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On the existence and uniqueness of (N,λ)-periodic solutions to a class of Volterra difference equations

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Abstract

In this paper we introduce the class of (N,λ)-periodic vector-valued sequences and show several notable properties of this new class. This class includes periodic, anti-periodic, Bloch and unbounded sequences. Furthermore, we show the existence and uniqueness of (N,λ)-periodic solutions to the following class of Volterra difference equations with infinite delay:

$$u(n+1)=\alpha\sum_{j=-\infty}^{n}a(n-j)u(j)+f(n,u(n)),\quad n\in\mathbb{Z},\alpha\in\mathbb{C},$$

where the kernel a and the nonlinear term f satisfy suitable conditions.

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1 Introduction

In this paper, we define and investigate a new class of vector-valued functions defined on $\mathbb{Z}$ called (N,λ)-periodic discrete functions. This type of sequences is the discrete version of the vector-valued (ω,c)-periodic functions introduced in [5]. Thus, we say that a function f is (N,λ)-periodic discrete function if there exist $N\in\mathbb{Z}_+$ and $\lambda\in\mathbb{C}\setminus\{0\}$ such that $f(n+N)=\lambda f(n)$ for all integers n. This definition includes: discrete periodic functions ($\lambda=1$), discrete anti-periodic functions ($\lambda=-1$), discrete Bloch-periodic functions ($\lambda=e^{ikN}$) and unbounded functions ($|\lambda|\neq 1$). Additionally, we establish a criterion of the existence and uniqueness of (N,λ)-periodic discrete solutions to the linear and nonlinear Volterra difference equations in Banach spaces.

The real-valued (ω,c)-periodic functions were introduced and studied by G. Floquet in [12]. He called this set of functions as periodic functions of the second kind. In that paper, Floquet considered a linear system with time-periodic coefficients with some periodicity T and a given initial condition

$$x'(t)=A(t)x(t),\quad t\in\mathbb{R};$$

$$A(t) = A(t+T), \quad t \in \mathbb{R};$$
$$x(0) = x_0,$$

where $x : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ is continuous. He proved that, if $\Phi(t)$ is a principal fundamental matrix (which exists by elementary theory of ODEs), then $\Phi(t+T) = \Phi(t)C$ where $C = \Phi(T)$ is a constant nonsingular matrix which is know as the monodromy matrix. Moreover, for a matrix L such that $e^{LT} = \Phi(T)$, there is a periodic matrix function $t \mapsto Z(t)$ such that $x(t) = Z(t)e^{LT}x_0$ where $L, Z(t) \in \mathbb{C}^{n\times n}$ and $Z(t) = z(t+T)$. An application of this result to the model of predator–prey interactions can be seen in [7]. For additional related applications, see [9, 11] and the references therein. On the other hand, in the discrete (real-valued) case, there are some analogous results to the Floquet's theory. For example, for a system of difference equations

$$X(n+1) = B(n)X(n), \quad n \in \mathbb{Z}_+,$$

where B is a periodic matrix with period N, Kelley and Peterson [15], Agarwal [2] and Elaydi [13] studied analogous results to the continuous case. Although several authors have worked with such real-valued sequences, so far none has mentioned the vector-valued case. Moreover, characterizations in terms of the N-periodic functions, Banach space structure of the set of (N,λ)-periodic discrete functions, as well as composition and convolution results, are problems that have not been studied, yet. These aspects are our main motivation for this work.

