The Annals of Statistics
1977, Vol. 5, No. 3, 561-564

ON THE EXISTENCE AND CONSTRUCTION OF A COMPLETE SET OF ORTHOGONAL $F(4t; 2t, 2t)$-SQUARES DESIGN

By Walter T. Federer

Cornell University

The purpose of this paper is to demonstrate the existence via construction of a complete set of mutually orthogonal F-squares of order $n = 4t$, t a positive integer, with two distinct symbols. The proof assumes that all Hadamard matrices of order $4t$ exist; they are known to exist for all $1 \leqq t \leqq 50$ and for 2^p. Two methods of construction, that is, Hadamard matrix theory and factorial design theory, are given; the methods are related, but the approaches differ.

1. Introduction and definitions. In an effort to conserve space, the reader is referred to Hedayat and Seiden (1970) for details and definitions concerning F-squares and orthogonality of F-squares design. The symbol OL (n, t) has been used frequently to denote a set of t mutually orthogonal Latin squares of order n, when $t = n - 1$, the set of orthogonal Latin squares is complete. For a set of t mutually orthogonal F-squares of order n and of the form $F(n; \lambda_1, \cdots, \lambda_m)$, m a constant, we use the following definition:

Definition 1.1. A set of $2 \leqq t \leqq N_m = (n-1)^2/(m-1)$, an integer, mutually orthogonal $F(n; \lambda_1, \cdots, \lambda_m)$-squares is denoted as $\mathrm{O}F(n; \lambda_1, \cdots, \lambda_m; t)$. When $t = N_m$, the set of mutually orthogonal $F(n; \lambda_1, \cdots, \lambda_m)$-squares, is said to be complete.

Hedayat et al. (1975) have shown how to construct an $\mathrm{O}F(n = s^p; \lambda_1, \cdots, \lambda_s; (s^p - 1)/(s - 1))$ set for s a prime power. In extending their results, Mandeli (1975) has shown how to obtain a complete set of F-squares with variable numbers of symbols, that is, $m = s^k$ for $1 \leqq k \leqq p$, instead of for a constant number s; he also demonstrated a one-to-one correspondence with factorial design theory and a way to make s^k a maximum when Latin squares are not used in the set. In this paper we show how to construct an $\mathrm{O}F(4t; 2t, 2t; (4t - 1)^2)$ set and hence prove their existence, on the presumption that all Hadamard matrices of order $4t$ exist; they are known to exist for all $1 \leqq t \leqq 50$ and for all $n = 2^p$. Thus, if all Hadamard matrices exist, a complete set of mutually orthogonal F-squares with the minimum number of symbols, $m = 2$, has been shown to exist for one-fourth of all positive integers, a considerable extension over the set of prime numbers. The results are expected to have application in zero-one graph theory, in orthogonal arrays, in coding theory, and other areas. It illustrates how

Received February 1976; revised October 1976.
AMS 1970 *subject classifications*. Primary 62K99, 62K15, 62J10.
Key words and phrases. F-square design, orthogonal F-squares, Hadamard matrix, analysis of variance, factorial design.

analysis of variance and factorial design theory can be used to construct a complete set of mutually orthogonal F-squares, and thus provides a new tool for construction purposes.

2. Construction of a complete set of $F(4t; 2t, 2t)$-squares design. The use of orthogonal contrasts in the analysis of variance for factorial experiments to construct Latin squares was indicated by Federer et al. (1971). Mandeli (1975) also used this procedure. It would appear that there is considerable potential in using the orthogonality of single degree of freedom contrasts from the row by column interaction to construct F-squares and Latin squares. The following theorem represents one such example:

THEOREM 2.1. *For all values of t for which a Hadamard matrix exists, there exists a complete set of* $(4t-1)^2$ *mutually orthogonal* $F(4t; 2t, 2t)$*-squares design.*

Two proofs are given for the theorem, the first for the more mathematically inclined reader and the second proof for the more statistically oriented individual. The second proof is considered important in that linear model theory for factorial experiments and for the analysis of variance is being utilized for a statistical design construction problem.

