ON THE EXCEPTIONAL SET OF TRANSCENDENTAL ENTIRE FUNCTIONS IN SEVERAL VARIABLES

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ABSTRACT. In this paper, among other things, we prove that any subset of $\overline{\mathbb{Q}}^m$ (closed under complex conjugation and which contains the origin) is the exceptional set of uncountable many transcendental entire functions over \mathbb{C}^m with rational coefficients. This result solves a several variables version of a question posed by Mahler for transcendental entire functions.

1. INTRODUCTION

An analytic function f over a domain $\Omega \subseteq \mathbb{C}$ is said to be an *algebraic* function over $\mathbb{C}(z)$, if there exists a non-zero polynomial $P \in \mathbb{C}[X, Y]$ for which P(z, f(z)) = 0, for all $z \in \Omega$. A function which is not algebraic is called a *transcendental function*.

The study of the possible arithmetic behavior of a transcendental function started, in 1886, with a letter of Weierstrass to Strauss, who proved the existence of such functions taking \mathbb{Q} into itself. Weierstrass also conjectured the existence of a transcendental entire function f for which $f(\overline{\mathbb{Q}}) \subseteq \overline{\mathbb{Q}}$ (as usual, $\overline{\mathbb{Q}}$ denotes the field of all algebraic numbers). Motivated by this kind of study, he still defined the *exceptional set* of an analytic function $f: \Omega \to \mathbb{C}$ as

$$S_f = \{ \alpha \in \overline{\mathbb{Q}} \cap \Omega : f(\alpha) \in \overline{\mathbb{Q}} \}.$$

Thus, Weierstrass' conjecture can be rephrased as: there exists a transcendental entire function f such that $S_f = \overline{\mathbb{Q}}$? This conjecture was settled in 1895, by Stäckel [4], who proved, in particular, that: for any $\Sigma \subseteq \overline{\mathbb{Q}}$, there exists a transcendental entire function f for which $\Sigma \subseteq S_f$.

In his classical book, Mahler [1] introduces the problem of the study of S_f for various classes of functions. After discussing a number of examples, Mahler raised several problems about the admissible exceptional sets for analytic functions, one of them is:

Problem C. Let $\rho \in (0, \infty]$ be a real number. Does there exists for any choice of $S \subseteq \overline{\mathbb{Q}} \cap B(0, \rho)$ (closed under complex conjugation and such that $0 \in S$) a transcendental analytic function $f \in \mathbb{Q}[[z]]$, with radius of convergence ρ , for which $S_f = S$?

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In 2016, Marques and Ramirez [3] proved that the answer of the previous question is 'yes' provided that $\rho = \infty$ (i.e., for entire functions). Indeed, they proved the a more general result about the arithmetic behavior of some entire functions. Let us state their result as a lemma (since we shall use it in what follows):

Lemma 1 (Cf. Theorem 1.3 of [3]). Let A be a countable set and let \mathbb{K} be a dense subset of \mathbb{C} . For each $\alpha \in A$, fix a dense subset $E_{\alpha} \subseteq \mathbb{C}$. Then there exist uncountably many transcendental entire function $f \in \mathbb{K}[[z]]$ such that $f(\alpha) \in E_{\alpha}$ for all $\alpha \in A$.

Their result was improved by Marques and Moreira in [2] who provided an affirmative answer to Mahler's Problem C for any $\rho \in (0, \infty]$.

In this paper, we consider the Problem C of Mahler in the context of transcendental entire functions of several variables. Although the previous definitions extend to the context of several variables in a very natural way, we shall include them here for the sake of completeness.

An analytic function f over a domain $\Omega \subseteq \mathbb{C}^m$ (we also say that f is *entire* if $\Omega = \mathbb{C}^m$) is said to be *algebraic* over $\mathbb{C}(z_1, \ldots, z_m)$ if it is a solution of a polynomial functional equation

