} H^p(R^1 \otimes R^{q-1}) & \xleftarrow{\text{proj}} & H^p\tilde{R}^q & \xleftarrow{\Sigma^*} & H^pR^q \\ \downarrow \lambda_1=0 & & \downarrow d_2^{pq}(\tilde{\mathscr{X}}) & & \downarrow d_2^{pq}(\mathscr{X}) \\ H^{p+2}(R^0 \otimes R^{q-1}) & \xleftarrow[\text{proj}]{} & H^{p+2}\tilde{R}^{q-1} & \xleftarrow[\Sigma^*]{} & H^{p+2}R^{q-1} . \end{array}$$

As the bottom row is the identity, we deduced that $d_2^{pq}(\mathscr{X})=0$.

(1.12) Proof of (1.4): The induction step. Suppose that the statement is true up to $s-1\geq 1$, and that $[\mathscr{X}]=m^s[\mathscr{X}_0]$ for some torsor $\mathscr{X}_0$. Put $[\mathscr{X}_1]=m^{s-1}[\mathscr{X}_0]$, so that $[\mathscr{X}]=m[\mathscr{X}_1]$. We apply the induction hypothesis to the $\tilde{\mathscr{A}}$-torsors $\tilde{\mathscr{X}}_1$ ($[\tilde{\mathscr{X}}_1]=m^{s-1}[\tilde{\mathscr{X}}_0]$) and $\tilde{\mathscr{X}}$ ($[\tilde{\mathscr{X}}]=m[\tilde{\mathscr{X}}_1]$), as well as to $\mathscr{X}$. We have therefore $E_{s+1}^{pq}(\mathscr{X})=H^pR^q$, $E_{s+1}^{pq}(\tilde{\mathscr{X}})=H^p\tilde{R}^q$, and $E_{s+1}^{pq}(\tilde{\mathscr{X}}_1)=H^p\tilde{R}^q$. As in (1.9), one sees that $d_{s+1}^{pq}(\tilde{\mathscr{X}})=\bigoplus \lambda_i^{(s+1)}$ for suitable morphisms $\lambda_i^{(s+1)}: H^p(R^i\otimes R^{q-i})\to H^{p+s+1}(R^{i-s}\otimes R^{q-i})$. Then, as in (1.10), one proves that $\lambda_s^{(s+1)}=0$. Finally, as in (1.11), we deduce a commutative diagram, in which the composite map of the bottom row is the identity:

$$\begin{array}{ccccc} H^p(R^s \otimes R^{q-s}) & \xleftarrow{\text{proj}} & H^p\tilde{R}^q & \xleftarrow{\Sigma^*} & H^pR^q \\ \downarrow \lambda_s^{(s+1)}=0 & & \downarrow d_{s+1}^{pq}(\tilde{\mathscr{X}}) & & \downarrow d_{s+1}^{pq}(\mathscr{X}) \\ H^{p+s+1}(R^0 \otimes R^{q-s}) & \xleftarrow[\text{proj}]{} & H^{p+s+1}\tilde{R}^{q-s} & \xleftarrow[\Sigma^*]{} & H^{p+s+1}R^{q-s} , \end{array}$$

thereby concluding that $d_{s+1}^{pq}(\mathscr{X})=0$.

The rest of this section is devoted to the proof of the following fact, which will be used in Section 3 below. (Note that this is a step in the direction of (1.5) (ii), for more general m-divisible torsors).

(1.13) Proposition. *Let B be an arbitrary scheme, and let $\alpha: \mathscr{A}\to B$ be an abelian scheme. Let $m\geq 2$ be an integer relatively prime to the residual characteristics of B, and such that m contains no prime factors that are less than or equal to $2\dim_B\mathscr{A}$. Let $\chi: \mathscr{X}\to B$ be a torsor for $\mathscr{A}$. If $\mathscr{X}$ is $4\dim_B\mathscr{A}$ times m-divisible, then the exact sequences of $(\boldsymbol{Z}/m\boldsymbol{Z})$-modules arising from the degeneration of the Leray spectral sequence for $(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}}$ (cf. (1.4)):*

$$(1.14)\quad 0\to H^{p+1}R^{q-1}\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}\to \frac{F^pH^{p+q}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}}}{F^{p+2}H^{p+q}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}}}\to H^pR^q\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}\to 0$$

are split exact sequences.

(1.15) Proof of (1.13): Case $q=1$. Write $[\mathscr{X}]=m[\mathscr{X}_0]$, with $\mathscr{X}_0$ a torsor which is $2\dim_B\mathscr{A}$ times m-divisible. By (1.4), the Leray spectral sequences for $(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}}$ and

$(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}_0}$ degenerate. Let $f: \mathscr{X}_0 \to \mathscr{X}$ be an equivariant map for $m: \mathscr{A} \to \mathscr{A}$. This induces a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & H^{p+1}R^0\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}} & \to & F^pH^{p+1}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}_0} & \to & H^pR^1\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}} & \to & 0 \\ & & \| & & \uparrow f^* & & \uparrow 0 & & \\ 0 & \to & H^{p+1}R^0\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}} & \to & F^pH^{p+1}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}} & \to & H^pR^1\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}} & \to & 0, \end{array}$$

hence the bottom exact sequence splits.

(1.16) We shall need case $q=1$ in a slightly more general form. Let $\mathscr{M}$ be a flat $(\boldsymbol{Z}/m\boldsymbol{Z})_B$-module and consider its inverse image $\mathscr{M}_{\mathscr{X}}$ in $\mathscr{X}$. By the projection formula (cf. e.g., [4]), we have, canonically, for all $q \geq 0$: $R^q\chi_*\mathscr{M}_{\mathscr{X}} = (R^q\chi_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}}) \otimes \mathscr{M} = (R^q\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}) \otimes \mathscr{M} = R^q\alpha_*\mathscr{M}_{\mathscr{A}}$. We write, for all $r \geq 2$, $d_r(\mathscr{X}, \mathscr{M}) = \bigoplus d_r^{pq}(\mathscr{X}, \mathscr{M})$ for the differentials of the Leray spectral sequence for $\mathscr{M}_{\mathscr{X}}$ with respect to χ:

$$H^pR^q\alpha_*\mathscr{M}_{\mathscr{A}} \Rightarrow H^{p+q}\mathscr{M}_{\mathscr{X}}.$$

Keeping in mind the projection formula, one can mimic the proof of Proposition (1.4) and show that, if $\mathscr{X}$ is s times m-divisible, $s \geq 1$, then $d_r(\mathscr{X}, \mathscr{M}) = 0$ for all $2 \leq r \leq s+1$. Hence, if $s \geq 2\dim_B \mathscr{A}$, the Leray spectral sequence for $\mathscr{M}_{\mathscr{X}}$ degenerates. Secondly, the same argument as in (1.15) above then shows that, for $\mathscr{X}$ as in (1.13), the sequence of $(\boldsymbol{Z}/m\boldsymbol{Z})$-modules

(1.17) $$0 \to H^{p+1}R^0\alpha_*\mathscr{M}_{\mathscr{A}} \to F^pH^{p+1}\mathscr{M}_{\mathscr{X}} \to H^pR^1\alpha_*\mathscr{M}_{\mathscr{A}} \to 0$$

is a split exact sequence.

(1.18) PROOF OF (1.13): General case. Let $\tilde{\mathscr{A}} = \mathscr{A} \times_B \mathscr{A}$, $\tilde{\mathscr{X}} = \mathscr{X} \times_B \mathscr{A}$ as before (cf. (1.6)). Denote by $\tilde{\chi}: \tilde{\mathscr{X}} \to B$ the structure map for this $\tilde{\mathscr{A}}$-torsor. Let moreover ρ and η be the projection maps in the product diagram

(1.19) $$\begin{array}{ccc} \tilde{\mathscr{X}} & \xrightarrow{\eta} & \mathscr{A} \\ \rho\downarrow & & \downarrow\alpha \\ \mathscr{X} & \xrightarrow[\chi]{} & B \end{array}$$

$(\tilde{\chi} = \chi\rho = \alpha\eta)$. By means of the projection map ρ, the scheme $\tilde{\mathscr{X}}$ is an abelian scheme over $\mathscr{X}$. By our assumption on $\mathscr{X}$, the Leray spectral sequence for $(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}}$ with respect to χ and the Leray spectral sequence for $(\boldsymbol{Z}/m\boldsymbol{Z})_{\tilde{\mathscr{X}}}$ with respect to $\tilde{\chi}$ degenerate, as well as, more generally, the Leray spectral sequence for $\mathscr{M}_{\mathscr{X}}$ with respect to χ, for any flat $(\boldsymbol{Z}/m\boldsymbol{Z})_B$-module $\mathscr{M}$. (Actually, since the lower bound for the prime factors of m is $2\dim_B \mathscr{A} + 1$ and not $2\dim_B \tilde{\mathscr{A}} + 1$, a slight change is needed in the proof of (1.4), to cover the case of $(\boldsymbol{Z}/m\boldsymbol{Z})_{\tilde{\mathscr{X}}}$. It suffices to replace, in (1.9), the map $1 \times_B d$ by the maps $1 \times_B (1 \times_B d)$ and $1 \times_B (d \times_B 1)$). Furthermore, the Leray spectral sequence for $(\boldsymbol{Z}/m\boldsymbol{Z})_{\tilde{\mathscr{X}}}$

with respect to ρ degenerates, too (cf. (1.5)). In order to distinguish between the filtrations induced by the Leray spectral sequences with respect to $\tilde{\chi}$ and ρ, we shall write them $F^{\cdot}_{\tilde{\mathscr{X}}/B}$ and $F^{\cdot}_{\tilde{\mathscr{X}}/\mathscr{X}}$ respectively. To simplify the notation, $F^{i}_{\tilde{\mathscr{X}}/B}$ (resp. $F^{i}_{\tilde{\mathscr{X}}/\mathscr{X}}$) will stand for $F^{i}_{\tilde{\mathscr{X}}/B}H^n(\mathbf{Z}/m\mathbf{Z})_{\tilde{\mathscr{X}}}$ (resp. $F^{i}_{\tilde{\mathscr{X}}/\mathscr{X}}H^n(\mathbf{Z}/m\mathbf{Z})_{\tilde{\mathscr{X}}}$), when the degree n is clear from the context. Also $F^{\cdot}_{\mathscr{X}/B}$ will indicate filtrations induced by the Leray spectral sequence with respect to χ (formerly referred to without subscript), and $F^{i}_{\mathscr{X}/B}$ will stand for $F^{i}_{\mathscr{X}/B}H^n(\mathbf{Z}/m\mathbf{Z})_{\mathscr{X}}$. Finally, as before, we write $R^i = R^i\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}$ and $\tilde{R}^i = R^i\tilde{\alpha}_*(\mathbf{Z}/m\mathbf{Z})_{\tilde{\mathscr{A}}}$.

We aim at constructing a canonical morphism of exact sequences

$$
(1.20)\qquad
\begin{array}{ccccccccc}
0 & \to & H^{p+1}(R^0\otimes R^{q-1}) & \to & F^{p}_{\tilde{\mathscr{X}}/B}H^{p+1}(\chi^*R^{q-1}) & \to & H^p(R^1\otimes R^{q-1}) & \to & 0\\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \to & H^{p+1}R^{q-1} & \to & F^{p}_{\mathscr{X}/B}/F^{p+2}_{\mathscr{X}/B} & \to & H^pR^q & \to & 0,
\end{array}
$$

where the bottom row is (1.14) and the top row is (1.17) with $\mathscr{M} = R^{q-1}$. By (1.16), the top row splits, and so will do then the bottom row, too, as has to be proved.

The morphism (1.20) is obtained as the composition of

$$
(1.21)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & H^{p+1}\tilde{R}^{q-1} & \longrightarrow & F^{p}_{\tilde{\mathscr{X}}/B}/F^{p+2}_{\tilde{\mathscr{X}}/B} & \longrightarrow & H^p\tilde{R}^q & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle \Sigma^*} & & \uparrow{\scriptstyle \Sigma^*} & & \uparrow{\scriptstyle \Sigma^*} & & \\
0 & \longrightarrow & H^{p+1}R^{q-1} & \longrightarrow & F^{p}_{\mathscr{X}/B}/F^{p+2}_{\mathscr{X}/B} & \longrightarrow & H^pR^q & \longrightarrow & 0
\end{array}
$$

(cf. (1.6)) with a morphism

$$
(1.22)\qquad
\begin{array}{ccccccccc}
0 & \to & H^{p+1}(R^0\otimes R^{q-1}) & \to & F^{p}_{\tilde{\mathscr{X}}/B}H^{p+1}(\chi^*R^{q-1}) & \to & H^p(R^1\otimes R^{q-1}) & \to & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & H^{p+1}\tilde{R}^{q-1} & \to & F^{p}_{\tilde{\mathscr{X}}/B}/F^{p+2}_{\tilde{\mathscr{X}}/B} & \to & H^p\tilde{R}^q & \to & 0.
\end{array}
$$

It remains to explain the diagram (1.22). The left hand side vertical arrow is the projection map from the Künneth decomposition, whence the identity map on the left hand side in (1.20) (cf. also (1.36)).

Let $G^{\cdot}_{(\tilde{\mathscr{X}})} = (0 \to G^{0}_{(\tilde{\mathscr{X}})} \to G^{1}_{(\tilde{\mathscr{X}})} \to \cdots)$ be a Godement resolution for the sheaf $(\mathbf{Z}/m\mathbf{Z})_{\tilde{\mathscr{X}}}$ (cf., e.g., [7]). The cohomology of $(\mathbf{Z}/m\mathbf{Z})_{\tilde{\mathscr{X}}}$ can be computed as the hypercohomology of the complex $\rho_*G^{\cdot}_{(\tilde{\mathscr{X}})}$ and also as the hypercohomology of the complex $\tilde{\chi}_*G^{\cdot}_{(\tilde{\mathscr{X}})} = \chi_*(\rho_*G^{\cdot}_{(\tilde{\mathscr{X}})})$. The respective second spectral sequences of hypercohomology are the Leray spectral sequences for $(\mathbf{Z}/m\mathbf{Z})_{\tilde{\mathscr{X}}}$ with respect to ρ and $\tilde{\chi}$ respectively. The standard map $\chi^*(\chi_*(\rho_*G^{\cdot}_{(\tilde{\mathscr{X}})})) \to \rho_*G^{\cdot}_{(\tilde{\mathscr{X}})}$ defines a morphism between these spectral sequences, that converges to the identity of $H^{\cdot}(\mathbf{Z}/m\mathbf{Z})_{\tilde{\mathscr{X}}}$. It yields an inclusion of filtrations $F^{\cdot}_{\tilde{\mathscr{X}}/B} \subset F^{\cdot}_{\tilde{\mathscr{X}}/\mathscr{X}}$ and it yields furthermore, at the E_2-level, the map $\bigoplus_{i=0}^{q} H^p(R^i\otimes R^{q-i}) = H^p\tilde{R}^q \to H^p(\chi^*R^q)$ which is projection onto the factor $H^p(R^0\otimes R^q) = H^pR^q$ followed by the obvious map $H^pR^q \to H^p(\chi^*R^q)$. We deduce in particular a morphism ($n=p+q$): $F^p_{\tilde{\mathscr{X}}/B}/F^{p+2}_{\tilde{\mathscr{X}}/B}\to F^p_{\tilde{\mathscr{X}}/\mathscr{X}}/F^{p+2}_{\tilde{\mathscr{X}}/\mathscr{X}}=M_q\oplus M_{q-1}$ (cf. (1.5)(ii)).

