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ON THE EXACT DISTRIBUTIONS OF VOTAW'S CRITERIA FOR TESTING COMPOUND SYMMETRY OF A COVARIANCE MATRIX

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1. Introduction. Let x_{ij}, for $i = 1, 2, 3, \cdots, n$, be n independent observations on $p + q$ stochastic variables X_j, where $j = 1, 2, 3, \cdots, (p + q)$, which are distributed normally. Also, let

$$(1.1) \qquad \bar{x}_j = n^{-1} \textstyle\sum_{i=1}^{n} x_{ij}, \qquad s_{jj'} = n^{-1} \sum_{i=1}^{n} (x_{ij} - \bar{x}_j)(x_{ij'} - \bar{x}_{j'}).$$

Also, let $S = ((s_{ij'}))$ be the sum of products (SP) matrix for X's and

$$(1.2) \qquad \begin{aligned} S_{aa} &= p^{-1} \textstyle\sum_{j=1}^{p} S_{jj}, & S_{aa'} &= 2(p^2 - p)^{-1} \textstyle\sum_{j>j'=1}^{p} S_{jj'}, \\ S_{bb} &= q^{-1} \textstyle\sum_{j=p+1}^{p+q} S_{jj}, & S_{bb'} &= 2(q^2 - q)^{-1} \textstyle\sum_{j>j'=p+1}^{p+q} S_{jj'}, \\ S_{ab} &= (pq)^{-1} \textstyle\sum_{j=1}^{p} \sum_{j'=p+1}^{p+q} S_{jj'}. \end{aligned}$$

To test the hypothesis H that the covariance matrix is of the bipolar form

$$(1.3) \qquad \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2' & \Sigma_3 \end{bmatrix}$$

where Σ_1 is a $p \times p$ matrix with diagonal elements equal to σ_{aa} and other elements to $\sigma_{aa'}$, Σ_2 is a $p \times q$ matrix with all elements equal to σ_{ab} and Σ_3 is a $q \times q$ matrix with diagonal elements σ_{bb} and other elements $\sigma_{bb'}$, the likelihood ratio statistic can be defined by

$$(1.4) \qquad L = |S|/\{[(S_{aa} + (p-1)S_{aa'})(S_{bb} + (q-1)S_{bb'}) - pqS_{ab}^2\} \cdot (S_{aa} - S_{aa'})^{p-1}(S_{bb} - S_{bb'})^{q-1}].$$

Votaw (1948) used Wilks' (1934) moment generating operator and derived the expected value $E(L^t \mid H)$ when the hypothesis H was true. By orthogonal transformation and by integrating over the range of different variates Roy (1951) proved that the expected value $E(L^t \mid H)$ i.e. the tth moment can be expressed in the form

$$(1.5) \qquad \begin{aligned} E(L^t \mid H) &= \{(p-1)^{p-1}(q-1)^{q-1}\}^t \\ &\cdot \left[\frac{\Gamma\{\frac{1}{2}(q-1)(n-1)\}\Gamma\{\frac{1}{2}(p-1)(n-1)\}}{\Gamma\{(p-1)(t+\frac{1}{2}(n-1))\}\Gamma\{(q-1)(t+\frac{1}{2}(n-1))\}}\right] \\ &\cdot \textstyle\prod_{r=0}^{p+q-3} \Gamma\{t + \frac{1}{2}(n-3) - \frac{1}{2}r\}[\Gamma\{\frac{1}{2}(n-3) - \frac{1}{2}r\}]^{-1} \end{aligned}$$

and obtained the distribution of L in the form of an infinite series. Later on,

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Roy (1951) further modified his series to get a better approximation of the distribution by taking a few terms.

A number of such criteria for testing different hypotheses were defined earlier by Wilks (1932). Box (1949) gave a general distribution theory for these criteria by which their distributions could be obtained in series. Anderson (1958) has given a nice exposition of these criteria and their use. Recently, Consul (1967) applied the method of asymptotic expansions to obtain another series approximation for the distribution of L.