$$P(z_1,\ldots,z_m,f(z_1,\ldots,z_m))=0, \ \forall (z_1,\ldots,z_m)\in\Omega,$$

for some nonzero polynomial $P \in \mathbb{C}[z_1, \ldots, z_m, z_{m+1}]$. A function which is not algebraic is called a transcendental function (we remark that an entire function in several variables is algebraic if and only if it is a polynomial function as well as in the case of one variable). Moreover, let \mathbb{K} be a subset of \mathbb{C} and let f be an analytic function on the polydisc $\Delta(0, \rho) := B(0, \rho_1) \times$ $\cdots \times B(0, \rho_m) \subseteq \mathbb{C}^m$, for some $\rho = (\rho_1, \ldots, \rho_m) \in (0, \infty]^m$, we say that $f \in \mathbb{K}[[z_1, \ldots, z_m]]$ if

$$f(z_1, \dots, z_m) = \sum_{(k_1, \dots, k_m) \in \mathbb{Z}_{\geq 0}^m} c_{k_1, \dots, k_m} z_1^{k_1} \cdots z_m^{k_m},$$

with $c_{k_1,\ldots,k_m} \in \mathbb{K}$, for all $(k_1,\ldots,k_m) \in \mathbb{Z}_{\geq 0}^m$ and for all $(z_1,\ldots,z_m) \in \Delta(0,\rho)$.

The exceptional set S_f of an analytic function $f: \Omega \subseteq \mathbb{C}^m \to \mathbb{C}$ is defined as

$$S_f := \{ (\alpha_1, \dots, \alpha_m) \in \Omega \cap \overline{\mathbb{Q}}^m : f(\alpha_1, \dots, \alpha_m) \in \overline{\mathbb{Q}} \}.$$

For example, let $f: \mathbb{C}^2 \to \mathbb{C}$ and $g: \mathbb{C}^2 \to \mathbb{C}$ be the transcendental entire functions given by

$$f(w,z) = e^{w+z}$$
 and $g(w,z) = e^{wz}$,

so, by Hermite-Lindemann's theorem, one has that

$$S_f = \{(\alpha, -\alpha) : \alpha \in \overline{\mathbb{Q}}\} \text{ and } S_g = (\overline{\mathbb{Q}} \times \{0\}) \cup (\{0\} \times \overline{\mathbb{Q}}).$$

In general, if $P_1(X,Y), \ldots, P_n(X,Y) \in \overline{\mathbb{Q}}[X,Y]$, then the function

$$f(w, z) = \exp\left(\prod_{k=1}^{n} P_k(w, z)\right)$$

has the exceptional set given by

$$S_f = \bigcup_{k=1}^n \{ (\alpha, \beta) \in \overline{\mathbb{Q}}^2 : P_k(\alpha, \beta) = 0 \}.$$

We refer the reader to [1, 5] (and references therein) for more about this subject.

In the main result of this paper, we shall prove that every subset S of $\overline{\mathbb{Q}}^m$ (under some mild conditions) is the exceptional set of uncountably many transcendental entire functions on several variables with rational coefficients. More precisely:

Theorem 1. Let *m* be a positive integer. Then, every subset *S* of $\overline{\mathbb{Q}}^m$, closed under complex conjugation and such that $(0, \ldots, 0) \in S$, is the exceptional set of uncountably many transcendental entire functions $f \in \mathbb{Q}[[z_1, \ldots, z_m]]$.

In order to prove this theorem, we shall provide a more general result about the possible arithmetic behavior of a transcendental entire function of several variables.

Theorem 2. Let X be a countable subset of \mathbb{C}^m and let \mathbb{K} be a dense subset of \mathbb{C} . For each $u \in X$, fix a dense subset $E_u \subseteq \mathbb{C}$ and suppose that if $(0, \ldots, 0) \in X$, then $E_{(0,\ldots,0)} \cap \mathbb{K} \neq \emptyset$. Then there exist uncountably many transcendental entire functions $f \in \mathbb{K}[[z_1, \ldots, z_m]]$ such that $f(u) \in E_u$, for all $u \in X$.

The previous result is a several variables extension of a result due to Marques and Ramirez [3, Theorem 1.3] (case m = 1).