Comparing this map with the projection onto M_{q-1} and following this by the canonical isomorphism $M_{q-1}\overset{\sim}{\to} H^{p+1}(\chi^*R^{q-1})$, we obtain a commutative diagram

$$\text{(1.23)}\qquad \begin{array}{ccc} H^{p+1}(R^0\otimes R^{q-1}) & \hookrightarrow & H^{p+1}(\chi^*R^{q-1}) \\ \uparrow & & \uparrow \\ H^{p+1}\tilde{R}^{q-1} & \hookrightarrow & F^p_{\tilde{\mathscr{X}}/B}/F^{p+2}_{\tilde{\mathscr{X}}/B}\,. \end{array}$$

(1.24) We whall show that the image of $F^p_{\tilde{\mathscr{X}}/B}/F^{p+2}_{\tilde{\mathscr{X}}/B}$ lies inside $F^p_{\mathscr{X}/B}H^{p+1}(\chi^*R^{q-1})$. The diagram (1.22) and hence the proof of Proposition (1.13) follows at once from this.

Unfortunately, we will have to do quite an extra work for this. In the first place, the maps d^* deduced from multiplication by $d\in \boldsymbol{Z}_{\geq 1}$ on the abelian scheme $\tilde{\mathscr{X}}/\mathscr{X}$ obviously preserve the filtrations $F^{\cdot}_{\tilde{\mathscr{X}}/\mathscr{X}}$ and $F^{\cdot}_{\tilde{\mathscr{X}}/B}$ of $H^n(\boldsymbol{Z}/m\boldsymbol{Z})_{\tilde{\mathscr{X}}}$. Thus, since the projection $F^p_{\tilde{\mathscr{X}}/\mathscr{X}}/F^{p+2}_{\tilde{\mathscr{X}}/\mathscr{X}}\to M_{q-1}$ is a linear combination of d^* and $(d^*)^2$ if d and $d-1$ are both invertible in $\boldsymbol{Z}/m\boldsymbol{Z}$, this projection maps the image of $F^p_{\tilde{\mathscr{X}}/B}/F^{p+2}_{\tilde{\mathscr{X}}/B}$ in $F^p_{\tilde{\mathscr{X}}/\mathscr{X}}/F^{p+2}_{\tilde{\mathscr{X}}/\mathscr{X}}$ into itself. Therefore:

(1.25) In order to prove (1.24), it is sufficient to find a $(\boldsymbol{Z}/m\boldsymbol{Z})$-submodule of $F^p_{\tilde{\mathscr{X}}/\mathscr{X}}/F^{p+2}_{\tilde{\mathscr{X}}/\mathscr{X}}$ intersecting $F^{p+1}_{\tilde{\mathscr{X}}/\mathscr{X}}/F^{p+2}_{\tilde{\mathscr{X}}/\mathscr{X}}=H^{p+1}(\chi^*R^{q-1})$ along $F^pH^{p+1}(\chi^*R^{q-1})$ and containing the image of $F^p_{\tilde{\mathscr{X}}/B}/F^{p+2}_{\tilde{\mathscr{X}}/B}$.

To this end, we replace the resolution $G^{\cdot}_{(\tilde{\mathscr{X}})}$ of $(\boldsymbol{Z}/m\boldsymbol{Z})_{\tilde{\mathscr{X}}}$ conveniently, in order to have a better hold on the machinery that underlies the Künneth formula in the present situation. Let $G^{\cdot}$ be a Godement resolution for $(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}$, and let $G^{\cdot}_{(\mathscr{X})}$ be a Godement resolution for $(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}}$. Write $Z^i=\ker(G^i\to G^{i+1})$, $Z^i_{(\mathscr{X})}=\ker(G^i_{(\mathscr{X})}\to G^{i+1}_{(\mathscr{X})})$, $B^i=\operatorname{Im}(R^0\alpha_*G^{i-1}\to R^0\alpha_*G^i)$ and $B^i_{(\mathscr{X})}=\operatorname{Im}(R^0\chi_*G^{i-1}_{(\mathscr{X})}\to R^0\chi_*G^i_{(\mathscr{X})})$. For a module over the ring $\boldsymbol{Z}/m\boldsymbol{Z}$, to be flat, projective or injective is one and the same thing (that is, locally free, if m is a prime power). It is then easily seen that G^i, $G^i_{(\mathscr{X})}$, Z^i, $Z^i_{(\mathscr{X})}$ are flat sheaves (i.e., that their fibres are flat $(\boldsymbol{Z}/m\boldsymbol{Z})$-modules) for all i, and that the same thing holds for $R^0\alpha_*G^i$ and $R^0\chi_*G^i_{(\mathscr{X})}$. From this one deduces by induction, by using that the R^i are locally free sheaves, that $R^0\alpha_*Z^i$, $R^0\chi_*Z^i_{(\mathscr{X})}$ and B^i, $B^i_{(\mathscr{X})}$ are, too, flat sheaves.

(1.26) The total complex $\tilde{G}^{\cdot}$ of the double complex $\rho^*G^{\cdot}_{(\mathscr{X})}\otimes\eta^*G^{\cdot}$ is a resolution of $(\boldsymbol{Z}/m\boldsymbol{Z})_{\tilde{\mathscr{X}}}$, which is $\tilde{\chi}$-acyclic and ρ-acyclic: The exactness of $\tilde{G}^{\cdot}$ follows from (either of) the spectral sequences of the double complex $\rho^*G^{\cdot}_{(\mathscr{X})}\otimes\eta^*G^{\cdot}$. The $\tilde{\chi}$-acyclicity follows from the Künneth formula (cf., e.g., [7, p. 258]) and the ρ-acyclicity follows from the proper base change theorem. By that same theorem, we have also (cf., e.g., [4, p. 101]):

$$\rho_*(\rho^*G^{\cdot}_{(\mathscr{X})}\otimes\eta^*G^{\cdot})=G^{\cdot}_{(\mathscr{X})}\otimes\rho_*\eta^*G^{\cdot}=G^{\cdot}_{(\mathscr{X})}\otimes\chi^*(R^0\alpha_*G^{\cdot})\,,$$

$$\tilde{\chi}_*(\rho^*G^{\cdot}_{(\mathscr{X})}\otimes\eta^*G^{\cdot})=\chi_*(G^{\cdot}_{(\mathscr{X})}\otimes\chi^*(R^0\alpha_*G^{\cdot}))=R^0\chi_*G^{\cdot}_{(\mathscr{X})}\otimes R^0\alpha_*G^{\cdot}\,.$$

(1.27) The double complex $G^{\cdot}_{(\mathscr{X})}\otimes\chi^*(R^0\alpha_*G^{\cdot})$ is a χ-acyclic resolution of the complex $\chi^*(R^0\alpha_*G^{\cdot})$, hence

$$H^n(\chi^*(R^0\alpha_*G^{\cdot}))=H^n(G^{\cdot}_{(\mathscr{X})}\otimes\chi^*(R^0\alpha_*G^{\cdot}))_{\text{tot}}=H^n(R^0\chi_*G^{\cdot}_{(\mathscr{X})}\otimes R^0\alpha_*G^{\cdot})_{\text{tot}}$$

($=H^n(\mathbf{Z}/m\mathbf{Z})_{\tilde{\mathscr{X}}}$) and the second spectral sequence of hypercohomology (SSH, for short) of the complex $(R^0\chi_* G^{\cdot}_{(\mathscr{X})} \otimes R^0\alpha_* G^{\cdot})_{\text{tot}}$ can be interpreted as the Leray spectral sequence (LSS, for short) for the complex $\chi^*(R^0\alpha_* G^{\cdot})$ with respect to the map χ:

$$H^p \boldsymbol{R}^q \chi_*(\chi^*(R^0\alpha_* G^{\cdot})) \Rightarrow \boldsymbol{H}^{p+q}(\chi^*(R^0\alpha_* G^{\cdot}))\,.$$

We recall that, by (1.26), the same SSH is also interpreted as the LSS for the sheaf $(\mathbf{Z}/m\mathbf{Z})_{\tilde{\mathscr{X}}}$ with repect to $\tilde{\chi}$. On the other hand, via the quasi-isomorphism between $\chi^*(R^0\alpha_* G^{\cdot})$ and $(G^{\cdot}_{(\mathscr{X})} \otimes \chi^*(R^0\alpha_* G^{\cdot}))_{\text{tot}}$, the second SSH for the complex $\chi^*(R^0\alpha_* G^{\cdot})$ coincides with the second SSH for the complex $(G^{\cdot}_{(\mathscr{X})} \otimes \chi^*(R^0\alpha_* G^{\cdot}))_{\text{tot}}$, which, by (1.26), is the LSS for the sheaf $(\mathbf{Z}/m\mathbf{Z})_{\tilde{\mathscr{X}}}$ with respect to ρ.

Note that, by (1.4) and our hypothesis in (1.13), all the spectral sequences mentioned in this paragraph (1.27) are degenerate.

(1.28) For a complex of sheaves $E^{\cdot}=(E^i)_{i\geq 0}$ we shall write $\mathscr{Z}^q(E^{\cdot})$ to denote the subcomplex $0 \to E^0 \to \cdots \to E^{q-1} \to Z^q \to 0$, and $\mathscr{B}^{q+1}(E^{\cdot})$ will indicate the quotient complex $0 \to B^q \to E^q \to E^{q+1} \to \cdots$ (the term B^q has degree $q-1$). One should take care not to confuse the complex $\mathscr{Z}^q(E^{\cdot})$ with the term $Z^q(E^{\cdot})=Z^q$, and, similarly, $\mathscr{B}^{q+1}(E^{\cdot})$ with $B^{q+1}(E^{\cdot})=B^{q+1}$. The filtration of the second SSH for $E^{\cdot}$ is given by $F^p\boldsymbol{H}^{p+q}E^{\cdot}=\operatorname{Im}(\boldsymbol{H}^{p+q}\mathscr{Z}^q(E^{\cdot}) \to \boldsymbol{H}^{p+q}E^{\cdot})$. If the second SSH for $E^{\cdot}$ degenerates, then the same thing happens for $\mathscr{Z}^q(E^{\cdot})$ and $\mathscr{B}^{q+1}(E^{\cdot})$ for all q, and we have short exact sequences, for all n and q: $0 \to \boldsymbol{H}^n\mathscr{Z}^q(E^{\cdot}) \to \boldsymbol{H}^n E^{\cdot} \to \boldsymbol{H}^n\mathscr{B}^{q+1}(E^{\cdot}) \to 0$ (hence $F^p\boldsymbol{H}^{p+q}E^{\cdot}=\boldsymbol{H}^{p+q}\mathscr{Z}^q(E^{\cdot})$ in this case).

(1.29) LEMMA. *We keep the preceding notation. One has, for all r, n, q:*

$$F^r_{\mathscr{X}/B}\boldsymbol{H}^n(\chi^*(R^0\alpha_* G^{\cdot})) \cap \boldsymbol{H}^n(\mathscr{Z}^q(\chi^*(R^0\alpha_* G^{\cdot}))) = F^r_{\mathscr{X}/B}\boldsymbol{H}^n(\mathscr{Z}^q(\chi^*(R^0\alpha_* G^{\cdot})))\,.$$

(1.30) REMARK. The filtrations that appear in this formula correspond to the Leray spectral sequence for the complexes $\chi^*(R^0\alpha_* G^{\cdot})$ and $\mathscr{Z}^q(\chi^*(R^0\alpha_* G^{\cdot}))$ respectively, with respect to the map χ. By (1.27) and (1.28), we may write this formula also as follows: For all r, n, p, setting $H^n=H^n(\mathbf{Z}/m\mathbf{Z})_{\tilde{\mathscr{X}}}$:

$$F^r_{\tilde{\mathscr{X}}/B}H^n \cap F^p_{\tilde{\mathscr{X}}/\mathscr{X}}H^n = F^r_{\mathscr{X}/B}(F^p_{\tilde{\mathscr{X}}/\mathscr{X}}H^n)\,.$$

In particular, for $r=p$ this gives, in view of the inclusion $F^p_{\tilde{\mathscr{X}}/B}H^n \subset F^p_{\tilde{\mathscr{X}}/\mathscr{X}}H^n$:

$$F^p_{\tilde{\mathscr{X}}/B}H^n = F^p_{\mathscr{X}/B}(F^p_{\tilde{\mathscr{X}}/\mathscr{X}}H^n)\,. \tag{1.31}$$

PROOF OF (1.29). The exact sequence

$$0 \to \mathscr{Z}^q(R^0\alpha_* G^{\cdot}) \to R^0\alpha_* G^{\cdot} \to \mathscr{B}^{q+1}(R^0\alpha_* G^{\cdot}) \to 0$$

yields, by the flatness of $R^0\chi_* G^{\cdot}_{(\mathscr{X})}$, an exact sequence of double complexes

$$\begin{aligned} 0 \to R^0\chi_* G^{\cdot}_{(\mathscr{X})} \otimes \mathscr{Z}^q(R^0\alpha_* G^{\cdot}) &\to R^0\chi_* G^{\cdot}_{(\mathscr{X})} \otimes R^0\alpha_* G^{\cdot} \\ &\to R^0\chi_* G^{\cdot}_{(\mathscr{X})} \otimes \mathscr{B}^{q+1}(R^0\alpha_* G^{\cdot}) \to 0\,. \end{aligned} \tag{1.32}$$

Let us introduce, for short, $0\to(\mathrm{I})\to(\mathrm{II})\to(\mathrm{III})\to 0$ as an alternative notation for the sequence of total complexes deduced from (1.32), just to be used in the next two diagrams. Putting $n=r+s$, we deduce from it an exact sequence of complexes

$$0\to\mathscr{Z}^s(\mathrm{I})\to\mathscr{Z}^s(\mathrm{II})\to\mathscr{Z}^s(\mathrm{III})\to 0\,. \tag{1.33}$$

(The only thing which has to be cared for is the surjectivity of the morphism $Z^s(R^0\chi_*G^{\boldsymbol{\cdot}}_{(\mathscr{X})}\otimes R^0\alpha_*G^{\boldsymbol{\cdot}})_{\mathrm{tot}}\to Z^s(R^0\chi_*G^{\boldsymbol{\cdot}}_{(\mathscr{X})}\otimes\mathscr{B}^{q+1}(R^0\alpha_*G^{\boldsymbol{\cdot}}))_{\mathrm{tot}}$. This is checked by direct inspection, and bearing in mind the flatness of the sheaf B^{q+1}).

