# ON THE EXCEPTIONAL SET OF TRANSCENDENTAL ENTIRE FUNCTIONS IN SEVERAL VARIABLES 

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#### Abstract

In this paper, among other things, we prove that any subset of $\overline{\mathbb{Q}}^{m}$ (closed under complex conjugation and which contains the origin) is the exceptional set of uncountable many transcendental entire functions over $\mathbb{C}^{m}$ with rational coefficients. This result solves a several variables version of a question posed by Mahler for transcendental entire functions.


## 1. Introduction

An analytic function $f$ over a domain $\Omega \subseteq \mathbb{C}$ is said to be an algebraic function over $\mathbb{C}(z)$, if there exists a non-zero polynomial $P \in \mathbb{C}[X, Y]$ for which $P(z, f(z))=0$, for all $z \in \Omega$. A function which is not algebraic is called a transcendental function.

The study of the possible arithmetic behavior of a transcendental function started, in 1886, with a letter of Weierstrass to Strauss, who proved the existence of such functions taking $\mathbb{Q}$ into itself. Weierstrass also conjectured the existence of a transcendental entire function $f$ for which $f(\overline{\mathbb{Q}}) \subseteq \overline{\mathbb{Q}}$ (as usual, $\overline{\mathbb{Q}}$ denotes the field of all algebraic numbers). Motivated by this kind of study, he still defined the exceptional set of an analytic function $f: \Omega \rightarrow \mathbb{C}$ as

$$
S_{f}=\{\alpha \in \overline{\mathbb{Q}} \cap \Omega: f(\alpha) \in \overline{\mathbb{Q}}\} .
$$

Thus, Weierstrass' conjecture can be rephrased as: there exists a transcendental entire function $f$ such that $S_{f}=\overline{\mathbb{Q}}$ ? This conjecture was settled in 1895, by Stäckel [4, who proved, in particular, that: for any $\Sigma \subseteq \overline{\mathbb{Q}}$, there exists a transcendental entire function $f$ for which $\Sigma \subseteq S_{f}$.

In his classical book, Mahler [1 introduces the problem of the study of $S_{f}$ for various classes of functions. After discussing a number of examples, Mahler raised several problems about the admissible exceptional sets for analytic functions, one of them is:
Problem C. Let $\rho \in(0, \infty]$ be a real number. Does there exists for any choice of $S \subseteq \overline{\mathbb{Q}} \cap B(0, \rho)$ (closed under complex conjugation and such that $0 \in S$ ) a transcendental analytic function $f \in \mathbb{Q}[[z]]$, with radius of convergence $\rho$, for which $S_{f}=S$ ?

[^0]In 2016, Marques and Ramirez [3] proved that the answer of the previous question is 'yes' provided that $\rho=\infty$ (i.e., for entire functions). Indeed, they proved the a more general result about the arithmetic behavior of some entire functions. Let us state their result as a lemma (since we shall use it in what follows):

Lemma 1 (Cf. Theorem 1.3 of [3). Let $A$ be a countable set and let $\mathbb{K}$ be a dense subset of $\mathbb{C}$. For each $\alpha \in A$, fix a dense subset $E_{\alpha} \subseteq \mathbb{C}$. Then there exist uncountably many transcendental entire function $f \in \mathbb{K}[[z]]$ such that $f(\alpha) \in E_{\alpha}$ for all $\alpha \in A$.

Their result was improved by Marques and Moreira in 2] who provided an affirmative answer to Mahler's Problem C for any $\rho \in(0, \infty]$.

In this paper, we consider the Problem C of Mahler in the context of transcendental entire functions of several variables. Although the previous definitions extend to the context of several variables in a very natural way, we shall include them here for the sake of completeness.