On the other hand, problems of existence and uniqueness of periodic and anti-periodic solutions of Volterra difference equations have been considered by several authors due to their various applications in mathematical biology, problems related to signals and systems, integral equations, the foundation of functional analysis and numerical methods for solving Volterra integral or integro-differential equations. The papers [1, 4, 8, 10, 14] cover many of these applications. Araya et al. in [6] studied the existence and uniqueness of almost automorphic discrete solutions for a class of nonlinear Volterra difference equations of convolution type on a Banach space $\mathbb{X}$ with norm $\|\cdot\|_{\mathbb{X}}$, namely

$$u(n+1) = \alpha \sum_{j=-\infty}^{n} a(n-j)u(j) + f\big(n, u(u)\big), \tag{1.1}$$

where α is a complex number or a bounded linear operator defined on $\mathbb{X}$, $a : \mathbb{N}_0 \to \mathbb{C}$ is summable, i.e., $\sum_{n=0}^{\infty} |a(n)| < +\infty$ and f is almost automorphic discrete function. However, no-one has studied the problem of the existence and uniqueness of (N,λ)-periodic discrete solutions for (1.1). In this paper, we have successfully solved this problem by using fixed point techniques.

In order to obtain our results, first we show a characterization of the (N,λ)-periodic discrete functions, which says that $f : \mathbb{Z} \to \mathbb{X}$ is an (N,λ)-periodic discrete function if and only if there exists an N-periodic function u such that $f(n) = \lambda^{n/N}u(n)$ for all $n \in \mathbb{Z}$.

Using this characterization, we prove that the set of (N,λ)-periodic discrete functions is a Banach space with the norm

$$\|f\|_{N\lambda} := \max_{n\in[0,N]} \big\|\lambda^{-n/N} f(n)\big\|_{\mathbb{X}}.$$

Note that the norm is well defined by the characterization mentioned above. We shall denote this Banach space by $\mathbb{P}_{N\lambda}(\mathbb{Z},\mathbb{X})$. We prove that the convolution $(b * f)(n) := \sum_{j=-\infty}^{n} b(n-j)f(j)$, $n \in \mathbb{Z}$, is an (N,λ)-periodic discrete function whenever the sequence $b^{\smile}(n) := \lambda^{-n/N} b(n)$ is summable and f is an (N,λ)-periodic discrete function. Also, we prove that the Nemytskii operator $\mathcal{N}(\phi)(\cdot) := f(\cdot,\phi(\cdot))$ is an (N,λ)-periodic discrete function if and only if u is an (N,λ)-periodic discrete function and $f(n+N,\lambda x) = \lambda f(n,x)$ for all $n \in \mathbb{Z}$ and for all $x \in \mathbb{X}$. With these tools, we prove that equation (1.1) has a unique solution on $\mathbb{P}_{N\lambda}(\mathbb{Z},\mathbb{X})$.

This paper is organized as follows: In Sect. 2, we introduce the definition of vector-valued (N,λ)-periodic discrete functions and show essential properties of this class of sequences. Section 3 is devoted to studying the existence and uniqueness of (N,λ)-periodic discrete solutions to linear and semilinear Volterra equations of convolution type. Also, some enlightening examples are given throughout the paper to illustrate the theory.

2 (N,λ)-Periodic discrete functions

In this section we introduce the concept of an (N,λ)-periodic discrete vector-valued function and show some remarkable properties of this class of vector-valued sequences.

Notation. Let $\mathbb{X}$ be a complex Banach space equipped with the norm $\|\cdot\|_{\mathbb{X}}$ and, $\mathbb{N}_0$, $\mathbb{Z}_+$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural numbers with 0, positive integers, integers, and rational, real and complex numbers, respectively.

We recall that a sequence $a : \mathbb{N}_0 \to \mathbb{C}$ is said to be summable (or complex summable) if $\sum_{n=0}^{\infty} |a(n)| < \infty$. The space of these sequences is denoted by $\ell_1(\mathbb{N}_0)$ and it is equipped with the norm

$$\|a\|_{\ell_1} = \sum_{n=0}^{\infty} \bigl|a(n)\bigr|.$$

The forward difference operator of the first-order is denoted by Δ, that is, $\Delta u(n) := u(n+1) - u(n)$. Additionally, $Z[f(n)]$ will denote the Z-transform of a given sequence $f : \mathbb{Z} \to \mathbb{X}$ defined as identically zero for negative integers n and