Consul (1966), (1967) has also given another method based upon operational calculus to obtain the exact distributions of all such criteria. In this paper we use that method on the expression of moments (1.5) to obtain the exact and cumulative distribution functions of the criteria L for a number of values of p and q given by

(i) $p = q = 2$ (ii) $p = q = 3$ (iii) $p = 3, \quad q = 2$

(iv) $p = 5, \quad q = 2$ and (v) $p = 5, \quad q = 3$.

2. Notations and some known results. We use the following symbols in the sense as indicated below:

(i) $(a)_0 = 1, \qquad (a)_1 = a, \qquad$ and

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1).$$

Also

$$\{(a)_n\}^{-1} = [a(a+1)(a+2)\cdots(a+n-1)]^{-1}.$$

(ii) Erdelyi and others ([9], page 102, (22)) have given the formula

$$(d^n/dz^n)[z^{c-1}F(a,b;c;z) = (c-n)_n z^{c-1-n}F(a,b;c-n;z)]. \tag{2.1}$$

(iii) Consul ([3], page 533–558, (5.1) and (4.1) has obtained the inverse Mellin transform

$$\begin{aligned}(2\pi i)^{-1}\textstyle\int_{c-i\infty}^{c+i\infty} & x^{-s}\Gamma(ps+a)\Gamma(qs+b)/\{\Gamma(ps+a+m)\Gamma(qs+b+n)\}\,ds\\ &= p^{-1}x^{a/p}(1-x^{1/q})^n\{\Gamma(m)\Gamma(n+1)\}^{-1}\textstyle\sum_{r=0}^{m-1}\binom{m-1}{r}(-x^{1/p})^r\\ &\quad\cdot F(n, 1-b+q(a+r)/p; n+1; 1-x^{1/q}), \qquad \text{for } p \neq q.\\ &= x^{a/p}(1-x^{1/p})^{m+n-1}\{2\Gamma(m+n)\}^{-1}\\ &\quad\cdot {}_2F_1(n, a-b+m; m+n; 1-x^{1/p}), \qquad \text{for } p = q.\end{aligned} \tag{2.2}$$

(iv) Consul ([6], (4.1)) has proved another inverse Mellin transform

$$\begin{aligned}(2\pi i)^{-1}\textstyle\int_{k-i\infty}^{k+i\infty} & x^{-s}\Gamma(ps+a)\Gamma(ps+b)\Gamma(qs+c)\\ &\cdot\{\Gamma(ps+a+m)\Gamma(ps+b+n)\Gamma(qs+c+r)\}^{-1}\,ds\\ &= x^{(a-1)/p}(1-x^{1/p})^{m+n}[q\Gamma(r)]^{-1}\textstyle\sum_{j=0}^{\infty}(n)_j(a+m-b)_j\\ &\cdot\{\Gamma(m+n+j+1)\}^{-1}(1-x^{1/p})^j(j!)^{-1}\textstyle\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\\ &\cdot F(1, 1-a+p(c+i)/q; m+n+j+1; -x^{-1/p}(1-x^{1/p})).\end{aligned} \tag{2.3}$$

When $p = q = 1$, the above can be simplified and reduced into

$$\begin{aligned}&(2\pi i)^{-1} \int_{k-i\infty}^{k+i\infty} x^{-s}\Gamma(s + a)\Gamma(s + b)\Gamma(s + c)\\ &\quad \{\Gamma(s + a + m)\Gamma(s + b + n)\Gamma(s + c + r)\}^{-1} ds\\ (2.4)\quad &= x^{a-r}(1 - x)^{m+n+r-1}\{\Gamma(m + n + r)\}^{-1} \sum_{j=0}^{\infty} (n)_j(a + m - b)_j\\ &\quad \cdot[j!\,(m + n + r)_j]^{-1} (1 - x)^j\\ &\quad \cdot F(c - a + r, r; m + n + r + j; -x^{-1}(1 - x)).\end{aligned}$$