An analytic function $f$ over a domain $\Omega \subseteq \mathbb{C}^{m}$ (we also say that $f$ is entire if $\left.\Omega=\mathbb{C}^{m}\right)$ is said to be algebraic over $\mathbb{C}\left(z_{1}, \ldots, z_{m}\right)$ if it is a solution of a polynomial functional equation

$$
P\left(z_{1}, \ldots, z_{m}, f\left(z_{1}, \ldots, z_{m}\right)\right)=0, \forall\left(z_{1}, \ldots, z_{m}\right) \in \Omega
$$

for some nonzero polynomial $P \in \mathbb{C}\left[z_{1}, \ldots, z_{m}, z_{m+1}\right]$. A function which is not algebraic is called a transcendental function (we remark that an entire function in several variables is algebraic if and only if it is a polynomial function as well as in the case of one variable). Moreover, let $\mathbb{K}$ be a subset of $\mathbb{C}$ and let $f$ be an analytic function on the polydisc $\Delta(0, \rho):=B\left(0, \rho_{1}\right) \times$ $\cdots \times B\left(0, \rho_{m}\right) \subseteq \mathbb{C}^{m}$, for some $\rho=\left(\rho_{1}, \ldots, \rho_{m}\right) \in(0, \infty]^{m}$, we say that $f \in \mathbb{K}\left[\left[z_{1}, \ldots, z_{m}\right]\right]$ if

$$
f\left(z_{1}, \ldots, z_{m}\right)=\sum_{\left(k_{1}, \ldots, k_{m}\right) \in \mathbb{Z}_{\geq 0}^{m}} c_{k_{1}, \ldots, k_{m}} z_{1}^{k_{1}} \cdots z_{m}^{k_{m}}
$$

with $c_{k_{1}, \ldots, k_{m}} \in \mathbb{K}$, for all $\left(k_{1}, \ldots, k_{m}\right) \in \mathbb{Z}_{\geq 0}^{m}$ and for all $\left(z_{1}, \ldots, z_{m}\right) \in$ $\Delta(0, \rho)$.

The exceptional set $S_{f}$ of an analytic function $f: \Omega \subseteq \mathbb{C}^{m} \rightarrow \mathbb{C}$ is defined as

$$
S_{f}:=\left\{\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in \Omega \cap \overline{\mathbb{Q}}^{m}: f\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in \overline{\mathbb{Q}}\right\} .
$$

For example, let $f: \mathbb{C}^{2} \rightarrow \mathbb{C}$ and $g: \mathbb{C}^{2} \rightarrow \mathbb{C}$ be the transcendental entire functions given by

$$
f(w, z)=e^{w+z} \quad \text { and } \quad g(w, z)=e^{w z},
$$

so, by Hermite-Lindemann's theorem, one has that

$$
S_{f}=\{(\alpha,-\alpha): \alpha \in \overline{\mathbb{Q}}\} \quad \text { and } \quad S_{g}=(\overline{\mathbb{Q}} \times\{0\}) \cup(\{0\} \times \overline{\mathbb{Q}}) .
$$

In general, if $P_{1}(X, Y), \ldots, P_{n}(X, Y) \in \overline{\mathbb{Q}}[X, Y]$, then the function

$$
f(w, z)=\exp \left(\prod_{k=1}^{n} P_{k}(w, z)\right)
$$

has the exceptional set given by

$$
S_{f}=\bigcup_{k=1}^{n}\left\{(\alpha, \beta) \in \overline{\mathbb{Q}}^{2}: P_{k}(\alpha, \beta)=0\right\} .
$$

We refer the reader to [1, 5] (and references therein) for more about this subject.

In the main result of this paper, we shall prove that every subset $S$ of $\overline{\mathbb{Q}}^{m}$ (under some mild conditions) is the exceptional set of uncountably many transcendental entire functions on several variables with rational coefficients. More precisely:
Theorem 1. Let $m$ be a positive integer. Then, every subset $S$ of $\overline{\mathbb{Q}}^{m}$, closed under complex conjugation and such that $(0, \ldots, 0) \in S$, is the exceptional set of uncountably many transcendental entire functions $f \in \mathbb{Q}\left[\left[z_{1}, \ldots, z_{m}\right]\right]$.

In order to prove this theorem, we shall provide a more general result about the possible arithmetic behavior of a transcendental entire function of several variables.