2. Proofs

2.1. Proof that Theorem 2 implies Theorem 1. In the statement of Theorem 2, choose $X = \overline{\mathbb{Q}}^m$ and $\mathbb{K} = \mathbb{Q}^* + i\mathbb{Q}$. Write $S = \{u_1, u_2, \ldots\}$ and $\overline{\mathbb{Q}}^m/S = \{v_1, v_2, \ldots\}$ (one of them may be finite). So, we define

(2.1)
$$E_u := \begin{cases} \overline{\mathbb{Q}} & \text{if } u \in S, \\ \mathbb{K} \cdot \pi^n & \text{if } u = v_n \end{cases}$$

By Theorem 2, there exist uncountably many transcendental entire functions

$$f(z_1, \dots, z_m) = \sum_{k_1 \ge 0, \dots, k_m \ge 0} c_{k_1, \dots, k_m} z_1^{k_1} \cdots z_m^{k_m} \in \mathbb{K}[[z_1, \dots, z_m]]$$

such that $f(u) \in E_u$ for all $u \in \overline{\mathbb{Q}}^m$.

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Let us define $\psi(z_1,\ldots,z_m)$ as

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$$\psi(z_1,\ldots,z_m) := \frac{f(z_1,\ldots,z_m) + \overline{f(\overline{z_1},\ldots,\overline{z_m})}}{2}$$

Thus, by properties of the conjugation of power series, we infer that

$$\psi(z_1, \dots, z_m) = \sum_{(k_1, \dots, k_m) \in \mathbb{Z}_{\geq 0}^m} \Re(c_{k_1, \dots, k_m}) z_1^{k_1} \cdots z_m^{k_m} \in \mathbb{Q}[[z_1, \dots, z_m]]$$

is a transcendental entire function (since $\Re(c_{k_1,\ldots,k_m}) \neq 0$, for all vector $(k_1,\ldots,k_m) \in \mathbb{Z}_{\geq 0}^m$) with rational coefficients (here, as usual, $\Re(z)$ denotes the real part of the complex number z).

Therefore, it suffices to prove that $S_{\psi} = S$. In fact, since S is closed under complex conjugation, one has that if $u \in S$, then $\overline{u} \in S$ and thus f(u) and $\overline{f(\overline{u})}$ are algebraic numbers and so is $\psi(u)$ (observe also that $f(0, \ldots, 0) = c_{0,\ldots,0} \in \overline{\mathbb{Q}}$). In the case in which $u = v_n$, for some n, we can distinguish two cases: when $v_n \in \mathbb{R}^m$, then $\psi(u) = \Re(f(v_n))$ is transcendental, since $f(v_n) \in \mathbb{K} \cdot \pi^n$. For the case $v_n \notin \mathbb{R}^m$, we have that $\overline{v_n} = v_l$ for some $l \neq n$. Thus, there exist non-zero algebraic numbers γ_1, γ_2 such that

$$\psi(v_n) = \frac{\gamma_1 \pi^n + \gamma_2 \pi^l}{2},$$

which is transcendental, since $\overline{\mathbb{Q}}$ is algebraically closed and π is transcendental. In conclusion, $\psi \in \mathbb{Q}[[z_1, \ldots, z_m]]$ is a transcendental entire function whose exceptional set is S.

2.2. **Proof of Theorem 2.** Let us proceed by induction on m. The case m = 1 being covered by lemma 1 proved by Marques and Ramirez [3]. So, suppose that the theorem holds for all positive integer $k \in [1, m - 1]$, i.e., if \mathbb{K} is a dense subset of \mathbb{C} and X is a countable subset of \mathbb{C}^k such that for each $u \in X$, we fix a dense subset $E_u \subseteq \mathbb{C}$, then there exist uncountably many transcendental entire functions $f \in \mathbb{K}[[z_1, \ldots, z_k]]$ such that $f(u) \in E_u$ for all $u \in X$, for any integer $k \in [1, m - 1]$.

Now, let X be a countable subset of \mathbb{C}^m such that E_u is a fixed dense subset of \mathbb{C} for all $u \in X$. Without loss of generality, we can assume that $(0, \ldots, 0) \in X$. In this case, by hypothesis, we have that $\mathbb{K} \cap E_{(0,\ldots,0)} \neq \emptyset$. In order to apply the induction hypothesis, we consider the partition of X given by

$$X = \bigcup_{S \in \mathcal{P}_m} X_S,$$

where \mathcal{P}_m denotes the powerset of $[1, m] = \{1, \ldots, m\}$ and X_S denotes the set of all $z = (z_1, \ldots, z_m)$ in $X \subseteq \mathbb{C}^m$ such that $z_i \neq 0$ if and only if $i \in S$. In particular, we have that $X_{\emptyset} = \{(0, \ldots, 0)\}$ and $X_{[1,m]}$ is given by $X \cap (\mathbb{C} \setminus \{0\})^m$.