$$Z\bigl[f(n)\bigr] = \tilde{f}(z) = \sum_{j=0}^{\infty} f(j) z^{-j}, \tag{2.1}$$

valid for all $z \in \mathbb{C}$ with $|z|$ sufficiently large. For instance, for a given complex number a we have

$$Z\bigl[a^n\bigr] = \frac{z}{z-a} \quad \text{for all } |z| > |a|,$$
$$Z\bigl[f(n+1)\bigr] = z\tilde{f}(z) - zf(0) \quad \text{for all } |z| > R,$$

where R denotes the radius of convergence of the series $\tilde{f}(z)$. For more information about the standard definitions and properties, see [13].

We introduce the following definition.

Definition 2.1 A vector-valued function $f : \mathbb{Z} \to \mathbb{X}$ is called an (N,λ)-periodic discrete function (or (N,λ)-periodic sequence) if there exist $N \in \mathbb{Z}_+$ and $\lambda \in \mathbb{C} \setminus \{0\}$ such that

$f(n+N) = \lambda f(n)$ for all $n \in \mathbb{Z}$; N is called the λ-period of f. The collection of these sequences with the same λ-period N will be denoted by $\mathbb{P}_{N\lambda}(\mathbb{Z},\mathbb{X})$.

In case $\lambda = 1$, we denote simply by $\mathbb{P}_N(\mathbb{Z},\mathbb{X})$ the set of all N-periodic sequences.

The following property gives a useful characterization of (N,λ)-periodic discrete functions.

Proposition 2.2 *A function f is an (N,λ)-periodic discrete function if and only if there exists $u \in \mathbb{P}_N(\mathbb{Z},\mathbb{X})$ such that*

$$f(n) = \lambda^\wedge(n)u(n), \quad \text{for all } n \in \mathbb{Z}, \tag{2.2}$$

where $\lambda^\wedge(n) := \lambda^{n/N}$.

Proof First, we assume that $f \in \mathbb{P}_{N\lambda}(\mathbb{Z},\mathbb{X})$ and define $u(n) := \lambda^\wedge(-n)f(n)$. Then,

$$u(n+N) = \lambda^\wedge\bigl(-(n+N)\bigr)f(n+N) = \lambda^\wedge(-n)f(n) = u(n).$$

Hence $u \in \mathbb{P}_N(\mathbb{Z},\mathbb{X})$ and $f(n) = \lambda^\wedge(n)u(n)$. Conversely, we suppose $f(n) = \lambda^\wedge(n)u(n)$. Then

$$f(n+N) = \lambda^\wedge(n+N)u(n+N) = \lambda \cdot \lambda^\wedge(n)u(n) = \lambda f(n). \qquad \square$$

Example 2.3 The function $f(n) = \cos(\pi n/6)$ is a $(6,-1)$-periodic discrete function. It follows from Proposition 2.2 that f has decomposition $f(n) = \lambda^\wedge(n)u(n)$ where

$$\lambda^\wedge(n) = (-1)^{n/6} = \cos(n\pi/6) + i\sin(n\pi/6),$$

and

$$u(n) = (-1)^{-n/6}f(n) = \cos(n\pi/6)\bigl[\cos(n\pi/6) - i\sin(n\pi/6)\bigr].$$