From (1.32) and (1.33) we deduce a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\to & H^n\mathscr{Z}^s(I) & \to & H^n\mathscr{Z}^s(\mathrm{II}) & \to & H^n\mathscr{Z}^s(\mathrm{III}) & \to & & \\
 & \downarrow & & \downarrow & & \downarrow & & & \\
\to & H^n(\mathrm{I}) & \to & H^n(\mathrm{II}) & \to & H^n(\mathrm{III}) & \to & & \\
 & \| & & \| & & \| & & & \\
0\to & H^n(\mathscr{Z}^q(\chi^*(R^0\alpha_*G^{\boldsymbol{\cdot}}))) & \to & H^n(\chi^*(R^0\alpha_*G^{\boldsymbol{\cdot}})) & \to & H^n(\mathscr{B}^{q+1}(\chi^*(R^0\alpha_*G^{\boldsymbol{\cdot}}))) & \to & 0\,. &
\end{array}$$

By (1.28), the lemma will follow, if we show that the third vertical arrow is injective. Actually all three are so, this being due, again by (1.28), to the degeneration of the second SSH of the corresponding complex appearing in the middle row. This is clear for the complex $(R^0\chi_*G^{\boldsymbol{\cdot}}_{(\mathscr{X})}\otimes R^0\alpha_*G^{\boldsymbol{\cdot}})_{\mathrm{tot}}$ (cf. the last remark in (1.27)). To see this for the complex $(R^0\chi_*G^{\boldsymbol{\cdot}}_{(\mathscr{X})}\otimes\mathscr{B}^{q+1}(R^0\alpha_*G^{\boldsymbol{\cdot}}))_{\mathrm{tot}}$, observe that the morphism of complexes $(R^0\chi_*G^{\boldsymbol{\cdot}}_{(\mathscr{X})}\otimes R^0\alpha_*G^{\boldsymbol{\cdot}})_{\mathrm{tot}}\to(R^0\chi_*G^{\boldsymbol{\cdot}}_{(\mathscr{X})}\otimes\mathscr{B}^{q+1}(R^0\alpha_*G^{\boldsymbol{\cdot}}))_{\mathrm{tot}}$ induces a split surjection between their cohomology sheaves:

$$\bigoplus_{i+j=t} R^i\otimes R^j\to\bigoplus_{\substack{i+j=t\\ j\geq q+1}} R^i\otimes R^j\,.$$

Therefore the morphism induced between their second SSH yields surjective maps at the E_2-level, and we conclude inductively that $d_2=d_3=\cdots=0$ for the second SSH of $(R^0\chi_*G^{\boldsymbol{\cdot}}_{(\mathscr{X})}\otimes\mathscr{B}^{q+1}(R^0\alpha_*G^{\boldsymbol{\cdot}}))_{\mathrm{tot}}$. (A similar argument, with injective maps replacing surjective ones, proves the analogous statement for the complex $(R^0\chi_*G^{\boldsymbol{\cdot}}_{(\mathscr{X})}\otimes\mathscr{Z}^q(R^0\alpha_*G^{\boldsymbol{\cdot}}))_{\mathrm{tot}}$, but we shall not need this). This completes the proof of Lemma (1.29).

We turn finally to the proof of the facts announced in (1.25), which will finish the proof of Proposition (1.13). Consider the exact sequence of complexes

$$
(1.34)\qquad
\begin{array}{ccccccccc}
 & & & & 0 & & & & 0\\
 & & & & \uparrow & & & & \uparrow\\
 & & 0 & \to & B^q & \to & R^0\alpha_*Z^q & \to 0 & \mathscr{Z}^\bullet\\
 & & \uparrow & & \uparrow & & \| & & \uparrow\\
0 & \to & B^{q-1} & \to & R^0\alpha_*G^{q-1} & \to & R^0\alpha_*Z^q & \to 0 & \mathscr{M}^\bullet\\
 & & \| & & \uparrow & & \uparrow & & \uparrow\\
0 & \to & B^{q-1} & \to & R^0\alpha_*Z^{q-1} & \to & 0 & & \mathscr{B}^\bullet\\
 & & & & \uparrow & & & & \uparrow\\
 & & & & 0 & & & & 0
\end{array}
$$

(degree$=q$ for the term $R^0\alpha_*Z^q$). Writing $n=p+q$, the exact sequences

$$
\begin{array}{ccccccc}
0 \to & F^{p+1}_{\tilde{\mathscr{X}}/\mathscr{X}}/F^{p+2}_{\tilde{\mathscr{X}}/\mathscr{X}} & \to & F^{p}_{\tilde{\mathscr{X}}/\mathscr{X}}/F^{p+2}_{\tilde{\mathscr{X}}/\mathscr{X}} & \to & F^{p}_{\tilde{\mathscr{X}}/\mathscr{X}}F^{p+1}_{\tilde{\mathscr{X}}/\mathscr{X}} & \to 0\\
 & \| & & & & \| & \\
 & H^{p+1}\chi^*R^{q-1} & & & & H^p\chi^*R^q &
\end{array}
$$

are part of the hypercohomology sequence of the inverse image of (1.34) by χ:

$$0\to \boldsymbol{H}^n(\chi^*\mathscr{B}^\bullet)\to \boldsymbol{H}^n(\chi^*\mathscr{M}^\bullet)\to \boldsymbol{H}^n(\chi^*\mathscr{Z}^\bullet)\to 0$$

(cf. (1.27) and (1.28)).

(1.35) We claim that the submodule $F^p_{\mathscr{X}/B}\boldsymbol{H}^n(\chi^*\mathscr{M}^\bullet)$ of $\boldsymbol{H}^n(\chi^*\mathscr{M}^\bullet)=F^p_{\tilde{\mathscr{X}}/\mathscr{X}}/F^{p+2}_{\tilde{\mathscr{X}}/\mathscr{X}}$ satisfies the requirements of (1.25). It contains the image of $F^p_{\tilde{\mathscr{X}}/B}/F^{p+2}_{\tilde{\mathscr{X}}/B}$, since we have seen in (1.31) that $F^p_{\tilde{\mathscr{X}}/B}$ is contained in (in fact it equals) $F^p_{\tilde{\mathscr{X}}/B}\boldsymbol{H}^n(\mathscr{Z}^q(\chi^*(R^0\alpha_*G^\bullet)))$ and, on the other hand, this is mapped into $F^p_{\mathscr{X}/B}\boldsymbol{H}^n(\chi^*\mathscr{M}^\bullet)$.

It remains to show that $F^p_{\mathscr{X}/B}\boldsymbol{H}^n(\chi^*\mathscr{M}^\bullet)\cap \boldsymbol{H}^n(\chi^*\mathscr{B}^\bullet)=F^p_{\mathscr{X}/B}\boldsymbol{H}^n(\chi^*\mathscr{B}^\bullet)$. More generally, one has, for any r, n: $F^r_{\mathscr{X}/B}\boldsymbol{H}^n(\chi^*\mathscr{M}^\bullet)\cap \boldsymbol{H}^n(\chi^*\mathscr{B}^\bullet)=F^r_{\mathscr{X}/B}\boldsymbol{H}^n(\chi^*\mathscr{B}^\bullet)$. This is proved in a way similar to Lemma (1.29): From the exact sequence

$$0\to R^0\chi_*G^\bullet_{(\mathscr{X})}\otimes\mathscr{B}^\bullet\to R^0\chi_*G^\bullet_{(\mathscr{X})}\otimes\mathscr{M}^\bullet\to R^0\chi_*G^\bullet_{(\mathscr{X})}\otimes\mathscr{Z}^\bullet\to 0\,,$$

and denoting again for a while 0→(I)→(II)→(III)→0 the corresponding sequence of total complexes, we deduce exact sequences (writing $r+s=n$):

$$0\to \mathscr{Z}^s(\mathrm{I})\to \mathscr{Z}^s(\mathrm{II})\to \mathscr{Z}^s(\mathrm{III})\to 0\,,$$

and the result follows from their hypercohomology sequences:

$$
\begin{array}{ccccccc}
\to & \boldsymbol{H}^n\mathscr{Z}^s(\mathrm{I}) & \to & \boldsymbol{H}^n\mathscr{Z}^s(\mathrm{II}) & \to & \boldsymbol{H}^n\mathscr{Z}^s(\mathrm{III}) & \to\\
 & \downarrow & & \downarrow & & \downarrow & \\
\to & \boldsymbol{H}^n(\mathrm{I}) & \to & \boldsymbol{H}^n(\mathrm{II}) & \to & \boldsymbol{H}^n(\mathrm{III}) & \to\\
 & \| & & \| & & \| & \\
0 \to & \boldsymbol{H}^n(\chi^*\mathscr{B}^\bullet) & \to & \boldsymbol{H}^n(\chi^*\mathscr{M}^\bullet) & \to & \boldsymbol{H}^n(\chi^*\mathscr{Z}^\bullet) & \to 0\,,
\end{array}
$$

by showing, as before, that the third vertical arrow is injective (we ignore the other ones). This follows again from the degeneration of the second SSH of the complex $R^0\chi_*G^{\cdot}_{(\mathscr{X})}\otimes\mathscr{L}^{\cdot}$, which here follows because this complex is quasi-isomorphic with $R^0\chi_*G^{\cdot}_{(\mathscr{X})}\otimes R^q[q]$, and because, by the hypothesis of (1.13), the LSS for χ^*R^q with respect to χ degenerates (cf. (1.16)).

This proves the claim of (1.35) and hence ends the proof of Proposition (1.13).

(1.36) Remark. Throughout the above proof we did not care about the precise value of the right hand side vertical arrow in the diagram (1.22). As one may guess, this is (as it was the case for the left hand side vertical arrow, too) the projection map from the Künneth decomposition. We shall need this in the next section.

The proof is condensed in the following diagram, in which we use freely facts and items from the proof of (1.13), and which is self-explanatory except for the details that we list below.

$$\begin{array}{lll}
F^p_{\mathscr{X}/B}(F^p_{\mathscr{X}/\mathscr{X}}/F^{p+2}_{\mathscr{X}/\mathscr{X}}) = \boldsymbol{H}^n\mathscr{L}^q(R^0\chi_*G^{\cdot}_{(\mathscr{X})}\otimes\mathscr{M}^{\cdot})_{\mathrm{tot}} & \longrightarrow & H^p(R^0\otimes R^q)\oplus H^p(R^1\otimes R^{q-1}) \\
\quad j\uparrow & & \qquad\uparrow \\
F^p_{\mathscr{X}/B}(H^{p+1}\chi^*R^{q-1}) = \boldsymbol{H}^n\mathscr{L}^q(R^0\chi_*G^{\cdot}_{(\mathscr{X})}\otimes\mathscr{B}^{\cdot})_{\mathrm{tot}} & \longrightarrow & H^p(R^1\otimes R^{q-1}) \\
\quad g\uparrow & & \qquad\uparrow \\
F^p_{\mathscr{X}/B}(F^p_{\mathscr{X}/\mathscr{X}}/F^{p+2}_{\mathscr{X}/\mathscr{X}}) = \boldsymbol{H}^n\mathscr{L}^q(R^0\chi_*G^{\cdot}_{(\mathscr{X})}\otimes\mathscr{M}^{\cdot})_{\mathrm{tot}} & \longrightarrow & H^p(R^0\otimes R^q)\oplus H^p(R^1\otimes R^{q-1}) \\
\quad\ \uparrow & & \qquad\uparrow \\
F^p_{\mathscr{X}/B}(F^p_{\mathscr{X}/\mathscr{X}}) = \boldsymbol{H}^n\mathscr{L}^q(R^0\chi_*G^{\cdot}_{(\mathscr{X})}\otimes\mathscr{L}^q(R^0\alpha_*G^{\cdot}))_{\mathrm{tot}} & \longrightarrow & H^p\tilde{R}^q \\
\quad\ \| & & \qquad\| \\
F^p_{\mathscr{X}/B} = \boldsymbol{H}^n\mathscr{L}^q(R^0\chi_*G^{\cdot}_{(\mathscr{X})}\otimes R^0\alpha_*G^{\cdot})_{\mathrm{tot}} & \longrightarrow & H^p\tilde{R}^q\,.
\end{array}$$

(i) All horizontal mappings are constructed as follows: Starting with a complex $E^{\cdot}$ of sheaves, we take the morphism induced on $\boldsymbol{H}^n$ by the morphism of complexes $\mathscr{L}^q(E^{\cdot})\to\mathscr{H}^q(E^{\cdot})[q]$, where $\mathscr{H}^q(E^{\cdot})$ indicates the q-th cohomology sheaf of the complex $E^{\cdot}$.

(ii) $p+q=n$.

(iii) Other unnamed arrows are the obvious ones in each case. The morphism g is the restriction of the canonical projection map onto M_{q-1} (cf. before (1.23)), and j is the inclusion map. The composite map jg is induced by the morphism deduced functorially from the endomorphism of $\mathscr{M}^{\cdot}$ given by $(1/(d-1)d^{2q-2})(d^qd^*-d^{*2})$, where $d\in\boldsymbol{Z}_{\geq 1}$ is chosen such that $d, d-1$ are invertible (mod m), and where $d^*:\mathscr{M}^{\cdot}\to\mathscr{M}^{\cdot}$ is deduced from a morphism of complexes $d^*: R^0\alpha_*G^{\cdot}\to R^0\alpha_*G^{\cdot}$, obtained by the yoga of Godement resolutions from the multiplication by d maps on $\mathscr{A}$.

(iv) The diagram is commutative except, possibly, for the second rectangle from above. Since the upper right arrow is injective, it follows that this rectangle is commutative, too, and this completes the proof.

2. The E_2-term of the Leray spectral sequence.

(2.1) Throughout this section we assume that all schemes satisfy enough conditions ensuring that Čech cohomology agrees with derived functor cohomology on their etale site (cf. (0.14)).

Let B be a scheme. Let $m \geq 2$ be an integer relatively prime to the residual characteristics of B. Let $\alpha: \mathscr{A} \to B$ be an abelian scheme, and $\chi: \mathscr{X} \to B$ a torsor for $\mathscr{A}$. In this section we compute the differentials.

(2.2) $$d_2^{pq} = d_2^{pq}(\mathscr{X}) : H^p R^q \alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}} \to H^{p+1} R^{q-1} \alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}$$

for the E_2-term of the Leray spectral sequence for the sheaf $(\mathbf{Z}/m\mathbf{Z})_{\mathscr{X}}$ with respect to the map χ.

By means of the cohomology of the exact sequence

(2.3) $$0 \longrightarrow {}_m\mathscr{A} \longrightarrow \mathscr{A} \xrightarrow{m} \mathscr{A} \longrightarrow 0,$$

the class $[\mathscr{X}] \in H^1(B, \mathscr{A})$ provides us with an element $\delta[\mathscr{X}] \in H^2(B, {}_m\mathscr{A})$. On the other hand we have a duality pairing

(2.4) $${}_m\mathscr{A} \otimes R^1\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}} \to R^0\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}} = (\mathbf{Z}/m\mathbf{Z})_B$$

whence, by the canonical isomorphism $R^q\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}} = \Lambda^q R^1\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}$, an inner contraction morphism

(2.5) $${}_m\mathscr{A} \otimes R^q\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}} \to R^{q-1}\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}, \qquad u \otimes \zeta \mapsto \iota(u)\zeta.$$

This induces:

(2.6) $$H^2(B, {}_m\mathscr{A}) \otimes H^p R^q\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}} \to H^{p+2} R^{q-1}\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}, \qquad \alpha \otimes \beta \mapsto \iota(\alpha)\beta.$$

(2.7) PROPOSITION. *Assumptions are as in* (2.1). *We suppose furthermore that* m *has no prime factors that are less than or equal to* q. *The following formula holds*:

$$d_2^{pq}(\mathscr{X}) = (-1)^{p+1}\iota(\delta[\mathscr{X}]).$$

The proof will take the rest of this section. We start with a few preliminary remarks.