3. Exact distributions of the criterion L. As the moments determine a distribution uniquely for likelihood criteria, we apply Mellin's inversion theorem on the tth moment, given by the expression (1.5) of the criterion L for testing the hypothesis H so that the exact probability density function $f(L)$ is given by

$$\begin{aligned}f(L) &= T(n)\cdot (2\pi i)^{-1} \int_{-i\infty}^{+i\infty} L^{-t-1}\{(p - 1)^{p-1}(q - 1)^{q-1}\}^t\\ &\quad \cdot\prod_{r=0}^{p+q-3} \Gamma\{t + \tfrac{1}{2}(n - 3) - \tfrac{1}{2}r\}\\ &\quad \cdot\{\Gamma\{(p - 1)(t + \tfrac{1}{2}(n - 1))\}\Gamma\{(q - 1)(t + \tfrac{1}{2}(n - 1))\}\}^{-1} dt\end{aligned}$$

where

$$(3.0.1)\quad T(n) = \Gamma\{\tfrac{1}{2}(p - 1)(n - 1)\}\Gamma\{\tfrac{1}{2}(q - 1)(n - 1)\} \cdot[\prod_{r=0}^{p+q-3} \Gamma\{\tfrac{1}{2}(n - 3) - \tfrac{1}{2}r\}]^{-1}.$$

By setting the transformation $t + \frac{1}{2}(n - 1) = s$ and on further simplification the exact density function becomes

$$\begin{aligned}(3.0.2)\quad f(L) &= K(n)\cdot L^{\frac{1}{2}(n-3)}(2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} L^{-s}\{(p - 1)^{p-1}(q - 1)^{q-1}\}^s\\ &\quad \cdot\prod_{r=0}^{p+q-3} \Gamma(s - 1 - \tfrac{1}{2}r)\{\Gamma\{(p - 1)s\}\Gamma\{(q - 1)s\}\}^{-1} ds\end{aligned}$$

where

$$(3.0.3)\qquad K(n) = T(n)/\{(p - 1)^{p-1}(q - 1)^{q-1}\}^{\frac{1}{2}(n-1)}.$$

As the expression for the distribution $f(L)$ splits up into a factor $K(n)$, depending upon n, and an integral, which is independent of n, the above representation of the exact density function as an inverse Mellin transform is quite interesting. We use it to obtain the exact probability distributions $f(L)$ for some specific values of p and q.

3.1 *Exact distributions of L for $p = q = 2$.* On putting the values of p and q in (3.0.2) and (3.0.3) and on simplification it gives

$$f(L) = K_1(n)\cdot L^{\frac{1}{2}(n-3)}(2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} L^{-s}\Gamma(s - 1)\Gamma(s - \tfrac{3}{2})\{\Gamma(s)\}^{-2} ds$$

where

$$K_1(n) = [\Gamma\{\tfrac{1}{2}(n - 1)\}]^2/[\Gamma\{\tfrac{1}{2}(n - 3)\}\Gamma\{\tfrac{1}{2}(n - 4)\}].$$

Evaluating the integral in $f(L)$ by the use of formula (2.2) and on further simplification we obtain the exact distribution $f(L)$ as

$$f(L) = \{K_1(n)/\Gamma(\tfrac{5}{2})\}\cdot L^{\frac{1}{2}n-3}(1 - L)^{\frac{3}{2}}F(1, 1; \tfrac{5}{2}; 1 - L) \quad \text{for} \quad 0 \leqq L \leqq 1. \tag{3.1.1}$$