Theorem 2. Let $X$ be a countable subset of $\mathbb{C}^{m}$ and let $\mathbb{K}$ be a dense subset of $\mathbb{C}$. For each $u \in X$, fix a dense subset $E_{u} \subseteq \mathbb{C}$ and suppose that if $(0, \ldots, 0) \in X$, then $E_{(0, \ldots, 0)} \cap \mathbb{K} \neq \emptyset$. Then there exist uncountably many transcendental entire functions $f \in \mathbb{K}\left[\left[z_{1}, \ldots, z_{m}\right]\right]$ such that $f(u) \in E_{u}$, for all $u \in X$.

The previous result is a several variables extension of a result due to Marques and Ramirez [3, Theorem 1.3] (case $m=1$ ).

## 2. Proofs

2.1. Proof that Theorem 2 implies Theorem 1. In the statement of Theorem 2, choose $X=\overline{\mathbb{Q}}^{m}$ and $\mathbb{K}=\mathbb{Q}^{*}+i \mathbb{Q}$. Write $S=\left\{u_{1}, u_{2}, \ldots\right\}$ and $\overline{\mathbb{Q}}^{m} / S=\left\{v_{1}, v_{2}, \ldots\right\}$ (one of them may be finite). So, we define

$$
E_{u}:=\left\{\begin{array}{lll}
\overline{\mathbb{Q}} & \text { if } & u \in S,  \tag{2.1}\\
\mathbb{K} \cdot \pi^{n} & \text { if } & u=v_{n} .
\end{array}\right.
$$

By Theorem 2, there exist uncountably many transcendental entire functions

$$
f\left(z_{1}, \ldots, z_{m}\right)=\sum_{k_{1} \geq 0, \ldots, k_{m} \geq 0} c_{k_{1}, \ldots, k_{m}} z_{1}^{k_{1}} \cdots z_{m}^{k_{m}} \in \mathbb{K}\left[\left[z_{1}, \ldots, z_{m}\right]\right]
$$

such that $f(u) \in E_{u}$ for all $u \in \overline{\mathbb{Q}}^{m}$.

Let us define $\psi\left(z_{1}, \ldots, z_{m}\right)$ as

$$
\psi\left(z_{1}, \ldots, z_{m}\right):=\frac{f\left(z_{1}, \ldots, z_{m}\right)+\overline{f\left(\overline{z_{1}}, \ldots, \overline{z_{m}}\right)}}{2}
$$

Thus, by properties of the conjugation of power series, we infer that

$$
\psi\left(z_{1}, \ldots, z_{m}\right)=\sum_{\left(k_{1}, \ldots, k_{m}\right) \in \mathbb{Z}_{\geq 0}^{m}} \Re\left(c_{k_{1}, \ldots, k_{m}}\right) z_{1}^{k_{1}} \cdots z_{m}^{k_{m}} \in \mathbb{Q}\left[\left[z_{1}, \ldots, z_{m}\right]\right]
$$

is a transcendental entire function (since $\Re\left(c_{k_{1}, \ldots, k_{m}}\right) \neq 0$, for all vector $\left.\left(k_{1}, \ldots, k_{m}\right) \in \mathbb{Z}_{\geq 0}^{m}\right)$ with rational coefficients (here, as usual, $\Re(z)$ denotes the real part of the complex number $z$ ).

Therefore, it suffices to prove that $S_{\psi}=S$. In fact, since $S$ is closed under complex conjugation, one has that if $u \in S$, then $\bar{u} \in S$ and thus $f(u)$ and $\overline{f(\bar{u})}$ are algebraic numbers and so is $\psi(u)$ (observe also that $f(0, \ldots, 0)=$ $\left.c_{0, \ldots, 0} \in \overline{\mathbb{Q}}\right)$. In the case in which $u=v_{n}$, for some $n$, we can distinguish two cases: when $v_{n} \in \mathbb{R}^{m}$, then $\psi(u)=\Re\left(f\left(v_{n}\right)\right)$ is transcendental, since $f\left(v_{n}\right) \in \mathbb{K} \cdot \pi^{n}$. For the case $v_{n} \notin \mathbb{R}^{m}$, we have that $\overline{v_{n}}=v_{l}$ for some $l \neq n$. Thus, there exist non-zero algebraic numbers $\gamma_{1}, \gamma_{2}$ such that