(2.8) For a sheaf G of abelian groups on (the etale site of) a B-scheme $\eta: \mathscr{Y} \to B$, the E_2-term for the Leray spectral sequence of G with respect to η can be computed explicitly from a resolution $G^\cdot$ of G, acyclic with respect to η, as follows. From the complex $R^0\eta_* G^\cdot$ we deduce exact commutative diagrams

(2.9) $$\begin{array}{ccccccccc} 0 \to & R^0\eta_* Z^{q-1} & \to & R^0\eta_* G^{q-1} & \to & R^0\eta_* Z^q & \to & R^q\eta_* G & \to 0 \\ & \downarrow & & \downarrow & & \| & & \| & \\ 0 \to & R^{q-1}\eta_* G & \to & (\cdots) & \to & R^0\eta_* Z^q & \to & R^q\eta_* G & \to 0, \end{array}$$

where $Z^i = \ker(G^i \to G^{i+1})$, and the left hand side square is a pushout diagram. Then $d_2^{pq} = (-1)^p \partial\partial : H^p R^q \eta_* G \to H^{p+2} R^{q-1} \eta_* G$, with $\partial\partial$ the iterated connecting homomorphism of the bottom row in (2.9).

For example, one can use a Godement resolution $G^{\cdot}$ of G. In this setting, given a morphism $f: \mathscr{Y}_1 \to \mathscr{Y}_2$ of B-schemes and a morphism of sheaves $f^*G_2 \to G_1$, the yoga of Godement resolutions (cf. [7, p. 90]), provides us with a blueprint (depending on choices) for a morphism of complxes $f^*: R^0\eta_{2*}G_2^{\cdot} \to R^0\eta_{1*}G_1^{\cdot}$ inducing the maps $f^*: R^q\eta_{2*}G_2 \to R^q\eta_{1*}G_1$, and also, more generally, the morphism of spectral sequences converging to $f^*: H^{\cdot}(\mathscr{Y}_2, G_2) \to H^{\cdot}(\mathscr{Y}_1, G_1)$.

In the particular case where G_2 is a constant sheaf, the map $f^*: f^*G_2^0 \to G_1^0$ is independent of choices, because the fibres of G_2 are then trivial Galois modules. Moreover, for $q=1$ the diagram (2.9) is reduced to the sequence

$$0 \to R^0\eta_*G \to R^0\eta_*G^0 \to R^0\eta_*Z^1 \to R^1\eta_*G \to 0 \tag{2.10}$$

so that, for G_2 a constant sheaf, $f^*G_2 \to G_1$ induces a canonical morphism between the sequences for G_2 and G_1 respectively:

$$\begin{array}{ccccccccc} 0 & \to & R^0\eta_{1*}G_1 & \to & R^0\eta_{1*}G_1^0 & \to & R^0\eta_{1*}Z_1^1 & \to & R^1\eta_{1*}G_1 & \to & 0 \\ & & \uparrow f^* & & \uparrow f^* & & \uparrow f^* & & \uparrow f^* & & \\ 0 & \to & R^0\eta_{2*}G_2 & \to & R^0\eta_{2*}G_2^0 & \to & R^0\eta_{2*}Z_2^1 & \to & R^1\eta_{2*}G_2 & \to & 0\,. \end{array} \tag{2.11}$$

This will depend functorially on f, when sticking to constant sheaves G_1, G_2.

(2.12) We give an explicit description of the morphism (2.4). Setting $\mathscr{Y} = \mathscr{A}$ and $G = (\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}$ in (2.8), the multiplication by m map on $\mathscr{A}$ leads to a diagram (2.11) that now reads:

$$\begin{array}{ccccccccc} 0 & \to & R^0\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}} & \to & R^0\alpha_*G^0 & \to & R^0\alpha_*Z^1 & \to & R^1\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}} & \to & 0 \\ & & \| & & \uparrow m^* & & \uparrow m^* & & \uparrow 0 & & \\ 0 & \to & R^0\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}} & \to & R^0\alpha_*G^0 & \to & R^0\alpha_*Z^1 & \to & R^1\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}} & \to & 0\,. \end{array} \tag{2.13}$$

Write (cf. (1.25)) $B^1 = \mathrm{Im}(R^0\alpha_*G^0 \to R^0\alpha_*Z^1)$.

Let $u \in \Gamma(U, {}_m\mathscr{A})$ and $\zeta \in \Gamma(U, R^1\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}})$ be local sections. We describe $\imath(u)\zeta \in \Gamma(U, (\mathbf{Z}/m\mathbf{Z})_B)$. The question being local, we may assume, by restricting U if necessary, that ζ is the image of some $z \in \Gamma(U, R^0\alpha_*Z^1)$. Then $m^*z \in \Gamma(U, B^1)$, and we may suppose again that m^*z is the image of some $h \in \Gamma(U, R^0\alpha_*G^0)$. Then, writing $\tau_u: \mathscr{A}_U \to \mathscr{A}_U$ for the translation by u, we have: $\imath(u)\zeta = \tau_u^*h - h \in \Gamma(U, R^0\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}) = \Gamma(U, (\mathbf{Z}/m\mathbf{Z})_B)$.

(2.14) PROOF OF (2.7): Case $q=1$.

(2.15) Let $\mathscr{U} = (U_i)_{i \in I}$ be a covering of B in the etale topology, trivializing the torsor $\mathscr{X}$. Write $U_{ij}=U_i\times_B U_j$, $\mathscr{X}_i=\mathscr{X}_{U_i}$, $\mathscr{A}_i=\mathscr{A}_{U_i}$, $\mathscr{X}_{ij}=\mathscr{X}_{U_{ij}}$ and $\mathscr{A}_{ij}=\mathscr{A}_{U_{ij}}$. We have isomorphisms of $\mathscr{A}_i$-torsors, $\varphi_i\colon \mathscr{A}_i \xrightarrow{\sim} \mathscr{X}_i$, and commutative diagrams, for each i, j:

$$\begin{array}{ccc} \mathscr{A}_{ij} & \xrightarrow{\varphi_j} & \mathscr{X}_{ij} \\ {\scriptstyle \tau_{s_{ij}}}\downarrow & & \| \\ \mathscr{A}_{ij} & \xrightarrow{\varphi_i} & \mathscr{X}_{ij}, \end{array}$$

with $s_{ij}\in\Gamma(U_{ij}, \mathscr{A})$. The class $[\mathscr{X}]\in H^1(B, \mathscr{A})$ is represented by $\{s_{ij}\}\in H^1(\mathscr{U}, \mathscr{A})$. Refining $\mathscr{U}$ if necessary, we may assume that the sections s_{ij} lift by the multiplication by m map (cf. (2.3)): $s_{ij}=m\tilde{s}_{ij}$ for suitable $\tilde{s}_{ij}\in\Gamma(U_{ij}, \mathscr{A})$. Then $\delta[\mathscr{X}]\in H^2(B, {}_m\mathscr{A})$ is represented by $\{\sigma_{ijk}\}\in H^2(\mathscr{U}, {}_m\mathscr{A})$, with $\sigma_{ijk}=\tilde{s}_{jk}-\tilde{s}_{ik}+\tilde{s}_{ij}\in\Gamma(U_{ijk}, {}_m\mathscr{A})$.

(2.16) As in (1.25), write $G^{\cdot}_{(\mathscr{X})}$ for a Godement resolution of $(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}}$, and $Z^i_{(\mathscr{X})}=\ker(G^i_{(\mathscr{X})}\to G^{i+1}_{(\mathscr{X})})$, $B^i_{(\mathscr{X})}=\mathrm{Im}(R^0\chi_*G^{i-1}_{(\mathscr{X})}\to R^0\chi_*G^i_{(\mathscr{X})})$. The sequence (2.10) for $(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}}$,

$$0\to R^0\chi_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}}\to R^0\chi_*G^0_{(\mathscr{X})}\to R^0\chi_*Z^1_{(\mathscr{X})}\to R^1\chi_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}}\to 0\,,$$

can be thought of as obtained from the analogous sequence (2.10) for $(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}$,

$$0\to R^0\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}\to R^0\alpha_*G^0\to R^0\alpha_*Z^1\to R^1\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}\to 0\,,$$

by glueing its restrictions to the U_i:

$$0\to R^0\alpha_{i*}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}_i}\to R^0\alpha_{i*}G^0_{\mathscr{A}_i}\to R^0\alpha_{i*}Z^1_{\mathscr{A}_i}\to R^1\alpha_{i*}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}_i}\to 0\,,$$

by means of transition maps

$$\begin{array}{ccccccccc} 0 & \to & R^0\alpha_{ij*}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}_{ij}} & \to & R^0\alpha_{ij*}G^0_{\mathscr{A}_{ij}} & \to & R^0\alpha_{ij*}Z^1_{\mathscr{A}_{ij}} & \to & R^1\alpha_{ij*}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}_{ij}} & \to & 0 \\ & & \| & & \simeq\uparrow\tau^*_{s_{ij}} & & \simeq\uparrow\tau^*_{s_{ij}} & & \| & & \\ 0 & \to & R^0\alpha_{ij*}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}_{ij}} & \to & R^0\alpha_{ij*}G^0_{\mathscr{A}_{ij}} & \to & R^0\alpha_{ij*}Z^1_{\mathscr{A}_{ij}} & \to & R^1\alpha_{ij*}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}_{ij}} & \to & 0\,. \end{array}$$

The identification is given by the isomorphisms

$$\begin{array}{ccccccccc} 0 & \to & R^0\alpha_{i*}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}_i} & \to & R^0\alpha_{i*}G^0_{\mathscr{A}_i} & \to & R^0\alpha_{i*}Z^1_{\mathscr{A}_i} & \to & R^1\alpha_{i*}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}_i} & \to & 0 \\ & & \simeq\uparrow\varphi_i^* & & \simeq\uparrow\varphi_i^* & & \simeq\uparrow\varphi_i^* & & \simeq\uparrow\varphi_i^* & & \\ 0 & \to & R^0\chi_{i*}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}_i} & \to & R^0\chi_{i*}G^0_{(\mathscr{X})\mathscr{X}_i} & \to & R^0\chi_{i*}Z^1_{(\mathscr{X})\mathscr{X}_i} & \to & R^1\chi_{i*}(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}_i} & \to & 0\,. \end{array}$$

(2.17) Let $\xi\in H^pR^1\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}$, represented by a cocycle $\{\xi_{i_0\cdots i_p}\}\in Z^p(\mathscr{U}, R^1\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}})$. By the canonical isomorphism $R^1\chi_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}}\xrightarrow{\sim} R^1\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}$, given by the φ_i^* (cf. (1.2)), this becomes an element of $Z^p(\mathscr{U}, R^1\chi_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}})$. Refining $\mathscr{U}$ if necessary, (actually it is not), we may assume that there exist $t'_{i_0\cdots i_p}\in\Gamma(U_{i_0\cdots i_p}, R^0\chi_*Z^1_{(\mathscr{X})})$ such that the cochain $t'=\{t'_{i_0\cdots i_p}\}$ maps to the given cocycle. Then $\delta t'\in Z^{p+1}(\mathscr{U}, B^1_{(\mathscr{X})})$ and, refining $\mathscr{U}$ again if necessary, there exist $\lambda'_{i_0\cdots i_{p+1}}\in\Gamma(U_{i_0\cdots i_{p+1}}, R^0\chi_*G^0_{(\mathscr{X})})$ mapping to $(\delta t')_{i_0\cdots i_{p+1}}=\sum_{\nu=0}^{p+1}(-1)^\nu t_{i_0\cdots\hat{i}_\nu\cdots i_{p+1}}$. The cocycle $\delta\lambda'\in Z^{p+2}(\mathscr{U}, R^0\chi_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}})$ carries over to a cocycle $\eta=\{\eta_{i_0\cdots i_{p+2}}\}\in Z^{p+2}(\mathscr{U}, R^0\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}})$, and $d_2^{p1}(\mathscr{X})(\xi)$ is given by $(-1)^p\eta$.

(2.18) We compute $\imath(\delta[\mathscr{X}])\xi$. In the notation of (2.15) and (2.17), $\imath(\delta[\mathscr{X}])\xi$ is represented by the cocycle $\{\imath(\sigma_{ijk})\xi_{kk_1\cdots k_p}\}\in Z^{p+2}(\mathscr{U}, R^0\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}})$. We compute this, according to (2.12). The elment $t_{i_0\cdots i_p}:=\varphi_{i_0}^*t'_{i_0\cdots i_p}\in\Gamma(U_{i_0\cdots i_p}, R^0\alpha_*Z^1)$ maps to $\xi_{i_0\cdots i_p}$. Refining $\mathscr{U}$ if necessary, there exists $h_{i_0\cdots i_p}\in\Gamma(U_{i_0\cdots i_p}, R^0\alpha_*G^0)$ mapping to $m^*t_{i_0\cdots i_p}$, and $\imath(\sigma_{ijk})\xi_{kk_1\cdots k_p}=\tau^*_{\sigma_{ijk}}h_{kk_1\cdots k_p}-h_{kk_1\cdots k_p}$.

(2.19) We compare both results. We have, in $\Gamma(U_{i_0\cdots i_{p+2}}, R^0\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}})$: $\eta_{i_0\cdots i_{p+2}}=m^*\varphi_{i_0}^*\sum_{\nu=0}^{p+2}(-1)^\nu\lambda'_{i_0\cdots\hat{i}_\nu\cdots i_{p+2}}$. The image in $R^0\alpha_*G^0$ of the element $m^*\varphi_{j_0}^*\lambda'_{j_0\cdots j_{p+1}}$ is $\sum_{\mu=1}^{p+1}(-1)^\mu m^*t_{j_0\cdots\hat{j}_\mu\cdots j_{p+1}}+\tau^*_{\tilde{s}_{j_1j_0}}m^*t_{j_1\cdots j_{p+1}}$. It follows that there exists an element $w_{j_0\cdots j_{p+1}}\in\Gamma(U_{j_0\cdots j_{p+1}}, R^0\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}})$ such that $m^*\varphi_{j_0}^*\lambda'_{j_0\cdots j_{p+1}}=(\delta h)_{j_0\cdots j_{p+1}}-h_{j_1\cdots j_{p+1}}+\tau^*_{\tilde{s}_{j_1j_0}}h_{j_1\cdots j_{p+1}}+w_{j_0\cdots j_{p+1}}$. Writing $w=\{w_{j_0\cdots j_{p+1}}\}$ for the $(p+1)$-cochain so obtained, one finds that $\eta_{i_0\cdots i_{p+2}}=(\delta w)_{i_0\cdots i_{p+2}}+\tau^*_{\tilde{s}_{i_2i_0}}(\tau^*_{(-\sigma_{i_0i_1i_2})}h_{i_2\cdots i_{p+2}}-h_{i_2\cdots i_{p+2}})=(\delta w)_{i_0\cdots i_{p+2}}+\tau^*_{(-\sigma_{i_0i_1i_2})}h_{i_2\cdots i_{p+2}}-h_{i_2\cdots i_{p+2}}$. This shows that η represents $\imath(-\delta[\mathscr{X}])\xi$, hence that $(-1)^p\eta$ represents $(-1)^{p+1}\imath(\delta[\mathscr{X}])\xi$, as claimed.