To obtain the probability that $0 \leqq L \leqq x(\leqq 1)$, for different values of n, we can integrate $f(L)$ with respect to L between the limits 0 to x and the integral will represent the cumulative distribution function Pr $(L \leqq x)$. Since the integral is not directly available, we first change $F(1, 1; \frac{5}{2}; 1 - L)$ in (3.1.1) to $L^{\frac{1}{2}}F(\frac{3}{2}, \frac{3}{2}; \frac{5}{2}; 1 - L)$, then integrate (3.1.1) by parts, treating $L^{\frac{1}{2}(n-5)}$ as second function and the rest expression as first function, and use the formula (2.1). Thus, on rearrangement and simplification, the cumulative distribution function becomes

$$\Pr(L \leqq x) = I_x(\tfrac{1}{2}n - 2, \tfrac{3}{2}) + 2K_1(n)\{(n-3)\Gamma(\tfrac{5}{2})\}^{-1} \cdot x^{\frac{1}{2}(n-3)}(1 - x)^{\frac{3}{2}}F(\tfrac{3}{2}, \tfrac{3}{2}; \tfrac{5}{2}; 1 - x) \tag{3.1.2}$$

where the incomplete beta function $I_x(a, b) = B^{-1}(a, b)\int_0^x t^{a-1}(1 - t)^{b-1}\,dt$ has been tabulated by Pearson (1932).

3.2 *Exact distributions of L for $p = q = 3$.* By simplifying (3.0.2) and (3.0.3) for the particular values of p and q and by using Legendre's duplication formula, the expression $f(L)$ is found to be

$$f(L) = K_2(n)\cdot L^{\frac{1}{2}(n-3)}(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} L^{-s}\Gamma(2s - 3)\Gamma(2s - 5)\{\Gamma(2s)\Gamma(2s)\}^{-1}\,ds$$

where

$$K_2(n) = \{\Gamma(n-1)\}^2/\{(n-4)\Gamma(n-6)\}. \tag{3.2.1}$$

By using the formula (2.2) to evaluate the above integral and by simplifying it, we obtain the exact probability distribution $f(L)$ as

$$f(L) = \{K_2(n)/7!\}\tfrac{1}{2}L^{\frac{1}{2}n-3}(1 - L^{\frac{1}{2}})^7F(5, 5; 8; 1 - L^{\frac{1}{2}}) \quad \text{for} \quad 0 \leqq L \leqq 1. \tag{3.2.2}$$

By considering $L^{\frac{1}{2}n-3}$ as second function and the remaining expression as first function to integrate (3.2.2) from 0 to x by parts, with the help of result (2.1), and by repeating this process three times the cumulative distribution function becomes

$$\begin{aligned}\Pr(L \leqq x) = {} & I_{x^{\frac{1}{2}}}(n - 6, 5) + \{K_2(n)/7!\}\cdot x^{\frac{1}{2}(n-4)}[(n-4)^{-1} \\ & \cdot(1 - x^{\frac{1}{2}})^7F(5, 5; 8; 1 - x^{\frac{1}{2}}) \\ & + 7\{(n-4)_2\}^{-1}x^{\frac{1}{2}}(1 - x^{\frac{1}{2}})^6F(5, 5; 7; 1 - x^{\frac{1}{2}}) \\ & + 7.6\{(n-4)_3\}^{-1}x(1 - x^{\frac{1}{2}})^5\cdot F(5, 5; 6; 1 - x^{\frac{1}{2}})]\end{aligned} \tag{3.2.3}$$

where $I_{x^{\frac{1}{2}}}(n - 6, 5)$ is the incomplete beta function as tabulated by Pearson.

3.3 *Exact distribution of L for $p = 3$, $q = 2$.* Under the given values of p and q, the expressions (3.0.2) and (3.0.3) can be simplified by the use of Legendre's duplication formula. Thus the exact density function $f(L)$ is obtained as

$$f(L) = K_3(n)\cdot L^{\frac{1}{2}(n-3)}\cdot (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} L^{-s}\Gamma(2s - 3)\Gamma(s - 2)\{\Gamma(2s)\Gamma(s)\}^{-1}\, ds$$

where

$$(3.3.1) \qquad K_3(n) = [\Gamma(n-1)\Gamma\{\tfrac{1}{2}(n-1)\}/\Gamma(n-4)\Gamma\{\tfrac{1}{2}(n-5)\}].$$