PROOF 1. The first proof utilizes Hadamard matrix theory (e.g., see Hall (1967), Raghavarao (1971), etc.). A normalized Hadamard matrix H_n is one in which there are all plus ones in the first row and in the first column with the remaining elements being plus and minus ones. A necessary condition for the existence of H_n is that $n = 2$ or $n \equiv 0 \pmod 4$. Hadamard matrices are known to exist for $n = 4t$ for $1 \leqq t \leqq 50$ and presumed to exist for all t. All but the first row of H_{4t} will contain an equal number of plus ones and of minus ones. The Kronecker product of two normalized Hadamard matrices, i.e., $H_{4t} \otimes H_{4t} = H_{16t^2} = A$, is also a normalized Hadamard matrix of order $n^2 = 16t^2$. Denote the ith row of A by A_i, $i = 1, 2, \cdots, n, n+1, \cdots, n^2$, with elements $a_{i1}, a_{i2}, \cdots, a_{in}, a_{in+1}, \cdots, a_{in^2-n+1}, \cdots, a_{in^2}$, and make the correspondence of A_i to the following $n \times n$ square:

$$\begin{matrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{in+1} & a_{in+2} & \cdots & a_{i2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{in^2-n+1} & a_{in^2-n+2} & \cdots & a_{in^2} \end{matrix} .$$

If we do not consider the first $4t$ rows, the $4t+1$st row, the $8t+1$st row, $\cdots$, $16t^2-4t+1$st row ($8t-1$ rows in all), there will remain $16^2 - 8t + 1 = (4t-1)^2$ rows of A under consideration. When each of the rows are arranged in an $n \times n$ square as above, the sum of the coefficients within any row or within any column will be zero, with the number of plus ones being equal to the number of minus ones. This follows from the orthogonality of each of these rows with the $8t-1$ rows not being considered. In each of the $(4t-1)^2$ rows of A under consideration, let the plus ones be replaced by the symbol a_1 and let the minus ones be replaced by the symbol a_2 to form $(4t - 1)^2 F(4t; 2t, 2t)$-squares. This method of construction produces the complete set, O $F(4t; 2t, 2t; (4t - 1)^2)$, of mutually orthogonal F-squares with the minimum number, two, of symbols.

PROOF 2. The second proof of the existence is by construction through single degree of freedom contrasts in the analysis of variance procedure. The n^2 observations from an n-row by n-column square may be partitioned in such a manner as to have n^2 orthogonal single degree of freedom contrasts. The resulting sums of squares are also orthogonal. For $n = 4t$, the particular sets of row and column contrasts we shall use are those from a normalized Hadamard matrix of side $4t$. There will be $2t$ plus ones and $2t$ minus ones in all rows and columns of a normalized Hadamard matrix, say H_{4t}, except the first row and column. The complete set of n^2 orthogonal single degree of freedom contrasts is obtained as $H_{4t} \otimes H_{4t} = H_{16t^2}$. Designate the $4t$ row contrasts as $H_{4t} = (R_0', R_1', \cdots, R_{4t-1}')'$ and the $4t$ column contrasts as $H_{4t} = (C_0', C_1', \cdots, C_{4t-1}')'$. Let the elements of the row vector R_i be r_{ig}, $g = 1, 2, \cdots, 4t$, and let the elements of the column vector C_j be c_{jh}, $h = 1, 2, \cdots, 4t$. Then a single degree of freedom contrast of the n^2 observations is obtained as the Kronecker product of R_i and C_j, that is $R_i \otimes C_j$, with the $igjh$th element in this $1 \times 16t^2$ row vector being $r_{ig} c_{jh}$. Note that this element will be either a plus one or a minus one and that there are $2t$ of each since this is a contrast. The one exception is $R_0 \otimes C_0$ which has $16t^2$ plus ones and corresponds to the correction for the mean in the analysis of variance table.

From the above we may now construct the following analysis of variance table for the $n^2 = 16t^2$ observations.