(2.20) To deal with the general case, we shall need a slight generalization of the case $q=1$. Let $\mathscr{M}$ be a flat $(\boldsymbol{Z}/m\boldsymbol{Z})_B$-module (in our application, this will be actually a locally free module of finite rank). Then, in the Leray spectral sequence for $\mathscr{M}_{\mathscr{X}}$ with respect to χ (cf. (1.16)) the differentials

$$d_2^{p1}=d_2^{p1}(\mathscr{X},\mathscr{M}):\ H^pR^1\alpha_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}\otimes\mathscr{M}\to H^{p+2}\mathscr{M}$$

are given by $(-1)^{p+1}\imath(\delta[\mathscr{X}])$, as before. Above we have considered the case $\mathscr{M}=(\boldsymbol{Z}/m\boldsymbol{Z})_B$. Replacing all sheaves by their tensor product with $\mathscr{M}$, and all morphisms by their tensor product with $1_{\mathscr{M}}$, the proof holds verbatim in this situation.

(2.21) Proof of (2.7): General case. In the notation of (1.6), we claim that $d_2^{pq}(\tilde{\mathscr{X}})=\bigoplus d_2^{pi}(\mathscr{X}, R^{q-i})$. This sharpens the statement of (1.9). We will not use that statement, but prove instead directly that the following diagram is commutative, for all i:

$$\begin{array}{ccc} H^p(R^i\otimes R^{q-i}) & \hookrightarrow & H^p(\tilde{R}^q)\\ {\scriptstyle d_2^{pi}(\mathscr{X},R^{q-i})}\downarrow & & \downarrow{\scriptstyle d_2^{pq}(\tilde{\mathscr{X}})}\\ H^{p+2}(R^{i-1}\otimes R^{q-i}) & \hookrightarrow & H^{p+2}\tilde{R}^{q-1}. \end{array}$$

Let $G^\cdot_{(\mathscr{X})}$, $G^\cdot$ and $G^\cdot_{(\tilde{\mathscr{X}})}$, respectively, be Godement resolutions for $(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}}$, $(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{A}}$ and $(\boldsymbol{Z}/m\boldsymbol{Z})_{\tilde{\mathscr{X}}}$. Let $R^0\chi_*G^\cdot_{(\mathscr{X})}\otimes R^0\alpha_*G^\cdot\to R^0\tilde{\chi}_*G^\cdot_{(\tilde{\mathscr{X}})}$ be a morphism of complexes inducing the Künneth maps $R^{ij}=R^i\otimes R^j\hookrightarrow\tilde{R}^{i+j}$. We derive from it the following commutative diagram with exact rows:

$$
\begin{array}{cccccccccc}
0\to & R^{i-1}\otimes R^{q-i} & \to & (\cdots)\otimes R^{q-i} & \to & R^0\chi_*Z^i_{(\mathscr{X})}\otimes R^{q-i} & \to & R^i\otimes R^{q-i} & \to 0 \\
 & \uparrow & & \uparrow & & \| & & \| & \\
0\to & R^0\chi_*Z^{i-1}_{(\mathscr{X})}\otimes R^{q-i} & \to & R^0\chi_*G^{i-1}_{(\mathscr{X})}\otimes R^{q-i} & \to & R^0\chi_*Z^i_{(\mathscr{X})}\otimes R^{q-i} & \to & R^i\otimes R^{q-i} & \to 0 \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
0\to & \ker & \to & R^0\chi_*G^{i-1}_{(\mathscr{X})}\otimes R^0\alpha_*Z^{q-i} & \to & R^0\chi_*Z^i_{(\mathscr{X})}\otimes R^0\alpha_*Z^{q-i} & \to & R^i\otimes R^0\alpha_*Z^{q-i} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0\to & R^0\tilde{\chi}_*Z^{q-1}_{(\tilde{\mathscr{X}})} & \to & R^0\tilde{\chi}_*G^{q-1}_{(\tilde{\mathscr{X}})} & \to & R^0\tilde{\chi}_*Z^q_{(\tilde{\mathscr{X}})} & \to & \tilde{R}^q & \to 0 \\
 & \downarrow & & \downarrow & & \| & & \| & \\
0\to & \tilde{R}^{q-1} & \to & (\cdots) & \to & R^0\tilde{\chi}_*Z^q_{(\tilde{\mathscr{X}})} & \to & \tilde{R}^q & \to 0\,.
\end{array}
$$

The top and bottom squares on the left hand side are pushout diagrams. Moreover, if we add the Künneth maps on the outer left and right hand sides, $R^{i-1}\otimes R^{q-i}\to\tilde{R}^{q-1}$, $R^i\otimes R^{q-i}\to\tilde{R}^q$, the resulting diagram stays commutative. It follows that we have a diagram

$$
\begin{array}{ccc}
H^p(R^i\otimes R^0\alpha_*Z^{q-i}) & \xrightarrow{\partial\partial} & H^{p+2}(\ker) \\
\downarrow\psi & & \downarrow \\
H^p(R^i\otimes R^{q-i}) & \xrightarrow{\partial\partial} & H^{p+2}(R^{i-1}\otimes R^{q-i}) \\
\downarrow & & \downarrow \\
H^p\tilde{R}^q & \xrightarrow{\partial\partial} & H^{p+2}\tilde{R}^{q-1}\,,
\end{array}
$$

in which the top square and the outer rectangle are known to commute (as they stem from morphisms between complexes). In order to prove that the bottom square commutes, it will be sufficient to show that the map ψ is surjective. This follows from the next result:

(2.22) LEMMA. *Let, as above, $G^\bullet$ be a Godement resolution of $(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}$. Write, as usual, $B^i=\operatorname{Im}(R^0\alpha_*G^{i-1}\to R^0\alpha_*G^i)$. Suppose that the integer m does not have prime factors less than or equal to q. Then the exact sequence of $(\mathbf{Z}/m\mathbf{Z})_B$-modules $0\to B^q\to R^0\alpha_*Z^q\to R^q\to 0$ splits.*

PROOF. We prove this by induction on q. For $q=0$ it is obviously true. As in (1.9) above, our tool in this proof is the following one (and variations of it): Suppose that we have a natural morphism $\phi\colon R^i\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}\to R^j\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}$ $i\neq j$, in the sense that it depends functorially on $\mathscr{A}$. Then, by using the multiplication maps by integers $d\geq 1$ on $\mathscr{A}$, we get $(d^j-d^i)\phi=0$. If i, j are bound to stay less than or equal to q, and m has no prime factors less than or equal to q, then there exists d such that d^j-d^i is a unit in $\mathbf{Z}/m\mathbf{Z}$, and hence $\phi=0$ follows.

The splitting of the sequence of the lemma is equivalent to the vanishing of the map $H^0(R^q \otimes (R^q)^\vee) \to H^1(B^q \otimes (R^q)^\vee)$. Write $\mathscr{M} = (R^q)^\vee$ for this locally free $(\mathbf{Z}/m\mathbf{Z})_B$-module. The sheaf $(R^0\alpha_* G^i) \otimes \mathscr{M} = R^0\alpha_*(G^i \otimes \mathscr{M}_{\mathscr{A}})$ is acyclic, since it is a flabby sheaf (in the terminology of [7, p. 87]). Therefore $H^1(B^q \otimes \mathscr{M}) \xrightarrow{\sim} H^2((R^0\alpha_* Z^{q-1}) \otimes \mathscr{M})$ canonically. By the induction hypothesis we have an exact sequence $0 \to H^2(B^{q-1} \otimes \mathscr{M}) \to H^2((R^0\alpha_* Z^{q-1}) \otimes \mathscr{M}) \to H^2(R^{q-1} \otimes \mathscr{M}) \to 0$. The morphism given by the composition $H^0(R^q \otimes \mathscr{M}) \to H^1(B^q \otimes \mathscr{M}) \xrightarrow{\sim} H^2((R^0\alpha_* Z^{q-1}) \otimes \mathscr{M}) \to H^2(R^{q-1} \otimes \mathscr{M})$ is natural with respect to the first factor, hence, by (an obvious variation of) the remark at the beginning of this proof, it is zero. So the map we started with can be viewed as a morphism $H^0(R^q \otimes \mathscr{M}) \to H^2(B^{q-1} \otimes \mathscr{M})$. Iterating the procedure, this morphism is reduced succesively to a morphism $H^0(R^q \otimes \mathscr{M}) \to H^{i+1}(B^{q-i} \otimes \mathscr{M})$ as i increases, and it becomes eventually zero, as claimed.

The ends the proof of Lemma (2.22) and hence of the claim at the beginning of (2.21).

Application of (1.6) now yields the right hand side square of the following commutative diagram

$$\begin{array}{ccccc}
H^p(R^1 \otimes R^{q-1}) & \xleftarrow{\text{proj}} & H^p\tilde{R}^q & \xleftarrow{\Sigma^*} & H^pR^q \\
\downarrow{\scriptstyle d_2^{p1}(\mathscr{X},\, R^{q-1})} & & \downarrow{\scriptstyle d_2^{pq}(\tilde{\mathscr{X}})} & & \downarrow{\scriptstyle d_2^{pq}(\mathscr{X})} \\
H^{p+2}(R^0 \otimes R^{q-1}) & \xleftarrow[\text{proj}]{} & H^{p+2}\tilde{R}^{q-1} & \xleftarrow[\Sigma^*]{} & H^{p+2}R^{q-1}\,.
\end{array}$$

The composite map of the bottom row is the identity. And the composition of the upper row with $d_2^{p1}(\mathscr{X}, R^{q-1}) = (-1)^{p+1}\iota(\delta[\mathscr{X}])$ is $(-1)^{p+1}\iota(\delta[\mathscr{X}])\colon H^pR^q \to H^{p+2}R^{q-1}$. This completes the proof of Proposition (2.7).

(2.23) REMARK. Proposition (2.7) thus gives an explicit computation of the natural mappings $H^1(B, \mathscr{A})_{\text{rep}} \to \operatorname{Hom}(H^pR^q\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}, H^{p+2}R^{q-1}\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}})$ given by $\mathscr{X} \mapsto d_2^{pq}(\mathscr{X})$. In a different direction, one has also natural mappings $H^0(B, \mathscr{A}) \to \operatorname{Hom}(H^pR^q\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}, H^{p+1}R^{q-1}\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}})$, again at least under certain restrictions on the integer m. Namely, if m has no prime factors less than or equal to $2\dim_B \mathscr{A}$, then the Leray spectral sequence for $(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}$ degenerates (cf. (1.4)). Then, given $s \in H^0(B, \mathscr{A})$, one has commutative diagrams (τ_s = translation by s)

$$\begin{array}{ccccccccc}
0 & \to & H^{p+1}R^{q-1}\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}} & \to & \dfrac{F^pH^{p+q}(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}}{F^{p+2}H^{p+q}(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}} & \to & H^pR^q\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}} & \to & 0 \\
 & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle \tau_s^*-1} & & \uparrow{\scriptstyle 0} & & \\
0 & \to & H^{p+1}R^{q-1}\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}} & \to & \dfrac{F^pH^{p+q}(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}}{F^{p+2}H^{p+q}(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}} & \to & H^pR^q\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}} & \to & 0
\end{array}$$

and hence induced morphisms $\tau_s^* - 1\colon H^pR^q\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}} \to H^{p+1}R^{q-1}\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}$. By arguments similar to the preceding ones in this section, one can show that this map is given by $(-1)^{p+1}\iota(\delta s)$, $\delta s \in H^1(B, {}_m\mathscr{A})$.

This allows one then to compute the endomorphism τ_s^* of $H^n(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}$ in terms of the canonical direct sum decomposition $H^n(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}=\bigoplus_{p+q=n} H^pR^q\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}$, where now we are supposing that m has no prime factors which are less than or equal to $2\dim_B\mathscr{A}+1$ (cf. (1.5) (ii)). Writing $\tau_s^*=\sum_{i\geq j}\theta_{ij}^{(s)}$ with $\theta_{ij}^{(s)}\colon H^{n-i}R^i\to H^{n-j}R^j$, we have: $\theta_{ii}^{(s)}=1$ and $\theta_{i,i-1}^{(s)}=(-1)^{n-i+1}\iota(\delta s)$. Then, using the relations $\tau_{s+t}^*=\tau_t^*\tau_s^*$, $s, t\in H^0(B,\mathscr{A})$ and $\tau_s^*r^*=r^*\tau_{rs}^*$, $r\in\mathbf{Z}$, $s\in H^0(B,\mathscr{A})$, one finds that

$$\theta_{i,i-h}^{(s)}=(-1)^{(n-i)h+(h+1)h/2}(1/h!)\iota(\delta s)^h\,.$$

3. Extension classes.

(3.1) Throughout this section we keep the assumptions of (2.1) on the validity of the equivalence between Čech cohomology and derived functor cohomology. Let k be a finite field, p_0 its characteristic, and B a k-scheme. Let $\alpha\colon\mathscr{A}\to B$ be an abelian scheme and let $\chi\colon\mathscr{X}\to B$ be a torsor for $\mathscr{A}$. By base extension from k to $\bar{k}$ we obtain a torsor $\bar{\chi}\colon\bar{\mathscr{X}}\to\bar{B}$ for the abelian scheme $\bar{\alpha}\colon\bar{\mathscr{A}}\to\bar{B}$. Suppose that this torsor is $4\dim_B\mathscr{A}$ times m-divisible, for a given $m\in\mathbf{Z}_{\geq 1}$, prime to p_0, and such that m has no prime factors less than or equal to $2\dim_B\mathscr{A}$. By Proposition (1.4), the Leray spectral sequence for $(\mathbf{Z}/m\mathbf{Z})_{\bar{\mathscr{X}}}$ degenerates, and by Proposition (1.13) the corresponding exact sequences

$$\text{(3.2)}\qquad 0\to H^{p+1}R^{q-1}\bar{\alpha}_*(\mathbf{Z}/m\mathbf{Z})_{\bar{\mathscr{A}}}\to\frac{F^pH^{p+q}(\mathbf{Z}/m\mathbf{Z})_{\bar{\mathscr{X}}}}{F^{p+2}H^{p+q}(\mathbf{Z}/m\mathbf{Z})_{\bar{\mathscr{X}}}}\to H^pR^q\bar{\alpha}_*(\mathbf{Z}/m\mathbf{Z})_{\bar{\mathscr{A}}}\to 0$$

of discrete $(\mathbf{Z}/m\mathbf{Z})[G]$-modules ($G=\mathrm{Gal}(\bar{k}/k)$) are split sequences of $(\mathbf{Z}/m\mathbf{Z})$-modules. Our aim is to compute the class of the extension (3.2). The result is stated below, in Proposition (3.7). We need to define three maps, before:

(3.3) Let $H^1(B,\mathscr{A})_{[m]}\subset H^1(B,\mathscr{A})$ be the subgroup consisting of those elements whose image in $H^1(\bar{B},\bar{\mathscr{A}})$ is m-divisible. There is a natural group homomorphism

$$\beta_1\colon H^1(B,\mathscr{A})_{[m]}\to H^1(k, H^1(\bar{B},{}_m\bar{\mathscr{A}}))$$

obtained as follows. We have a commutative exact diagram

$$\begin{array}{ccccccc}
\longrightarrow & H^1(\bar{B},\bar{\mathscr{A}}) & \xrightarrow{m} & H^1(\bar{B},\bar{\mathscr{A}}) & \longrightarrow & H^2(\bar{B},{}_m\bar{\mathscr{A}}) & \longrightarrow\\
& \uparrow & & \uparrow & & \uparrow & \\
\longrightarrow & H^1(B,\mathscr{A}) & \xrightarrow{m} & H^1(B,\mathscr{A}) & \xrightarrow{\delta} & H^2(B,{}_m\mathscr{A}) & \longrightarrow\\
& & & & & \uparrow & \\
& & & & & H^1(k, H^1(\bar{B},{}_m\bar{\mathscr{A}})) & \\
& & & & & \uparrow & \\
& & & & & 0\;, &
\end{array}$$

the horizontal sequences coming from the multiplication by m map sequences for $\bar{\mathscr{A}}$ and $\mathscr{A}$, respectively, and the vertical sequence coming from the Hochschild-Serre spectral sequence. It follows that $H^1(B, \mathscr{A})_{[m]}$ is the inverse image of $H^1(k, H^1(\bar{B}, {}_m\bar{\mathscr{A}}))$ by the connecting homomorphism δ, and this map induces β_1.