$$
\psi\left(v_{n}\right)=\frac{\gamma_{1} \pi^{n}+\gamma_{2} \pi^{l}}{2}
$$

which is transcendental, since $\overline{\mathbb{Q}}$ is algebraically closed and $\pi$ is transcendental. In conclusion, $\psi \in \mathbb{Q}\left[\left[z_{1}, \ldots, z_{m}\right]\right]$ is a transcendental entire function whose exceptional set is $S$.
2.2. Proof of Theorem 2, Let us proceed by induction on $m$. The case $m=1$ being covered by lemma 1 proved by Marques and Ramirez [3]. So, suppose that the theorem holds for all positive integer $k \in[1, m-1]$, i.e., if $\mathbb{K}$ is a dense subset of $\mathbb{C}$ and $X$ is a countable subset of $\mathbb{C}^{k}$ such that for each $u \in X$, we fix a dense subset $E_{u} \subseteq \mathbb{C}$, then there exist uncountably many transcendental entire functions $f \in \mathbb{K}\left[\left[z_{1}, \ldots, z_{k}\right]\right]$ such that $f(u) \in E_{u}$ for all $u \in X$, for any integer $k \in[1, m-1]$.

Now, let $X$ be a countable subset of $\mathbb{C}^{m}$ such that $E_{u}$ is a fixed dense subset of $\mathbb{C}$ for all $u \in X$. Without loss of generality, we can assume that $(0, \ldots, 0) \in X$. In this case, by hypothesis, we have that $\mathbb{K} \cap E_{(0, \ldots, 0)} \neq \emptyset$. In order to apply the induction hypothesis, we consider the partition of $X$ given by

$$
X=\bigcup_{S \in \mathcal{P}_{m}} X_{S}
$$

where $\mathcal{P}_{m}$ denotes the powerset of $[1, m]=\{1, \ldots, m\}$ and $X_{S}$ denotes the set of all $z=\left(z_{1}, \ldots, z_{m}\right)$ in $X \subseteq \mathbb{C}^{m}$ such that $z_{i} \neq 0$ if and only if $i \in S$. In particular, we have that $X_{\emptyset}=\{(0, \ldots, 0)\}$ and $X_{[1, m]}$ is given by $X \cap(\mathbb{C} \backslash\{0\})^{m}$.

Given $S=\left\{i_{1}, \ldots, i_{k}\right\}$ in $\mathcal{Q}_{m}=\mathcal{P}_{m} \backslash\{\emptyset,[1, m]\}$ and $z=\left(z_{1}, \ldots, z_{m}\right)$ in $\mathbb{C}^{m}$, we denote by $z_{S}$ the element $\left(z_{i_{1}}, \ldots, z_{i_{k}}\right) \in \mathbb{C}^{k}$, in order to simplify
the exposition we will always assume that $i_{1}<\cdots<i_{k}$ for all $S \in \mathcal{Q}_{m}$. Our goal is to show that there exist uncountably many ways of to construct a transcendental entire function $f \in \mathbb{K}\left[\left[z_{1}, \ldots, z_{m}\right]\right]$, given by

$$
f\left(z_{1}, \ldots, z_{m}\right)=a_{0}+\left(\sum_{S \in \mathcal{Q}_{m}}\left(\prod_{i \in S} z_{i}\right) f_{S}\left(z_{S}\right)\right)+f^{*}\left(z_{1}, \ldots, z_{m}\right)
$$

where $a_{0} \in E_{(0, \ldots, 0)} \cap \mathbb{K}$ and for each $S=\left\{i_{1}, \ldots, i_{k}\right\} \in \mathcal{Q}_{m}$ we have that $f_{S}: \mathbb{C}^{k} \rightarrow \mathbb{C}$ is a transcendental entire function such that

$$
f_{S}\left(u_{S}\right) \in \frac{1}{\alpha_{i_{1}} \cdots \alpha_{i_{k}}} \cdot\left(E_{u}-\Theta_{S, u}\right),
$$

for all $u=\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in X_{S}$ with

$$
\Theta_{S, u}=a_{0}+\sum_{T \in \mathcal{Q}_{m}, T \neq S}\left(\prod_{i \in T} \alpha_{i}\right) f_{T}\left(u_{T}\right) \in \mathbb{C} .
$$