Given $S = \{i_1, \ldots, i_k\}$ in $\mathcal{Q}_m = \mathcal{P}_m \setminus \{\emptyset, [1, m]\}$ and $z = (z_1, \ldots, z_m)$ in \mathbb{C}^m , we denote by z_S the element $(z_{i_1}, \ldots, z_{i_k}) \in \mathbb{C}^k$, in order to simplify

the exposition we will always assume that $i_1 < \cdots < i_k$ for all $S \in \mathcal{Q}_m$. Our goal is to show that there exist uncountably many ways of to construct a transcendental entire function $f \in \mathbb{K}[[z_1, \ldots, z_m]]$, given by

$$f(z_1,\ldots,z_m) = a_0 + \left(\sum_{S \in \mathcal{Q}_m} \left(\prod_{i \in S} z_i\right) f_S(z_S)\right) + f^*(z_1,\ldots,z_m),$$

where $a_0 \in E_{(0,...,0)} \cap \mathbb{K}$ and for each $S = \{i_1, \ldots, i_k\} \in \mathcal{Q}_m$ we have that $f_S : \mathbb{C}^k \to \mathbb{C}$ is a transcendental entire function such that

$$f_S(u_S) \in \frac{1}{\alpha_{i_1} \cdots \alpha_{i_k}} \cdot (E_u - \Theta_{S,u}),$$

for all $u = (\alpha_1, \ldots, \alpha_m) \in X_S$ with

$$\Theta_{S,u} = a_0 + \sum_{T \in \mathcal{Q}_m, T \neq S} \left(\prod_{i \in T} \alpha_i\right) f_T(u_T) \in \mathbb{C}.$$

Note that, by induction hypothesis, f_S exists for all $S \in \mathcal{Q}_m$ (here we use that if E_u is a dense subset of \mathbb{C} , then $(\alpha_{i_1} \cdots \alpha_{i_k})^{-1} \cdot (E_u - \Theta_{S,u})$ is too). Moreover, we want that the function $f^*(z_1, \ldots, z_m) \in \mathbb{K}[[z_1, \ldots, z_m]]$ satisfies the condition

(2.2)
$$f^*(u) \in \left(E_u - a_0 - \sum_{S \in \mathcal{Q}_m} \left(\prod_{i \in S} \alpha_i\right) f_S(u_S)\right),$$

for all $u = (\alpha_1, \ldots, \alpha_m) \in X_{[1,m]}$ and $f^*(z_1, \ldots, z_m) = 0$ whenever $z_i = 0$ for some $1 \leq i \leq m$. Under these conditions, it is easy to see that if $S \in \mathcal{Q}_m$ and $u \in X_S$, then $f^*(u) = 0$ and $f(u) \in E_u$.

In order to construct the function $f^* : \mathbb{C}^m \to \mathbb{C}$ let us consider the enumeration $\{u_1, u_2, \ldots\}$ of $X_{[1,m]}$ where we denote $u_j = (\alpha_1^{(j)}, \ldots, \alpha_m^{(j)})$. Our function $f^* \in \mathbb{K}[[z_1, \ldots, z_m]]$ will be given by

$$f^*(z_1,\ldots,z_m) = \sum_{n=m}^{\infty} P_n(z_1,\ldots,z_m) = \sum_{i_1 \ge 1,\ldots,i_m \ge 1} c_{i_1,\ldots,i_m} z_1^{i_1} \cdots z_m^{i_m},$$

where P_n is a homogeneous polynomial of degree n and the coefficients $c_{i_1,\ldots,i_m} \in \mathbb{K}$ will be chosen conveniently so that f^* will satisfy the desired conditions.