On evaluating the integral in $f(L)$ with the help of formula (2.2) and on further simplification, the exact distribution $f(L)$ is given by

$$(3.3.2) \quad f(L) = K_3(n)\cdot 2^{-3}L^{\frac{1}{2}(n-6)}(1 - L)^2 \cdot \textstyle\sum_{r=0}^{2} \binom{2}{r}(-L^{\frac{1}{2}})^r F(2, \tfrac{3}{2} + \tfrac{1}{2}r; 3; 1 - L) \qquad \text{for} \quad 0 \leqq L \leqq 1.$$

Now, by integrating $f(L)$ in (3.3.2) by parts, taking $(1 - L)^2F(2, \frac{3}{2} + \frac{1}{2}r; 3; 1 - L)$ as first function and $L^{\frac{1}{2}(n+r-6)}$ as second function and using the formula (2.1), between the limits 0 to x we obtain the cumulative distribution function $\Pr(L \leqq x)$ as

$$(3.3.3) \quad \Pr(L \leqq x) = I_x(\tfrac{1}{2}(n-5), 2) + 2^{-2}K_3(n) \cdot x^{\frac{1}{2}(n-4)}(1 - x)^2 \textstyle\sum_{r=0}^{2} \binom{2}{r}(n + r - 4)^{-1}(-x^{\frac{1}{2}})^r F(2, \tfrac{3}{2} + \tfrac{1}{2}r; 3; 1 - x)$$

where $I_x(\frac{1}{2}(n - 5), 2)$ is the incomplete beta function tabulated by Pearson.

3.4 *Exact distribution of L for* $p = 5$, $q = 2$. By substituting the values of p and q in the expressions (3.0.2) and (3.0.3), and by using Legendre's duplication formula, the exact density function $f(L)$ can be put, on simplification, in the form

$$f(L) = K_4(n)\cdot L^{\frac{1}{2}(n-3)}(2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} L^{-s}\Gamma(s-1)\Gamma(2s-4)\Gamma(2s-6) \cdot\{\Gamma(s)\Gamma(2s)\Gamma(2s + \tfrac{1}{2})\}^{-1}\, ds$$

where

$$(3.4.1) \quad K_4(n) = 2^{-2n+2}\pi^{\frac{1}{2}}(n-3)\Gamma(2n-2)/\{\Gamma(n-5)\Gamma(n-7)\}.$$

Now, by using Consul's inverse Mellin transform (2.3) to evaluate the integral in $f(L)$ and by simplification, the exact probability distribution function of L becomes

$$(3.4.2) \quad f(L) = K_4(n)\cdot L^{\frac{1}{2}(n-5)} \textstyle\sum_{j=0}^{\infty} (4)_j(4\tfrac{1}{2})_j[j!\,\Gamma(11\tfrac{1}{2} + j)]^{-1} \cdot (1 - L^{\frac{1}{2}})^{10\frac{1}{2}+j}F(5, 10\tfrac{1}{2} + j; 11\tfrac{1}{2} + j; 1 - L^{\frac{1}{2}}) \qquad \text{for} \quad 0 \leqq L \leqq 1.$$

To obtain the cumulative distribution function $\Pr(L \leqq x)$ we integrate (3.4.2) by parts from 0 to x with the help of formula (2.1). On successive integration, rearrangement and simplification $\Pr(L \leqq x)$ is found to be