(3.4) We call

$$\beta_2 : H^1(k, H^1(\bar{B}, {}_m\bar{\mathscr{A}})) \to H^1(k, \mathrm{Hom}_{\mathbf{Z}/m\mathbf{Z}}(H^pR^q\bar{\alpha}_*(\mathbf{Z}/m\mathbf{Z})_{\bar{\mathscr{A}}}, H^{p+1}R^{q-1}\bar{\alpha}_*(\mathbf{Z}/m\mathbf{Z})_{\bar{\mathscr{A}}}))$$

the map obtained functorially from the morphism $H^1(\bar{B}, {}_m\bar{\mathscr{A}}) \to \mathrm{Hom}_{\mathbf{Z}/m\mathbf{Z}}(H^pR^q\bar{\alpha}_*(\mathbf{Z}/m\mathbf{Z})_{\bar{\mathscr{A}}}, H^{p+1}R^{q-1}\bar{\alpha}_*(\mathbf{Z}/m\mathbf{Z})_{\bar{\mathscr{A}}})$ defined by the inner contraction map (2.5) for $\bar{\mathscr{A}}$.

(3.5) For any pair of discrete $(\mathbf{Z}/m\mathbf{Z})[G]$-modules M and N, with M finite, the spectral sequence $H^p(k, \mathrm{Ext}^q_{\mathbf{Z}/m\mathbf{Z}}(M, N)) \Rightarrow \mathrm{Ext}^{p+q}_{(\mathbf{Z}/m\mathbf{Z})[G]}(M, N)$ gives, in low degrees, a canonical isomorphism

$$H^1(k, \mathrm{Hom}_{\mathbf{Z}/m\mathbf{Z}}(M, N)) \xrightarrow{\sim} \mathrm{Ext}^1_{(\mathbf{Z}/m\mathbf{Z})[G]}(M, N)^{\mathrm{split}}, \tag{3.6}$$

where the right hand side member is the $(\mathbf{Z}/m\mathbf{Z})$-submodule of $\mathrm{Ext}^1_{(\mathbf{Z}/m\mathbf{Z})[G]}(M, N)$ consisting of the extensions which split as extensions of $(\mathbf{Z}/m\mathbf{Z})$-modules. Given such an extension $0 \to N \to (\cdots) \to M \to 0$, one forms the exact sequence of discrete $(\mathbf{Z}/m\mathbf{Z})[G]$-modules

$$0 \to \mathrm{Hom}_{\mathbf{Z}/m\mathbf{Z}}(M, N) \to \mathrm{Hom}_{\mathbf{Z}/m\mathbf{Z}}(M, (\cdots)) \to \mathrm{Hom}_{\mathbf{Z}/m\mathbf{Z}}(M, M) \to 0$$

and takes the image of 1_M by the connecting homomorphism of the associated cohomology sequence. This is the element of $H^1(k, \mathrm{Hom}_{\mathbf{Z}/m\mathbf{Z}}(M, N))$ to which the sequence corresponds, under the above isomorphism.

We let

$$\beta_3 : H^1(k, \mathrm{Hom}_{\mathbf{Z}/m\mathbf{Z}}(H^pR^q\bar{\alpha}_*(\mathbf{Z}/m\mathbf{Z})_{\bar{\mathscr{A}}}, H^{p+1}R^{q-1}\bar{\alpha}_*(\mathbf{Z}/m\mathbf{Z})_{\bar{\mathscr{A}}})) \hookrightarrow \mathrm{Ext}^1_{(\mathbf{Z}/m\mathbf{Z})[G]}(H^pR^q\bar{\alpha}_*(\mathbf{Z}/m\mathbf{Z})_{\bar{\mathscr{A}}}, H^{p+1}R^{q-1}\bar{\alpha}_*(\mathbf{Z}/m\mathbf{Z})_{\bar{\mathscr{A}}})$$

be the inclusion map given by (3.6).

Now we can state:

(3.7) PROPOSITION. *In the notation and under the assumptions of* (3.1), *the class of the extension* (3.2) *is given by* $(-1)^p\beta_3\beta_2\beta_1([\mathscr{X}])$.

The class of the extension (3.2) belongs to the image of β_3. Identifying by β_3, we may consider it therefore as an element of $H^1(k, \mathrm{Hom}_{\mathbf{Z}/m\mathbf{Z}}(H^pR^q\bar{\alpha}_*(\mathbf{Z}/m\mathbf{Z})_{\bar{\mathscr{A}}}, H^{p+1}R^{q-1}\bar{\alpha}_*(\mathbf{Z}/m\mathbf{Z})_{\bar{\mathscr{A}}}))$. We prove that this element coincides with $(-1)^p\beta_2\beta_1([\mathscr{X}])$.

(3.8) PROOF OF (3.7): Case $q=1$. Let $\chi' : \mathscr{X}' \to \bar{B}$ be a torsor for $\bar{\mathscr{A}}$ such that $[\bar{\mathscr{X}}] = m[\mathscr{X}']$ in $H^1(\bar{B}, \bar{\mathscr{A}})$. Let $f : \mathscr{X}' \to \bar{\mathscr{X}}$ be a $\bar{B}$-morphism, equivariant for $m : \bar{\mathscr{A}} \to \bar{\mathscr{A}}$. We may find a covering $\mathscr{U} = (U_i)_{i\in I}$ of B in the etale topology trivializing the torsor $\mathscr{X}$ and such that the covering $\bar{\mathscr{U}} = (\bar{U}_i)_{i\in I}$ of $\bar{B}$ deduced by base change trivializes $\mathscr{X}'$, in a way compatible with f. In the abreviated notation introduced in (2.15), this means that we have isomorphisms of $\mathscr{A}_i$-torsors $\varphi_i: \mathscr{A}_i \xrightarrow{\sim} \mathscr{X}_i$ and isomorphisms of $\bar{\mathscr{A}}_i$-torsors $\varphi'_i: \bar{\mathscr{A}}_i \xrightarrow{\sim} \mathscr{X}'_i$ making the following diagrams commutative:

$$\begin{array}{ccc} \mathscr{X}'_i & \xrightarrow{f} & \bar{\mathscr{X}}_i \\ \varphi'_i \uparrow \simeq & & \simeq \uparrow \bar{\varphi}_i = \varphi_i \otimes_k \bar{k} \\ \bar{\mathscr{A}}_i & \xrightarrow{m} & \bar{\mathscr{A}}_i . \end{array}$$

Over U_{ij} we have $\varphi_j = \varphi_i \tau_{s_{ij}}$ and, over $\bar{U}_{ij}$: $\varphi'_j = \varphi'_i \tau_{s'_{ij}}$, with $\{s_{ij}\} \in Z^1(\mathscr{U}, \mathscr{A})$, $\{s'_{ij}\} \in Z^1(\bar{\mathscr{U}}, \bar{\mathscr{A}})$, describing $[\mathscr{X}] \in H^1(B, \mathscr{A})$ and $[\mathscr{X}'] \in H^1(\bar{B}, \bar{\mathscr{A}})$ respectively. Moreover, $\bar{s}_{ij} = m s'_{ij}$, where $\{\bar{s}_{ij}\} \in Z^1(\bar{\mathscr{U}}, \bar{\mathscr{A}})$ is deduced from $\{s_{ij}\}$ by base change.

We have commutative exact diagrams of sheaves on (the etale site of) $\bar{B}$ (cf. (2.11))

$$(3.9)\qquad \begin{array}{ccccccccc} 0 \to & R^0\bar{\alpha}_*(Z/mZ)_{\bar{\mathscr{A}}} & \to & R^0\bar{\alpha}_* G^0 & \to & R^0\bar{\alpha}_* Z^1 & \to & R^1\bar{\alpha}_*(Z/mZ)_{\bar{\mathscr{A}}} & \to 0 \\ & \| & & \uparrow m^* & & \uparrow m^* & & \uparrow 0 & \\ 0 \to & R^0\bar{\alpha}_*(Z/mZ)_{\bar{\mathscr{A}}} & \to & R^0\bar{\alpha}_* G^0 & \to & R^0\bar{\alpha}_* Z^1 & \to & R^1\bar{\alpha}_*(Z/mZ)_{\bar{\mathscr{A}}} & \to 0 , \end{array}$$

$$(3.10)\qquad \begin{array}{ccccccccc} 0 \to & R^0\chi'_*(Z/mZ)_{\mathscr{X}'} & \to & R^0\chi'_* G^0_{(\mathscr{X}')} & \to & R^0\chi'_* Z^1_{(\mathscr{X}')} & \to & R^1\chi'_*(Z/mZ)_{\mathscr{X}'} & \to 0 \\ & \simeq \uparrow f^* & & \uparrow f^* & & \uparrow f^* & & \uparrow 0 & \\ 0 \to & R^0\bar{\chi}_*(Z/mZ)_{\bar{\mathscr{X}}} & \to & R^0\bar{\chi}_* G^0_{(\bar{\mathscr{X}})} & \to & R^0\bar{\chi}_* Z^1_{(\bar{\mathscr{X}})} & \to & R^1\bar{\chi}_*(Z/mZ)_{\bar{\mathscr{X}}} & \to 0 . \end{array}$$

The restriction of these two ladders to $\bar{U}_i$ are connected by morphisms $(\varphi'_{i_0})^*$ (top rows) and $\bar{\varphi}_i^*$ (bottom rows) giving a commutative diagram.

(3.11) We compute $\beta_2\beta_1([\mathscr{X}])$. Given $\gamma \in G$, we have $\bar{s}_{ij} = {}^\gamma\bar{s}_{ij} = m\,{}^\gamma s'_{ij}$, hence, putting $\sigma_{ij} = s'_{ij} - {}^\gamma s'_{ij}$: $m\sigma_{ij} = 0$ and $\{\sigma_{ij}\} \in Z^1(\bar{\mathscr{U}}, {}_m\bar{\mathscr{A}})$. The map $G \to H^1(\bar{B}, {}_m\bar{\mathscr{A}})$, $\gamma \mapsto \{\sigma_{ij}\}$ is a 1-cocycle defining $\beta_1([\mathscr{X}]) \in H^1(k, H^1(\bar{B}, {}_m\bar{\mathscr{A}}))$. In this notation, $\beta_2\beta_1([\mathscr{X}])$ is represented by the cocycle $G \to \mathrm{Hom}_{Z/mZ}(H^pR^1\bar{\alpha}_*(Z/mZ)_{\bar{\mathscr{A}}}, H^{p+1}R^0\bar{\alpha}_*(Z/mZ)_{\bar{\mathscr{A}}})$

$$(3.12)\qquad \gamma \mapsto \iota(\{\sigma_{ij}\}) .$$

By (2.12), $\iota(\{\sigma_{ij}\}): H^pR^1\bar{\alpha}_*(Z/mZ)_{\bar{\mathscr{A}}} \to H^{p+1}R^0\bar{\alpha}_*(Z/mZ)_{\bar{\mathscr{A}}}$ is described as follows (cf. also (2.18)). Given $\xi \in H^pR^1\bar{\alpha}_*(Z/mZ)_{\bar{\mathscr{A}}}$, refine successively $\mathscr{U}$, if necessary, in order that the following be feasible (cf. the diagram (3.9)): Represent ξ by a p-cocycle of $R^1\bar{\alpha}_*(Z/mZ)_{\bar{\mathscr{A}}}$ which is the image of a p-cochain $\{\tilde{z}_{i_0\cdots i_p}\} \in C^p(\bar{\mathscr{U}}, R^0\bar{\alpha}_*Z^1)$. Let $\{h_{i_0\cdots i_p}\} \in C^p(\bar{\mathscr{U}}, R^0\bar{\alpha}_*G^0)$ be a p-cochain mapping to $\{m^*\tilde{z}_{i_0\cdots i_p}\}$. Then $\iota(\{\sigma_{ij}\})\xi = \{\tau^*_{\sigma_{i_0i_1}} h_{i_1\cdots i_{p+1}} - h_{i_1\cdots i_{p+1}}\}$.