The first condition is

$$|c_{i_1,\dots,i_m}| < s_{i_1+\dots+i_m} := \frac{1}{\binom{i_1+\dots+i_m-1}{m-1}(i_1+\dots+i_m)!}$$

where $c_{i_1,\ldots,i_n} \neq 0$ for infinitely many *m*-tuples of integers $i_1 \geq 1 \ldots i_m \geq 1$. This conditions will be used to guarantee that f^* is an entire function. Indeed, if we denote by L(P) the length of polynomial $P(z_1,\ldots,z_m) \in \mathbb{C}[z_1,\ldots,z_m]$ given by the sum of absolute values of its coefficients. Since

$$|P_n(z_1,\ldots,z_m)| \le L(P_n) \max\{1,|z_1|,\ldots,|z_m|\}^n,$$

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we have that for all $n \ge m$ and (z_1, \ldots, z_m) belonging to the open ball B(0, R)

$$|P_n(z_1,\ldots,z_m)| < \frac{\binom{n-1}{m-1}}{\binom{n-1}{m-1}n!} \max\{1,R\}^n = \frac{\max\{1,R\}^n}{n!},$$

where we use that $P_n(z_1, \ldots, z_m)$ has at most $\binom{n-1}{m-1}$ monomials of degree n. Hence, we have that the series $\sum_{n \ge m} P_n(z_1, \ldots, z_m)$ converges uniformly in any of these balls. Thus, f^* is a transcendental entire function such that $f^*(0, z_2, \ldots, z_m) = f^*(z_1, 0, z_3, \ldots, z_m) = f^*(z_1, z_2, \ldots, 0) = 0$.

In order to obtain the coefficients $c_{i_1,\ldots,i_m} \in \mathbb{K}$ such that f^* satisfies the condition (2.2), we consider for each positive integer n and $1 \leq j \leq n$ a hyperplane

$$\pi(n,j): \mu_{n,1}^{(j)} z_1 + \dots + \mu_{n,m}^{(j)} z_m - \lambda_n^{(j)} = 0$$

such that if u_j , u_{n+1} and the origin are non-collinear, then $\pi(n, j)$ is a hyperplane containing u_j and parallel to the line passing through the origin and the point u_{n+1} and if u_j , u_{n+1} and the origin are collinear, then $\pi(n, j)$ is a hyperplane containing u_j and perpendicular o the line passing through the origin and the point u_{n+1} . Note that in both cases we have that $\lambda_n^{(j)} \neq 0$ and u_{n+1} does not belong to any hyperplane $\pi(n, j)$ with $1 \leq j \leq n$.

Now, we define the polynomials $A_0(z_1, \ldots, z_m) := z_1 \cdots z_m$ and

$$A_n(z_1,\ldots,z_m) := \prod_{j=1}^n (\mu_{n,1}^{(j)} z_1 + \cdots + \mu_{n,m}^{(j)} z_m - \lambda_n^{(j)}),$$

for all $n \geq 1$. By definition of $\pi(n, j)$, we have that $A_n(u_j) = 0$ for all $1 \leq j \leq n$. Since u_{n+1} and the origin does not belong to $\pi(n, j)$, we also have that $A_n(0, \ldots, 0) \neq 0$ and $A_n(u_{n+1}) \neq 0$, for all $n \geq 1$. Thus, we define the function

$$f_{1,0}^*(z_1,\ldots,z_m) := \delta_{1,0}A_0(z_1,\ldots,z_m) = \delta_{1,0}z_1\cdots z_m$$

such that $\Theta_1 + f_{1,0}^*(u_1) \in E_{u_1}$ and $0 < |\delta_{1,0}| < s_m/m$, where

$$\Theta_j := a_0 + \sum_{S \in \mathcal{Q}_m} \left(\prod_{i \in S} \alpha_i^{(j)} \right) f_S(u_{j,S})$$

and $u_{j,S} = (\alpha_{i_1}^{(j)}, \ldots, \alpha_{i_k}^{(j)})$ for $S = \{i_1, \ldots, i_k\}$, for all integer $j \ge 1$. Moreover, since \mathbb{K} is a dense subset of \mathbb{C} , we can choose $\delta_{1,1}$ such that the

Moreover, since K is a dense subset of \mathbb{C} , we can choose $\delta_{1,1}$ such that the coefficient $c_{1,1,\dots,1}$ of $z_1 \cdots z_m$ in the function

$$f_{1,1}^*(z_1,\ldots,z_m) := f_{1,0}^*(z_1,\ldots,z_m) + \delta_{1,1}z_1\cdots z_m A_1^{(1)}(z_1,\ldots,z_m)$$

belongs to \mathbb{K} with $|c_{1,1,\dots,1}| < s_m$. Therefore, we take

$$f_1^*(z_1,\ldots,z_m) := f_{1,1}^*(z_1,\ldots,z_m),$$

where $P_1(z_1, \ldots, z_m) = c_{1,1,\ldots,1} z_1 \cdots z_m$.