$$\begin{aligned}\Pr(L \leqq x) &= I_{x^{\frac{1}{2}}}(n-7, 6\tfrac{1}{2}) + \{2(n-3)^{-1}K_4(n)/\Gamma(10\tfrac{1}{4})\} \\ &\quad \cdot [\textstyle\sum_{r=1}^{4} (11\tfrac{1}{2} - r)_{r-1}\{(n-5)_r\}^{-1} \\ &\quad \cdot x^{\frac{1}{2}(n-6+r)}(1 - x^{\frac{1}{2}})^{10\frac{1}{2}-r}F(6, 6\tfrac{1}{2}; 11\tfrac{1}{2} - r; 1 - x^{\frac{1}{2}}) \\ &\quad + \textstyle\sum_{j=0}^{\infty} (4)_j(4\tfrac{1}{2})_j[j!\,\Gamma(11\tfrac{1}{2} + j)]^{-1} \\ &\quad \cdot x^{\frac{1}{2}(n-3)}(1 - x^{\frac{1}{2}})^{10\frac{1}{2}+j}F(5, 10\tfrac{1}{2} + j; 11\tfrac{1}{2} + j; 1 - x^{\frac{1}{2}})].\end{aligned} \tag{3.4.3}$$

3.5 *Exact distribution of L for* $p = 5, q = 3$. The relations (3.0.2) and (3.0.3) for the particular values of p and q, can be simplified by the use of Legendre's duplication formula and thus the probability density function $f(L)$ is given by

$$f(L) = K_5(n)\cdot L^{\frac{1}{2}(n-3)}(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} L^{-s}\Gamma(2s-3)\Gamma(2s-5)\Gamma(2s-7) \cdot\{\Gamma(2s)\Gamma(2s)\Gamma(2s+\tfrac{1}{2})\}^{-1}\,ds$$

where

$$K_5(n) = \{\Gamma(n-1)\}^2\Gamma(n-\tfrac{1}{2})/\{\Gamma(n-4)\Gamma(n-6)\Gamma(n-8)\}. \tag{3.5.1}$$

By evaluating the integral in the above expression with the help of the particular case of Consul's inverse Mellin transform (2.4), the exact distribution function of L becomes

$$\begin{aligned} f(L) &= K_5(n)\cdot\tfrac{1}{2}L^{\frac{1}{2}(n-15)}\textstyle\sum_{j=0}^{\infty}(3)_j(3\tfrac{1}{2})_j[j!\,\Gamma(15\tfrac{1}{2}+j)]^{-1} \\ &\quad \cdot (1 - L^{\frac{1}{2}})^{14\frac{1}{2}+j}F(7, 5; 15\tfrac{1}{2}+j; -L^{-\frac{1}{2}}(1 - L^{\frac{1}{2}})) \qquad \text{for } 0 \leqq L \leqq 1.\end{aligned} \tag{3.5.2}$$

To determine the probability of $0 \leqq L \leqq x(\leqq 1)$, for different values of n, the exact distribution function $f(L)$ in (3.5.2) can be integrated by parts from 0 to x with the help of (2.1). This is achieved by first transforming $F(7, 5; 15\frac{1}{2} + j; -L^{-\frac{1}{2}}(1 - L^{\frac{1}{2}}))$ to $F(7, 10\frac{1}{2} + j; 15\frac{1}{2} + j; 1 - L^{\frac{1}{2}})$, then integrating 5 times and then changing the infinite series into another hypergeometric function and integrating again. Thus the cumulative distribution function $\Pr(L \leqq x)$ is found to be

$$\begin{aligned}\Pr(L \leqq x) &= I_{x^{\frac{1}{2}}}(n-8, 7\tfrac{1}{2}) + K_5(n)\cdot\{\Gamma(10\tfrac{1}{2})(n-6)_5\}^{-1} \\ &\quad \cdot \textstyle\sum_{p=1}^{3}(11\tfrac{1}{2} - p)_{p-1}\{(n-4)_p\}^{-1} \\ &\quad \cdot x^{\frac{1}{2}(n-5+p)}(1 - x^{\frac{1}{2}})^{10\frac{1}{2}-p}F(7, 7\tfrac{1}{2}; 11\tfrac{1}{2} - p; 1 - x^{\frac{1}{2}}) \\ &\quad + K_5(n)\textstyle\sum_{p=1}^{5}\{(n-6)_p\}^{-1}x^{\frac{1}{2}(n-7+p)} \\ &\quad \cdot \textstyle\sum_{j=0}^{\infty}(3)_j(3\tfrac{1}{2})_j[j!\,\Gamma(15\tfrac{1}{2}+j)]^{-1}(16\tfrac{1}{2} + j - p)_{p-1} \\ &\quad \cdot (1 - x^{\frac{1}{2}})^{15\frac{1}{2}+j-p}\cdot F(7, 10\tfrac{1}{2} + j; 16\tfrac{1}{2} + j - p; 1 - x^{\frac{1}{2}}).\end{aligned} \tag{3.5.3}$$