(3.13) On the other hand, it follows from (3.5) and (1.15) that the element of $H^1(k, \mathrm{Hom}_{Z/mZ}(H^pR^1\bar{\alpha}_*(Z/mZ)_{\bar{\mathscr{A}}}, H^{p+1}R^0\bar{\alpha}_*(Z/mZ)_{\bar{\mathscr{A}}}))$ which is sent by β_3 to the extension class given by the exact sequence

$$(3.14)\qquad 0 \to H^{p+1}R^0\bar{\alpha}_*(Z/mZ)_{\bar{\mathscr{A}}} \to F^pH^{p+1}(Z/mZ)_{\bar{\mathscr{X}}} \to H^pR^1\bar{\alpha}_*(Z/mZ)_{\bar{\mathscr{A}}} \to 0$$

is described by he cocycle $G \to \mathrm{Hom}_{Z/mZ}(H^pR^1\bar{\alpha}_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{A}}}, H^{p+1}R^0\bar{\alpha}_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{A}}})$, $\gamma \mapsto f^* - {}^{\gamma}(f^*)$. (The right hand side member is an endomorphism of $F^pH^{p+1}(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{X}}}$, which actually corresponds to a map $H^pR^1\bar{\alpha}_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{A}}} \to H^{p+1}R^0\bar{\alpha}_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{A}}}$). Since ${}^{\gamma}(f^*) = \gamma f^* \gamma^{-1} = (\gamma^{-1})^* f^* \gamma^*$, this can be written in the form

$$\gamma \mapsto f^* - (\gamma^{-1})^* f^* \gamma^* . \tag{3.15}$$

(3.16) We compare the cocycles (3.12) and (3.15). First we recall that, modulo the canonical identifications (given on $\bar{U}_i$ by $\bar{\varphi}_i^*$) $R^i\bar{\chi}_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{X}}} \xrightarrow{\sim} R^i\bar{\alpha}_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{A}}}$, the exact sequence (3.14) comes from the exact sequence of hypercohomology of the short exact sequence of complexes (written vertically, and $\deg R^0\bar{\chi}_*Z^1_{(\bar{\mathscr{X}})} = 1$):

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \uparrow & & & & \uparrow & & \\
0 & \to & B^1_{(\bar{\mathscr{X}})} & \to & R^0\bar{\chi}_*Z^1_{(\bar{\mathscr{X}})} & \to & 0 & \mathscr{Z}^{\bullet} \\
 & & \uparrow & & \| & & \uparrow & & \\
0 & \to & R^0\bar{\chi}_*G^0_{(\bar{\mathscr{X}})} & \to & R^0\bar{\chi}_*Z^1_{(\bar{\mathscr{X}})} & \to & 0 & \mathscr{M}^{\bullet} \\
 & & \uparrow & & & & \uparrow & & \\
0 & \to & R^0\bar{\chi}_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{X}}} & \to & 0 & & & \mathscr{B}^{\bullet} \\
 & & \uparrow & & & & \uparrow & & \\
 & & 0 & & & & 0, & &
\end{array}$$

specifically:

$$0 \to \boldsymbol{H}^{p+1}\mathscr{B}^{\bullet} \to \boldsymbol{H}^{p+1}\mathscr{M}^{\bullet} \to \boldsymbol{H}^{p+1}\mathscr{Z}^{\bullet} \to 0 . \tag{3.17}$$

Now consider again $\xi \in H^pR^1\bar{\alpha}_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{A}}}$ as in (3.11). Write $\xi_{\bar{\mathscr{X}}} \in H^pR^1\bar{\chi}_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{X}}}$ for the element to which it corresponds under the natural identification. By the degeneration of the Leray spectral sequence for $(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{X}}}$, the map $H^pR^0\bar{\chi}_*Z^1_{(\bar{\mathscr{X}})} \to H^pR^1\bar{\chi}_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{X}}}$ is surjective (cf. (2.8)). Let $\{z_{i_0\cdots i_p}\} \in Z^p(\bar{\mathscr{U}}, R^0\bar{\chi}_*Z^1_{(\bar{\mathscr{X}})})$ be a p-cocycle mapping to $\xi_{\bar{\mathscr{X}}}$. Together with the zero element of $Z^{p+1}(\bar{\mathscr{U}}, R^0\bar{\chi}_*G^0_{(\bar{\mathscr{X}})})$, this describes a $(p+1)$-hypercocycle of $\mathscr{M}^{\bullet}$, defining an element of $\boldsymbol{H}^{p+1}\mathscr{M}^{\bullet} = F^pH^{p+1}(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{X}}}$ which maps to $\xi_{\bar{\mathscr{X}}}$ by (3.14) (=(3.17)). The image of this element by f^* lies in $H^{p+1}R^0\bar{\alpha}_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{A}}}$, and is obtained as follows: Refining $\mathscr{U}$ if necessary, let $g = \{g_{i_0\cdots i_p}\} \in C^p(\bar{\mathscr{U}}, R^0\chi'_*G^0_{(\mathscr{X}')})$ be a p-cochain mapping to $\{f^*z_{i_0\cdots i_p}\} \in Z^p(\bar{\mathscr{U}}, R^0\chi'_*Z^1_{(\mathscr{X}')})$. Then the image of $(-1)^{p+1}\delta g \in Z^{p+1}(\bar{\mathscr{U}}, R^0\chi'_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\mathscr{X}'})$ in $R^0\bar{\alpha}_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{A}}}$ gives the claimed image. That is, the class of the cocycle $(i_0\cdots i_{p+1}) \mapsto (-1)^{p+1}\sum_{\nu=0}^{p+1}(-1)^{\nu}(\varphi'_{i_0})^*g_{i_0\cdots\hat{i}_\nu\cdots i_{p+1}}$ of $R^0\bar{\alpha}_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{A}}}$. In this way, letting moreover $g' = \{g'_{i_0\cdots i_p}\} \in C^p(\bar{\mathscr{U}}, R^0\chi'_*G^0_{(\mathscr{X}')})$ map to $\{f^*\gamma^*z_{i_0\cdots i_p}\} \in Z^p(\bar{\mathscr{U}}, R^0\chi'_*Z^1_{(\mathscr{X}')})$, it follows that $(f^* - (\gamma^{-1})^*f^*\gamma^*)(\xi) \in H^{p+1}R^0\bar{\alpha}_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{A}}}$ is represented by the cocycle $(-1)^{p+1}\{\sum_{\nu=0}^{p+1}(-1)^{\nu}(\varphi'_{i_0})^*g_{i_0\cdots\hat{i}_\nu\cdots i_{p+1}} - (\gamma^{-1})^*\sum_{\nu=0}^{p+1}(-1)^{\nu}(\varphi'_{i_0})^*g'_{i_0\cdots\hat{i}_\nu\cdots i_{p+1}}\}$.

Now let us choose the cochains g and g' suitably: as $\{\bar{\varphi}^*_{i_0}z_{i_0\cdots i_p}\}$ and $\{\tilde{z}_{i_0\cdots i_p}\}$ both map to ξ, we may assume that there exists a cochain $\{e_{i_0\cdots i_p}\}\in C^p(\bar{\mathscr{U}}, R^0\bar{\alpha}_*G^0)$ mapping to $\{\bar{\varphi}^*_{i_0}z_{i_0\cdots i_p}-\tilde{z}_{i_0\cdots i_p}\}$. Then $\{m^*e_{i_0\cdots i_p}+h_{i_0\cdots i_p}\}$ maps to $\{(\varphi'_{i_0})^*f^*z_{i_0\cdots i_p}\}$. We choose $\{g_{i_0\cdots i_p}\}$ as determined by the condition $(\varphi'_{i_0})^*g_{i_0\cdots i_p}=m^*e_{i_0\cdots i_p}+h_{i_0\cdots i_p}$. Secondly, $\{g'_{i_0\cdots i_p}\}$ may be taken as determined by the condition $\{(\varphi'_{i_0})^*g'_{i_0\cdots i_p}\}=\{\gamma^*(\varphi'_{i_0})^*g_{i_0\cdots i_p}\}$. With these choices, one finds that $(f^*-(\gamma^{-1})^*f^*\gamma^*)(\xi)$ is represented by the cocycle $\{(-1)^p\tau^*_{\gamma s'_{i_0 i_1}}(\tau^*_{\sigma_{i_0 i_1}}h_{i_1\cdots i_{p+1}}-h_{i_1\cdots i_{p+1}})\}$, which equals $(-1)^p\{\tau^*_{\sigma_{i_0 i_1}}h_{i_1\cdots i_{p+1}}-h_{i_1\cdots i_{p+1}}\}$, since sections of $R^0\bar{\alpha}_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{A}}}$ are invariant under translations. Thus the cocycles (3.12) and (3.15) differ by a factor $(-1)^p$, and this proves Proposition (3.7) for $q=1$.

(3.18) As in previous sections (cf. (1.16), (2.20)), in order to deal with the general case we shall need the case $q=1$ in a more general setting. Notation being as in (3.1), let furthermore $\mathscr{M}$ be a locally free $(\boldsymbol{Z}/m\boldsymbol{Z})_B$-module of finite rank (in (3.8) we had $\mathscr{M}=(\boldsymbol{Z}/m\boldsymbol{Z})_B$). By (1.16), the Leray spectral sequence for $\mathscr{M}_{\bar{\mathscr{X}}}$ degenerates, and the exact sequence of discrete $(\boldsymbol{Z}/m\boldsymbol{Z})[G]$-modules

$$0\to H^{p+1}R^0\bar{\alpha}_*\mathscr{M}_{\bar{\mathscr{A}}}\to F^pH^{p+1}\mathscr{M}_{\bar{\mathscr{X}}}\to H^pR^1\bar{\alpha}_*\mathscr{M}_{\bar{\mathscr{A}}}\to 0 \tag{3.19}$$

splits as a sequence of $(\boldsymbol{Z}/m\boldsymbol{Z})$-modules. The arguments of (3.8) apply to show that the extension class of (3.19) is given by $(-1)^p\beta_{3,\mathscr{M}}\beta_{2,\mathscr{M}}\beta_1([\mathscr{X}])$, where $\beta_{2,\mathscr{M}}$ and $\beta_{3,\mathscr{M}}$ are as follows. The map $\beta_{2,\mathscr{M}}\colon H^1(k, H^1(\bar{B}, {}_m\mathscr{A}))\to H^1(k, \operatorname{Hom}_{\boldsymbol{Z}/m\boldsymbol{Z}}(H^pR^1\bar{\alpha}_*\mathscr{M}_{\bar{\mathscr{A}}}, H^{p+1}R^0\bar{\alpha}_*\mathscr{M}_{\bar{\mathscr{A}}}))$ is deduced functorially from the inner contraction map (2.5), and by using the projection formula $R^i\bar{\alpha}_*\mathscr{M}_{\bar{\mathscr{A}}}=R^i\bar{\alpha}_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{A}}}\otimes\mathscr{M}_{\bar{B}}$. The map $\beta_{3,\mathscr{M}}$ is deduced from (3.6), like β_3.

(3.20) Proof of (3.7): General case. We use freely the notation of (1.6) and, in particular, R^i stands now for $R^i\bar{\alpha}_*(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{A}}}$. We have a canonical commutative diagram of discrete $(\boldsymbol{Z}/m\boldsymbol{Z})[G]$-modules, deduced from (1.20) and (1.36):

$$\begin{array}{ccccccccc}
0 & \to & H^{p+1}(R^0\otimes R^{q-1}) & \to & F^p_{\bar{\mathscr{X}}/\bar{B}}H^{p+1}(\bar{\chi}^*R^{q-1}) & \to & H^p(R^1\otimes R^{q-1}) & \to & 0\\
 & & \| & & \uparrow & & \uparrow{\scriptstyle v} & & \\
0 & \to & H^{p+1}R^{q-1} & \to & \dfrac{F^pH^{p+q}(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{X}}}}{F^{p+2}H^{p+q}(\boldsymbol{Z}/m\boldsymbol{Z})_{\bar{\mathscr{X}}}} & \to & H^pR^q & \to & 0.
\end{array}$$

Here $v=(\mathrm{proj})\circ\Sigma^*$ is the composite of $\sum^*\colon H^pR^q\to H^p\tilde{R}^q$ with the Künneth projection $\mathrm{proj}\colon H^p\tilde{R}^q\to H^p(R^1\otimes R^{q-1})$. It is now elementary, that, calling

$$H^1(k, \operatorname{Hom}_{\boldsymbol{Z}/m\boldsymbol{Z}}(H^p(R^1\otimes R^{q-1}), H^{p+1}(R^0\otimes R^{q-1})))\xrightarrow{v^*} H^1(k, \operatorname{Hom}_{\boldsymbol{Z}/m\boldsymbol{Z}}(H^pR^q, H^{p+1}R^{q-1}))$$

$$\operatorname{Ext}^1_{(\boldsymbol{Z}/m\boldsymbol{Z})[G]}(H^p(R^1\otimes R^{q-1}), H^{p+1}(R^0\otimes R^{q-1}))\xrightarrow{v^*}\operatorname{Ext}^1_{(\boldsymbol{Z}/m\boldsymbol{Z})[G]}(H^pR^q, H^{p+1}R^{q-1})$$

the morphisms induced by v, one has, for $\mathscr{M}=R^{q-1}\alpha_*(\mathbf{Z}/m\mathbf{Z})_{\mathscr{A}}: v^*\beta_{2,\mathscr{M}}=\beta_2$ and $v^*\beta_{3,\mathscr{M}}=\beta_3 v^*$. Therefore, the formula of (3.7) follows from the one in (3.18), and this ends the proof of Proposition (3.7).

4. Abelian schemes over curves.

(4.1) Let k be a finite field of characteristic p_0. Let B be a smooth, geometrically connected (not necessarily projective) curve over k, and let $\alpha: \mathscr{A}\to B$ be an abelian scheme over B. It follows from [11, p. 178] (cf. also [7, p. 123]) that, in the notation of (1.1), $H^1(\bar{B}, \bar{\mathscr{A}})_{\mathrm{rep}}=H^1(\bar{B}, \bar{\mathscr{A}})$, and $H^1(B, \mathscr{A})_{\mathrm{rep}}=H^1(B, \mathscr{A})$. For all $m\in \mathbf{Z}_{\geq 1}$ prime to p_0, the groups $H^i(\bar{B}, {}_m\bar{\mathscr{A}})$ and $H^i(B, {}_m\mathscr{A})$ are finite. The groups $H^1(\bar{B}, \bar{\mathscr{A}})$ and $H^1(B, \mathscr{A})$ are torsion groups and, for all primes $l\neq p_0$, the primary components $H^1(\bar{B}, \bar{\mathscr{A}})(l)$ and $H^1(B, \mathscr{A})(l)$ are of cofinite type (cf., e.g., [6, p. 242]). The coranks of the groups $H^1(\bar{B}, \bar{\mathscr{A}})(l)$, $l\neq p_0$ are all equal—given by the Ogg-Shafarevich formula (cf. [10])—, and, for all but a finite set of primes $l\neq p_0$, the group $H^1(\bar{B}, \bar{\mathscr{A}})(l)$ is divisible. Similar facts are expected to hold for the groups $H^1(B, \mathscr{A})(l)$, with the common corank equal to zero. In other words, it is conjectured that the group $H^1(B, \mathscr{A})(\mathrm{non}\, p_0)$ is finite. This is equivalent (cf. [15]) to the fact that $H^1(B, \mathscr{A})(l)$ be finite for at least *one* prime $l\neq p_0$.

By the assumptions on B, the B-scheme $\mathscr{A}$ is projective (cf. [2, p. 153]). Let $\lambda: \mathscr{A}\to\hat{\mathscr{A}}$ be a polarization of $\mathscr{A}$ and fix an odd prime number l (cf. (4.8) (ii)) which is prime to the degree of λ, different from p_0, and such that the group $H^1(\bar{B}, \bar{\mathscr{A}})(l)$ is divisible. This excludes a finite number of choices.