TABLE 1

Source of Variation	Degrees of Freedom		Sum of Squares
Correction for the mean	1		$(R_0 C_0)^2/16t^2$
Rows	$4t - 1$		sum of following
R_1		1	$(R_1 C_0)^2/16t^2$
R_2		1	$(R_2 C_0)^2/16t^2$
$\vdots$		$\vdots$	$\vdots$
R_{4t-1}		1	$(R_{4t-1} C_0)^2/16t^2$
Columns	$4t - 1$		sum of following
C_1		1	$(R_0 C_1)^2/16t^2$
C_2		1	$(R_0 C_2)^2/16t^2$
$\vdots$		$\vdots$	$\vdots$
C_{4t-1}		1	$(R_0 C_{4t-1})^2/16t^2$
Row $\times$ Column interaction	$(4t - 1)^2$		sum of following
$R_1 C_1$		1	$(R_1 C_1)^2/16t^2$
$R_1 C_2$		1	$(R_1 C_2)^2/16t^2$
$\vdots$		$\vdots$	$\vdots$
$R_1 C_{4t-1}$		1	$(R_1 C_{4t-1})^2/16t^2$
$R_2 C_1$		1	$(R_2 C_1)^2/16t^2$
$\vdots$		$\vdots$	$\vdots$
$R_{4t-1} C_{4t-1}$		1	$(R_{4t-1} C_{4t-1})^2/16t^2$

In computing sums of squares in the above table, the sign for $r_{ig}c_{jh}$ is applied to the *igjh*th observation and a single sum is obtained; it is denoted as R_iC_j. This sum is squared and divided by $16t^2$ to obtain a sum of squares with one degree of freedom. From factorial and analysis of variance theory we know that these are orthogonal sums of squares.

In order to construct an $F(4t; 2t, 2t)$-square, we proceed by noting that any $R_i \otimes C_j$ contrast may be arranged as follows:

row		column number			
		1	2	...	$4t$
number	contrast coefficient	column contrast coefficient c_{j1}	c_{j2}	...	c_{j4t}
1	r_{i1}	$r_{i1}c_{j1}$	$r_{i1}c_{j2}$		$r_{i1}c_{j4t}$
2	r_{i2}	$r_{i2}c_{j1}$	$r_{i2}c_{j2}$		$r_{i2}c_{j4t}$
⋮					
$4t-1$	r_{i4t}	$r_{i4t}c_{j1}$	$r_{i4t}c_{j2}$		$r_{i4t}c_{j4t}$

In each row and in each column of the above $4t \times 4t$ square, there will be $2t$ plus ones and $2t$ minus ones. Let the symbol a_1 replace the minus ones and the symbol a_2 replace the plus ones to produce the $F(4t; 2t, 2t)$-square. Since there are $(4t-1)^2$ single degree of freedom contrasts in the row $\times$ column interaction, since each one may be used to construct an $F(4t; 2t, 2t)$-square, and since each contrast is orthogonal to the remaining $(4t-1)^2 - 1$ contrasts, $(4t-1)^2$ orthogonal $F(4t; 2t, 2t)$-squares are thus formed, and the theorem is proved.

Acknowledgment. The author appreciates a comment from Esther Seiden to the effect that the theorem be proved using Hadamard matrix theory, and some helpful comments by A. Hedayat and a referee.

REFERENCES

FEDERER, W. T., HEDAYAT, A., PARKER, E. T., RAKTOE, B. L., SEIDEN, E. and TURYN, R. J. (1971). Some techniques for constructing mutually orthogonal latin squares. MRC Technical Summary Report No. 1030, Mathematics Research Center, Univ. of Wisconsin. (A preliminary version of this report appeared in the Proc. of the Fifteenth Conference on the Design of Experiments in Army Research Development and Testing, ARO-D Report 70-2, July, 1970, The Office of the Chief of Research and Development, Durham, North Carolina.)

HALL, M., JR. (1967). *Combinatorial Theory*. Blaisdell, Waltham, Mass.

HEDAYAT, A., RAGHAVARAO, D. and SEIDEN, E. (1975). Further contributions to the theory of F-squares design. *Ann. Statist.* **3** 712–716.

HEDAYAT, A. and SEIDEN, E. (1970). F-square and orthogonal F-square design: A generalization of Latin square and orthogonal Latin squares design. *Ann. Math. Statist.* **41** 2035–2044.

MANDELI, J. P. (1975). Complete sets of orthogonal F-squares. Masters thesis, Cornell Univ.

RAGHAVARAO, D. (1971). *Constructions and Combinatorial Problems in Design of Experiments.* Wiley, New York.

GRADUATE FIELD AND CENTER OF STATISTICS
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853