Note that, by induction hypothesis, $f_{S}$ exists for all $S \in \mathcal{Q}_{m}$ (here we use that if $E_{u}$ is a dense subset of $\mathbb{C}$, then $\left(\alpha_{i_{1}} \cdots \alpha_{i_{k}}\right)^{-1} \cdot\left(E_{u}-\Theta_{S, u}\right)$ is too). Moreover, we want that the function $f^{*}\left(z_{1}, \ldots, z_{m}\right) \in \mathbb{K}\left[\left[z_{1}, \ldots, z_{m}\right]\right]$ satisfies the condition

$$
\begin{equation*}
f^{*}(u) \in\left(E_{u}-a_{0}-\sum_{S \in \mathcal{Q}_{m}}\left(\prod_{i \in S} \alpha_{i}\right) f_{S}\left(u_{S}\right)\right) \tag{2.2}
\end{equation*}
$$

for all $u=\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in X_{[1, m]}$ and $f^{*}\left(z_{1}, \ldots, z_{m}\right)=0$ whenever $z_{i}=0$ for some $1 \leq i \leq m$. Under these conditions, it is easy to see that if $S \in \mathcal{Q}_{m}$ and $u \in X_{S}$, then $f^{*}(u)=0$ and $f(u) \in E_{u}$.

In order to construct the function $f^{*}: \mathbb{C}^{m} \rightarrow \mathbb{C}$ let us consider the enumeration $\left\{u_{1}, u_{2}, \ldots\right\}$ of $X_{[1, m]}$ where we denote $u_{j}=\left(\alpha_{1}^{(j)}, \ldots, \alpha_{m}^{(j)}\right)$. Our function $f^{*} \in \mathbb{K}\left[\left[z_{1}, \ldots, z_{m}\right]\right]$ will be given by

$$
f^{*}\left(z_{1}, \ldots, z_{m}\right)=\sum_{n=m}^{\infty} P_{n}\left(z_{1}, \ldots, z_{m}\right)=\sum_{i_{1} \geq 1, \ldots, i_{m} \geq 1} c_{i_{1}, \ldots, i_{m}} z_{1}^{i_{1}} \cdots z_{m}^{i_{m}}
$$

where $P_{n}$ is a homogeneous polynomial of degree $n$ and the coefficients $c_{i_{1}, \ldots, i_{m}} \in \mathbb{K}$ will be chosen conveniently so that $f^{*}$ will satisfy the desired conditions.

The first condition is

$$
\left|c_{i_{1}, \ldots, i_{m}}\right|<s_{i_{1}+\cdots+i_{m}}:=\frac{1}{\binom{i_{1}+\cdots+i_{m}-1}{m-1}\left(i_{1}+\cdots+i_{m}\right)!},
$$

where $c_{i_{1}, \ldots, i_{n}} \neq 0$ for infinitely many $m$-tuples of integers $i_{1} \geq 1 \ldots i_{m} \geq 1$. This conditions will be used to guarantee that $f^{*}$ is an entire function. Indeed, if we denote by $L(P)$ the length of polynomial $P\left(z_{1}, \ldots, z_{m}\right) \in$ $\mathbb{C}\left[z_{1}, \ldots, z_{m}\right]$ given by the sum of absolute values of its coefficients. Since

$$
\left|P_{n}\left(z_{1}, \ldots, z_{m}\right)\right| \leq L\left(P_{n}\right) \max \left\{1,\left|z_{1}\right|, \ldots,\left|z_{m}\right|\right\}^{n}
$$

we have that for all $n \geq m$ and $\left(z_{1}, \ldots, z_{m}\right)$ belonging to the open ball $B(0, R)$

$$
\left|P_{n}\left(z_{1}, \ldots, z_{m}\right)\right|<\frac{\binom{n-1}{m-1}}{\binom{n-1}{m-1} n!} \max \{1, R\}^{n}=\frac{\max \{1, R\}^{n}}{n!}
$$