(4.2) Remark. The group $H^1(\bar{B}, \bar{\mathscr{A}})(\mathrm{non}\, p_0)$ is divisible, when B is not projective. On the other hand, when B is projective, the finite group $H^1(\bar{B}, \bar{\mathscr{A}})(l)/H^1(\bar{B}, \bar{\mathscr{A}})(l)_{\mathrm{div}}$ is dual to $H^0(\bar{B}, \hat{\bar{\mathscr{A}}})(l)/H^0(\bar{B}, \hat{\bar{\mathscr{A}}})(l)_{\mathrm{div}}$. This is the l-primary torsion subgroup of the finitely generated $\mathbf{Z}$-module $M=\hat{\bar{A}}(\bar{k}(\bar{B}))/\hat{\bar{A}}_0(\bar{k})$, where $\hat{\bar{A}}$ is the generic fibre of $\hat{\bar{\alpha}}: \hat{\bar{\mathscr{A}}}\to\bar{B}$ and $\hat{\bar{A}}_0$ is its $\bar{k}(\bar{B})/\bar{k}$-trace. Therefore, when B is projective, the primes $l\neq p_0$ such that $H^1(\bar{B}, \bar{\mathscr{A}})(l)$ is not divisible are precisely those primes $l\neq p_0$ which appear as torsion coefficients of M.

(4.3) Putting $m=l^r$, $r\in\mathbf{Z}_{\geq 1}$, in (3.3) and taking projective limits, we obtain a commutative diagram

$$\begin{array}{ccc} H^1(B, \mathscr{A}) & \xrightarrow{\ \delta\ } & H^2(B, T_l\mathscr{A}) \\ \downarrow & \searrow^{\beta_1} & \uparrow\cup \\ H^1(B, \mathscr{A})(l)/H^1(B, \mathscr{A})(l)_{\mathrm{div}} & \hookrightarrow & H^1(k, H^1(\bar{B}, T_l\bar{\mathscr{A}}))\,. \end{array} \tag{4.4}$$

Next, a procedure analogous to that in (3.4) for $m=l^r$, $r\in\mathbf{Z}_{\geq 1}$, carried out in the limit, yields a morphism

$$\beta_2: H^1(k, H^1(\bar{B}, T_l\bar{\mathscr{A}}))\to H^1(k, \mathrm{Hom}_{\mathbf{Z}_l}(H^pR^q\bar{\alpha}_*\mathbf{Z}_{l,\bar{\mathscr{A}}}, H^{p+1}R^{q-1}\bar{\alpha}_*\mathbf{Z}_{l,\bar{\mathscr{A}}}))\,.$$

Thirdly, proceeding as in (3.5), with continuous $Z_l[G]$-modules, we obtain similarly an inclusion

$$\beta_3 : H^1(k, \mathrm{Hom}_{Z_l}(H^p R^q \bar{\alpha}_* Z_{l,\bar{\mathscr{A}}}, H^{p+1} R^{q-1} \bar{\alpha}_* Z_{l,\bar{\mathscr{A}}})) \hookrightarrow \mathrm{Ext}^1_{Z_l[G]}(H^p R^q \bar{\alpha}_* Z_{l,\bar{\mathscr{A}}}, H^{p+1} R^{q-1} \bar{\alpha}_* Z_{l,\bar{\mathscr{A}}}) .$$

The composite $\beta_3\beta_2$ equals the projective limit of the corresponding composites from Section 3. Putting it all together, we obtain a group homomorphism

$$(4.5) \qquad \beta_3\beta_2\beta_1 : H^1(B, \mathscr{A}) \to \mathrm{Ext}^1_{Z_l[G]}(H^p R^q \bar{\alpha}_* Z_{l,\bar{\mathscr{A}}}, H^{p+1} R^{q-1} \bar{\alpha}_* Z_{l,\bar{\mathscr{A}}}) .$$

Section 3 then gives the following result, where we repeat the assumptions of (4.1), for the reader's convenience:

(4.6) PROPOSITION. *Let k be a finite field of characteristic p_0. Let B be a smooth, geometrically connected (not necessarily projective) curve over k, and let $\alpha : \mathscr{A} \to B$ be an abelian scheme over B. Let l be a prime number, different from 2 and from p_0, prime to the degree of a given polarization λ* of $\mathscr{A}$, *and such that the group $H^1(\bar{B}, \bar{\mathscr{A}})(l)$ is divisible (cf. (4.2)). For any torsor $\mathscr{X}$ for $\mathscr{A}$, the Leray spectral sequence $H^p R^q \bar{\alpha}_* Z_{l,\bar{\mathscr{A}}} \Rightarrow H^{p+q} Z_{l,\bar{\mathscr{X}}}$ degenerates. The extension class of the exact sequence of continuous $Z_l[G]$-modules*

$$(4.7) \quad 0 \to H^{p+1} R^{q-1} \bar{\alpha}_* Z_{l,\bar{\mathscr{A}}} \to F^p H^{p+q} Z_{l,\bar{\mathscr{X}}} / F^{p+2} H^{p+q} Z_{l,\bar{\mathscr{X}}} \to H^p R^q \bar{\alpha}_* Z_{l,\bar{\mathscr{A}}} \to 0$$

is given by $(-1)^p \beta_3\beta_2\beta_1([\mathscr{X}])$.

(4.8) REMARKS. (i) The groups $H^p R^q \bar{\alpha}_*(Z/mZ)_{\bar{\mathscr{A}}}$ are zero if $p \geq 3$ or if B is not projective and $p \geq 2$. Therefore the only meaningful sequences (4.7) are those with $p=0$ or with $p=1$, whenever B is projective.

(ii) By the same reason, $d_r(\bar{\mathscr{X}})=0$ for $r \geq 3$, and the degeneration of the Leray spectral sequence follows also from Proposition (2.7): Note that we have replaced the condition $l \geq 2\dim_B \mathscr{A} + 1$ that appears in (3.1) and in (1.4) by $l > 2$, since that condition was used fully only in the proof of (1.4), while in (1.13) and in (1.36) we only needed $l > 2$ (cf. (1.24) and (1.36)), and in the proof of (2.7) $l > 2$ is sufficient here (cf. (2.22)).

(iii) If B is not projective, then the Leray spectral sequence $H^p R^q \bar{\alpha}_* Z_{l,\bar{\mathscr{A}}} \Rightarrow H^{p+q} Z_{l,\bar{\mathscr{X}}}$ degenerates for any prime number $l \neq p_0$. If B is projective, then the maps $d_2^{0q}(\bar{\mathscr{X}}) : H^0 R^q \bar{\alpha}_* Z_{l,\bar{\mathscr{A}}} \to H^2 R^{q-1} \bar{\alpha}_* Z_{l,\bar{\mathscr{A}}}$ (and only these) might be non zero (cf. also (4.12)).

The left hand side member of (4.5) is a torsion abelian group, and the right hand side member is a finitely generated Z_l-module. Thus the morphism (4.5) vanishes on all primary components of $H^1(B, \mathscr{A})$ other than $H^1(B, \mathscr{A})(l)$ (hence the extensions (4.7) are trivial for the corresponding torsors), and the only meaningful part of (4.5) is the induced morphism

$$(4.9) \qquad \beta : H^1(B, \mathscr{A})(l) \to \mathrm{Tors}(\mathrm{Ext}^1_{Z_l[G]}(H^p R^q \bar{\alpha}_* Z_{l,\bar{\mathscr{A}}}, H^{p+1} R^{q-1} \bar{\alpha}_* Z_{l,\bar{\mathscr{A}}})) .$$

The right hand side group is finite. Thus, if β is injective, then $H^1(B, \mathscr{A})(l)$ is a finite group. The following result completes (4.6):

(4.10) PROPOSITION. *With the assumptions of* (4.6), *for* $(p, q)=(0, 2)$ *the kernel of the map* β *equals* $H^1(B, \mathscr{A})(l)_{\text{div}}$, *the divisible subgroup of* $H^1(B, \mathscr{A})(l)$. *Hence* $H^1(B, \mathscr{A})(l)$ *is finite if and only if this map is injective.*

PROOF. In view of diagram (4.4) and of the injectivity of β_3 (cf. (4.3)), it suffices to show that the map β_2 is injective. We show that the map $H^1(\bar{B}, T_l\bar{\mathscr{A}})\to \operatorname{Hom}_{\boldsymbol{Z}_l}(H^0R^2\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}}, H^1R^1\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}})$ inducing β_2 is a split injection of continuous $\boldsymbol{Z}_l[G]$-modules. The polarization $\lambda\in\operatorname{Hom}_B(\mathscr{A}, \hat{\mathscr{A}})$ is defined by a section $\lambda\in H^0(B, \mathscr{N}\mathscr{S}_{\mathscr{A}/B})$, and this section yields a cohomology class $\lambda\in H^0R^2\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}}(1)$. On the other hand, the Weil pairing $T_l\bar{\mathscr{A}}\otimes T_l\hat{\bar{\mathscr{A}}}\to\boldsymbol{Z}_{l,\bar{B}}(1)$ together with the pairing $T_l\bar{\mathscr{A}}\otimes R^1\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}}\to\boldsymbol{Z}_{l,\bar{B}}$ (cf. (2.4)) give a canonical isomorphism $T_l\hat{\bar{\mathscr{A}}}\xrightarrow{\sim} R^1\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}}(1)$. Consider then the following diagram of continuous $\boldsymbol{Z}_l[G]$-modules:

$$
\begin{array}{ccc}
H^1(\bar{B}, T_l\bar{\mathscr{A}}) & \longrightarrow & \operatorname{Hom}_{\boldsymbol{Z}_l}(H^0R^2\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}}, H^1R^1\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}}) \\
 & & \| \\
\Big\downarrow \lambda & & \operatorname{Hom}_{\boldsymbol{Z}_l}(H^0R^2\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}}(1), H^1R^1\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}}(1)) \\
 & & \Big\downarrow \operatorname{Hom}(\lambda, 1) \\
H^1(\bar{B}, T_l\hat{\bar{\mathscr{A}}}) & \xrightarrow{\sim} & H^1R^1\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}}(1)\ ,
\end{array}
\tag{4.11}
$$

where the isomorphism on the bottom row comes from the identification just mentioned, and the right hand side vertical arrow is evaluation at λ. The commutativity of this diagram is standard. Then, since the degree of λ is prime to l, λ induces an isomorphism $T_l\bar{\mathscr{A}}\xrightarrow{\sim} T_l\hat{\bar{\mathscr{A}}}$, and so the left hand side vertical arrow is an isomorphism, too, thereby ending this proof.

(4.12) REMARKS (cf. (4.8)). (i) Notation being as in (4.1), let now l be any odd prime number different from p_0 and prime to the degree of λ. For all torsors $\mathscr{X}$ for $\mathscr{A}$ one has: The Leray spectral sequence $H^pR^q\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}}\Rightarrow H^{p+q}\boldsymbol{Z}_{l,\bar{\mathscr{X}}}$ degenerates if and only if the torsor $\mathscr{X}$ is infinitely l-divisible. Indeed, by Proposition (2.7) (cf. (4.7) (ii)), the morphism $d_2^{pq}(\mathscr{X})\colon H^pR^q\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}}\to H^{p+2}R^{q-1}\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}}$ is given by $(-1)^{p+1}\imath(\delta[\mathscr{X}])$. And one has $\delta[\mathscr{X}]=0$ if and only if $\mathscr{X}$ is infinitely l-divisible. The if-case follows immediately from this (cf. (4.8)). For the converse, a diagram similar to (4.11) shows that the map $H^2(\bar{B}, T_l\bar{\mathscr{A}})\to\operatorname{Hom}_{\boldsymbol{Z}_l}(H^0R^2\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}}, H^2R^1\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}})$ given by $\xi\mapsto\imath(\xi)$ is injective. Hence, if $d_2^{02}(\mathscr{X})=0$, then $\mathscr{X}$ is infinitely l-divisible. (We note that, as remarked in (4.2) and in (4.8) (iii), the present equivalence is obvious in case B is not projective over k).

(ii) Keeping the notation of (i), a somewhat different version of the map $d_2^{02}(\mathscr{X})\colon H^0R^2\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}}\to H^2R^1\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}}$ is obtained by twisting with $\boldsymbol{Z}_l(1)$ (this actually corresponding to the Leray spectral sequence for the sheaf $\boldsymbol{Z}_{l,\bar{\mathscr{X}}}(1)$): As in (4.11), identify first $H^2R^1\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}}(1)=H^2(\bar{B}, T_l\hat{\bar{\mathscr{A}}})$. Then, since $d_2^{02}(\bar{\mathscr{X}})(1)$ varies additively with $\bar{\mathscr{X}}$, the image of this morphism lies in $\mathrm{Tors}(H^2(\bar{B}, T_l\hat{\bar{\mathscr{A}}}))$. This group is identified with the first term of the exact sequence (deduced from the sequences (2.3) for $\hat{\bar{\mathscr{A}}}$ and $m=l^r$, $r\geq 1$)

$$0\to H^1(\bar{B}, \hat{\bar{\mathscr{A}}})(l)/H^1(\bar{B}, \hat{\bar{\mathscr{A}}})(l)_{\mathrm{div}}\to H^2(\bar{B}, T_l\hat{\bar{\mathscr{A}}})\to T_l(H^2(\bar{B}, \hat{\bar{\mathscr{A}}}))\to 0\,.$$

Then, by using (2.7), one finds that the morphism

$$d_2^{02}(\bar{\mathscr{X}})(1)\colon H^0R^2\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}}(1)\to H^1(\bar{B}, \hat{\bar{\mathscr{A}}})(l)/H^1(\bar{B}, \hat{\bar{\mathscr{A}}})(l)_{\mathrm{div}}$$

is as follows: Consider the inclusion $H^0R^2\bar{\alpha}_*\boldsymbol{Z}_{l,\bar{\mathscr{A}}}(1)\subset\mathrm{Hom}_{\bar{B}}(\bar{\mathscr{A}}(l), \hat{\bar{\mathscr{A}}}(l))$, which underlies the inclusion $H^0(\bar{B}, \mathscr{N}\mathscr{S}_{\bar{\mathscr{A}}/\bar{B}})\subset\mathrm{Hom}_{\bar{B}}(\bar{\mathscr{A}}, \hat{\bar{\mathscr{A}}})$. Compose this with the obvious maps $\mathrm{Hom}_{\bar{B}}(\bar{\mathscr{A}}(l), \hat{\bar{\mathscr{A}}}(l))\to\mathrm{Hom}(H^1(\bar{B}, \bar{\mathscr{A}}(l)), H^1(\bar{B}, \hat{\bar{\mathscr{A}}}(l)))\to\mathrm{Hom}(H^1(\bar{B}, \bar{\mathscr{A}}(l))/H^1(\bar{B}, \bar{\mathscr{A}}(l))_{\mathrm{div}}, H^1(\bar{B}, \hat{\bar{\mathscr{A}}}(l))/H^1(\bar{B}, \hat{\bar{\mathscr{A}}}(l))_{\mathrm{div}})=\mathrm{Hom}(H^1(\bar{B}, \bar{\mathscr{A}})(l)/H^1(\bar{B}, \bar{\mathscr{A}})(l)_{\mathrm{div}}, H^1(\bar{B}, \hat{\bar{\mathscr{A}}})(l)/H^1(\bar{B}, \hat{\bar{\mathscr{A}}})(l)_{\mathrm{div}})$, and then evaluate at the image of $[\bar{\mathscr{X}}]$ in $H^1(\bar{B}, \bar{\mathscr{A}})(l)/H^1(\bar{B}, \bar{\mathscr{A}})(l)_{\mathrm{div}}$. The opposite of this map is $d_2^{02}(\bar{\mathscr{X}})(1)$.

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