The method of obtaining the distributions is very general and can be similarly extended to obtain the exact distributions of the criteria for other values of p and q. Of course, the form of the distributions become more and more complicated with the increase in the values of p and q.

4. Appendix. By using a number of relations between the hypergeometric functions (Consul, (1966)) the exact distribution functions $f(L)$ can be transformed by a complicated process into elementary functions. Their forms are given below for a few cases:

(i) When $p = q = 2$

$$f(L) = 4K_1(n) \cdot \pi^{-\frac{1}{2}} L^{\frac{1}{2}n-3}[(1 - L)^{\frac{1}{2}} - L^{\frac{1}{2}} \sin^{-1} (1 - L)^{\frac{1}{2}}] \qquad \text{for} \quad 0 \leqq L \leqq 1 \tag{4.1}$$

and

$$\Pr (L \leqq x) = I_x(\tfrac{1}{2}n - 2, \tfrac{3}{2}) + 8\pi^{-\frac{1}{2}}(n - 3)^{-1}K_1(n) \cdot x^{\frac{1}{2}n-2}[(1 - x)^{\frac{1}{2}} - x^{\frac{1}{2}} \sin^{-1} (1 - x)^{\frac{1}{2}}]; \tag{4.2}$$

(ii) When $p = q = 3$

$$f(L) = \{K_2(n)/(4!)^2\} \cdot \tfrac{1}{2}L^{\frac{1}{2}n-4}[1 - 16L^{\frac{1}{2}} - 108L + 80L^{\frac{3}{2}} + 43L^2 - (36L + 48L^{\frac{3}{2}} + 6L^2) \ln L] \qquad \text{for} \quad 0 \leqq L \leqq 1 \tag{4.3}$$

and

$$\begin{aligned} \Pr (L \leqq x) &= (4!)^{-2}K_2(n) \\ &\quad \cdot x^{\frac{1}{2}n-3}[(n - 6)^{-1} - 16x^{\frac{1}{2}}/(n - 5) - 36(3n - 14)x/(n - 4)^2 \\ &\quad + 16(5n - 9)x^{\frac{3}{2}}/(n - 3)^2 + (43n - 74)x^2/(n - 2)^2 \\ &\quad - \{36x/(n - 4) + 48x^{\frac{3}{2}}/(n - 3) + 6x^2/(n - 2)\} \ln x]; \end{aligned} \tag{4.4}$$

(iii) When $p = 3, q = 2$

$$f(L) = \tfrac{1}{2}K_3(n) \cdot L^{\frac{1}{2}(n-7)}[\tfrac{1}{3} - 2L^{\frac{1}{2}} + L + \tfrac{2}{3}L^{\frac{3}{2}} - L \ln L] \qquad \text{for} \quad 0 \leqq L \leqq 1 \tag{4.5}$$

and

$$\Pr (L \leqq x) = K_3(n) \cdot x^{(n-5)/2}[\{3(n - 5)\}^{-1} - 2x^{\frac{1}{2}}/(n - 4) + (n - 1)x/(n - 3)^2 + 2x^{\frac{3}{2}}/3(n - 2) - x \ln x(n - 3)\}]. \tag{4.6}$$

NOTE. The values of $K_1(n)$, $K_2(n)$ and $K_3(n)$, in the above expressions are given by (3.1.1), (3.2.1) and (3.3.1) respectively.

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