where we use that $P_{n}\left(z_{1}, \ldots, z_{m}\right)$ has at most $\binom{n-1}{m-1}$ monomials of degree $n$. Hence, we have that the series $\sum_{n \geq m} P_{n}\left(z_{1}, \ldots, z_{m}\right)$ converges uniformly in any of these balls. Thus, $f^{*}$ is a transcendental entire function such that $f^{*}\left(0, z_{2}, \ldots, z_{m}\right)=f^{*}\left(z_{1}, 0, z_{3}, \ldots, z_{m}\right)=f^{*}\left(z_{1}, z_{2}, \ldots, 0\right)=0$.

In order to obtain the coefficients $c_{i_{1}, \ldots, i_{m}} \in \mathbb{K}$ such that $f^{*}$ satisfies the condition (2.2), we consider for each positive integer $n$ and $1 \leq j \leq n$ a hyperplane

$$
\pi(n, j): \mu_{n, 1}^{(j)} z_{1}+\cdots+\mu_{n, m}^{(j)} z_{m}-\lambda_{n}^{(j)}=0
$$

such that if $u_{j}, u_{n+1}$ and the origin are non-collinear, then $\pi(n, j)$ is a hyperplane containing $u_{j}$ and parallel to the line passing through the origin and the point $u_{n+1}$ and if $u_{j}, u_{n+1}$ and the origin are collinear, then $\pi(n, j)$ is a hyperplane containing $u_{j}$ and perpendicular o the line passing through the origin and the point $u_{n+1}$. Note that in both cases we have that $\lambda_{n}^{(j)} \neq 0$ and $u_{n+1}$ does not belong to any hyperplane $\pi(n, j)$ with $1 \leq j \leq n$.

Now, we define the polynomials $A_{0}\left(z_{1}, \ldots, z_{m}\right):=z_{1} \cdots z_{m}$ and

$$
A_{n}\left(z_{1}, \ldots, z_{m}\right):=\prod_{j=1}^{n}\left(\mu_{n, 1}^{(j)} z_{1}+\cdots+\mu_{n, m}^{(j)} z_{m}-\lambda_{n}^{(j)}\right)
$$

for all $n \geq 1$. By definition of $\pi(n, j)$, we have that $A_{n}\left(u_{j}\right)=0$ for all $1 \leq j \leq n$. Since $u_{n+1}$ and the origin does not belong to $\pi(n, j)$, we also have that $A_{n}(0, \ldots, 0) \neq 0$ and $A_{n}\left(u_{n+1}\right) \neq 0$, for all $n \geq 1$. Thus, we define the function

$$
f_{1,0}^{*}\left(z_{1}, \ldots, z_{m}\right):=\delta_{1,0} A_{0}\left(z_{1}, \ldots, z_{m}\right)=\delta_{1,0} z_{1} \cdots z_{m}
$$

such that $\Theta_{1}+f_{1,0}^{*}\left(u_{1}\right) \in E_{u_{1}}$ and $0<\left|\delta_{1,0}\right|<s_{m} / m$, where

$$
\Theta_{j}:=a_{0}+\sum_{S \in \mathcal{Q}_{m}}\left(\prod_{i \in S} \alpha_{i}^{(j)}\right) f_{S}\left(u_{j, S}\right)
$$

and $u_{j, S}=\left(\alpha_{i_{1}}^{(j)}, \ldots, \alpha_{i_{k}}^{(j)}\right)$ for $S=\left\{i_{1}, \ldots, i_{k}\right\}$, for all integer $j \geq 1$.
Moreover, since $\mathbb{K}$ is a dense subset of $\mathbb{C}$, we can choose $\delta_{1,1}$ such that the coefficient $c_{1,1, \ldots, 1}$ of $z_{1} \cdots z_{m}$ in the function

$$
f_{1,1}^{*}\left(z_{1}, \ldots, z_{m}\right):=f_{1,0}^{*}\left(z_{1}, \ldots, z_{m}\right)+\delta_{1,1} z_{1} \cdots z_{m} A_{1}^{(1)}\left(z_{1}, \ldots, z_{m}\right)
$$

belongs to $\mathbb{K}$ with $\left|c_{1,1, \ldots, 1}\right|<s_{m}$. Therefore, we take

$$
f_{1}^{*}\left(z_{1}, \ldots, z_{m}\right):=f_{1,1}^{*}\left(z_{1}, \ldots, z_{m}\right),
$$

where $P_{1}\left(z_{1}, \ldots, z_{m}\right)=c_{1,1, \ldots, 1} z_{1} \cdots z_{m}$.