Recursively, we can construct a function $f_{n,0}^*(z_1,\ldots,z_m)$ given by

$$f_{n,0}^*(z_1,\ldots,z_m) := f_{n-1}^*(z_1,\ldots,z_m) + \delta_{n,0} z_1^n z_2 \cdots z_m A_{n-1}(z_1,\ldots,z_m)$$

where we take $\delta_{n,0} \neq 0$ in the ball $B(0, s_{n+m-1}/(n+m-1))$ such that

$$\Theta_n + f_{n,0}^*(u_n) \in E_{u_n},$$

which is possible since E_{u_n} is a dense subset of \mathbb{C} and all coordinates of u_n are non-zero.

Moreover, since \mathbb{K} is a dense subset of \mathbb{C} , if we consider the ordering of the monomials of degree n + m - 1 given by the lexicographical order of the exponents, then we can choose a $\delta_{n,l}$ such that the coefficient c_{j_1,\ldots,j_m} of *l*-th monomial $z_1^{j_1} \cdots z_m^{j_m}$ in

$$f_{n,l}^*(z_1,\ldots,z_m) := f_{n,l-1}^*(z_1,\ldots,z_m) + \delta_{n,l} z_1^{j_1} \cdots z_m^{j_m} A_n(z_1,\ldots,z_m)$$

belongs to \mathbb{K} with $|c_{j_1,\ldots,j_m}| < s_{n+m-1}$. Thus, we define

$$f_n^*(z_1, \dots, z_m) := f_{n,L}^*(z_1, \dots, z_m)$$

where $L = \binom{n+m-2}{m-1}$ is the number of distinct monomials of degree n+m-1. Therefore, we have that $f_n^*(z_1,\ldots,z_m)$ is a polynomial function such that $c_{j_1,\ldots,j_m} \in \mathbb{K}$, for all *m*-tuple (j_1,\ldots,j_m) such that $j_1+\cdots+j_m \leq n+m-1$.

Finally, this construction implies that the functions f_n^* converges for a transcendental entire function $f^* \in \mathbb{K}[[z_1, \ldots, z_m]]$ as $n \to \infty$ such that

$$f^*(u_j) = f^*_n(u_j) = f^*_j(u_j)$$

for all $n \ge j \ge 1$. Hence, let $f : \mathbb{C}^m \to \mathbb{C}$ be the entire functions given by

$$f(z_1,\ldots,z_m) = a_0 + \left(\sum_{S \in \mathcal{Q}_m} \left(\prod_{i \in S} z_i\right) f_S(z_S)\right) + f^*(z_1,\ldots,z_m),$$

we have that $f(u) \in E_u$ for all $u \in X \in \mathbb{C}^m$. Since f is an entire function that is not polynomial, we have that f is transcendental. Note that there are uncountably many ways to choose the constants $\delta_{n,j}$ and this completes the proof.

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References

- K. Mahler, Lectures on Transcendental Numbers, Lecture Notes in Math., vol. 546, Springer-Verlag, Berlin, 1976.
- D. Marques, C.G. Moreira, A note on a complete solution of a problem posed by Mahler, Bull. Aust. Math. Soc. 98 (2018) (1) 60–63.
- 3. D. Marques and J. Ramirez, On exceptional sets: the solution of a problem posed by K. Mahler, *Bull. Aust. Math. Soc.* 94 (2016), 15–19.
- 4. P. Stäckel, Ueber arithmetische Eingenschaften analytischer Functionen, Math. Ann. 46 (1895), 513–520.
- M. Waldschmidt, Algebraic values of analytic functions, J. Comput. Appl. Math. 160 (2003), 323–333.

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