Recursively, we can construct a function $f_{n, 0}^{*}\left(z_{1}, \ldots, z_{m}\right)$ given by

$$
f_{n, 0}^{*}\left(z_{1}, \ldots, z_{m}\right):=f_{n-1}^{*}\left(z_{1}, \ldots, z_{m}\right)+\delta_{n, 0} z_{1}^{n} z_{2} \cdots z_{m} A_{n-1}\left(z_{1}, \ldots, z_{m}\right)
$$

where we take $\delta_{n, 0} \neq 0$ in the ball $B\left(0, s_{n+m-1} /(n+m-1)\right)$ such that

$$
\Theta_{n}+f_{n, 0}^{*}\left(u_{n}\right) \in E_{u_{n}},
$$

which is possible since $E_{u_{n}}$ is a dense subset of $\mathbb{C}$ and all coordinates of $u_{n}$ are non-zero.

Moreover, since $\mathbb{K}$ is a dense subset of $\mathbb{C}$, if we consider the ordering of the monomials of degree $n+m-1$ given by the lexicographical order of the exponents, then we can choose a $\delta_{n, l}$ such that the coefficient $c_{j_{1}, \ldots, j_{m}}$ of $l$-th monomial $z_{1}^{j_{1}} \cdots z_{m}^{j_{m}}$ in

$$
f_{n, l}^{*}\left(z_{1}, \ldots, z_{m}\right):=f_{n, l-1}^{*}\left(z_{1}, \ldots, z_{m}\right)+\delta_{n, l} l_{1}^{j_{1}} \cdots z_{m}^{j_{m}} A_{n}\left(z_{1}, \ldots, z_{m}\right)
$$

belongs to $\mathbb{K}$ with $\left|c_{j_{1}, \ldots, j_{m}}\right|<s_{n+m-1}$. Thus, we define

$$
f_{n}^{*}\left(z_{1}, \ldots, z_{m}\right):=f_{n, L}^{*}\left(z_{1}, \ldots, z_{m}\right)
$$

where $L=\binom{n+m-2}{m-1}$ is the number of distinct monomials of degree $n+m-1$. Therefore, we have that $f_{n}^{*}\left(z_{1}, \ldots, z_{m}\right)$ is a polynomial function such that $c_{j_{1}, \ldots, j_{m}} \in \mathbb{K}$, for all $m$-tuple $\left(j_{1}, \ldots, j_{m}\right)$ such that $j_{1}+\cdots+j_{m} \leq n+m-1$.

Finally, this construction implies that the functions $f_{n}^{*}$ converges for a transcendental entire function $f^{*} \in \mathbb{K}\left[\left[z_{1}, \ldots, z_{m}\right]\right]$ as $n \rightarrow \infty$ such that

$$
f^{*}\left(u_{j}\right)=f_{n}^{*}\left(u_{j}\right)=f_{j}^{*}\left(u_{j}\right)
$$

for all $n \geq j \geq 1$. Hence, let $f: \mathbb{C}^{m} \rightarrow \mathbb{C}$ be the entire functions given by

$$
f\left(z_{1}, \ldots, z_{m}\right)=a_{0}+\left(\sum_{S \in \mathcal{Q}_{m}}\left(\prod_{i \in S} z_{i}\right) f_{S}\left(z_{S}\right)\right)+f^{*}\left(z_{1}, \ldots, z_{m}\right),
$$

we have that $f(u) \in E_{u}$ for all $u \in X \in \mathbb{C}^{m}$. Since $f$ is an entire function that is not polynomial, we have that $f$ is transcendental. Note that there are uncountably many ways to choose the constants $\delta_{n, j}$ and this completes the proof.

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