Example 2.4 Let $\mathcal{A}$ be a $k \times k$ matrix. Assume that there exists $N \in \mathbb{Z}_+$ (sufficiently large) such that $\mathcal{A}(n+N) = \mathcal{A}(n)$ for all $n \in \mathbb{Z}_+$. Let $\mathcal{B}$ be the $k \times k$ matrix defined as follows:

$$\mathcal{B} := \prod_{i=0}^{N-1} \mathcal{A}(i), \quad n \in \mathbb{Z}_+,$$

where $\prod_{i=0}^{N-1} \mathcal{A}(i) := \mathcal{A}(N-1)\mathcal{A}(N-2)\cdots\mathcal{A}(0)$. Furthermore, let $\lambda_0 \in \mathbb{C} \setminus \{0\}$ be any eigenvalue of $\mathcal{B}$ with corresponding eigenvector X_0. It can be proved that the solution of the system

$$\begin{aligned} U(n+1) &= \mathcal{A}(n)U(n), \quad \text{for } n \in \mathbb{Z}_+ \\ U(0) &= X_0, \end{aligned} \tag{2.3}$$

is given by

$$U(n) = \prod_{i=0}^{n-1} \mathcal{A}(i)X_0. \tag{2.4}$$

Moreover,

$$\begin{aligned} U(n+N) &= \prod_{i=0}^{n+N-1} \mathcal{A}(i)X_0 = \prod_{i=N}^{n+N-1} \mathcal{A}(i)\prod_{i=0}^{N-1}\mathcal{A}(i)X_0 \\ &= \prod_{i=N}^{n+N-1} \mathcal{A}(i)\mathcal{B}\lambda_0 = \prod_{i=0}^{n-1}\mathcal{A}(i)\lambda_0 X_0 = \lambda_0\prod_{i=0}^{n-1}\mathcal{A}(i)X_0 = \lambda_0 U(n). \end{aligned}$$

Hence system (2.3) has an (N,λ_0)-periodic solution given by (2.4). Moreover, $U(n) = \lambda_0^{\wedge}(n)P(n)$ where $P(n) := \lambda_0^{\wedge}(-n)\prod_{i=0}^{n-1}\mathcal{A}(i)X_0$ is a periodic sequence of period N. As a particular example, if

$$\mathcal{A}(n) = \begin{pmatrix} 0 & \frac{3+(-1)^n}{2} \\ \frac{3-(-1)^n}{2} & 0 \end{pmatrix},$$

we have that $N = 2$ and the eigenvalues of $\mathcal{B} := \mathcal{A}(1)\mathcal{A}(0)$ are $\lambda_1 = 1$ and $\lambda_2 = 4$ with the corresponding eigenvectors

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

respectively. If $X(0) = X_1$, then system (2.3) has a $(2,1)$-periodic solution and, if $X(0) = X_2$, it has a $(2,4)$-periodic solution.

Next, we present some algebraic properties of the (N,λ)-periodic discrete functions.

Theorem 2.5 *Let f and g be (N,λ)-periodic discrete functions, $c \in \mathbb{C}$ and $l \in \mathbb{Z}$. Then the following assertions are valid*:

(i) *$w := f + g$ is an (N,λ)-periodic discrete function.*
(ii) *$p := cf$ is an (N,λ)-periodic discrete function.*
(iii) *For each fixed l in $\mathbb{Z}$ the function $f_l : \mathbb{Z} \to \mathbb{X}$ defined by $f_l(n) := f(n+l)$ is an (N,λ) periodic discrete function.*

Proof The proof is immediate. Indeed, for all $n \in \mathbb{Z}$ we have that

(i) $w(n+N) = (f+g)(n+N) = \lambda w(n)$;
(ii) $p(n+N) = (cf)(n+N) = \lambda p(n)$;
(iii) $f_l(n+N) = f(n+N+l) = f(n_0+N) = \lambda f_l(n)$. □

Theorem 2.6 *If $f \in \mathbb{P}_{N\lambda}(\mathbb{Z},\mathbb{X})$, then $\Delta f \in \mathbb{P}_{N\lambda}(\mathbb{Z},\mathbb{X})$.*

Proof Since $\Delta f(n) = f(n+1) - f(n)$, then by (i) and (iii) of Theorem 2.5, we have that Δf is an (N,λ)-periodic discrete function. □

In order to give a Banach structure to the vector space $\mathbb{P}_{N\lambda}(\mathbb{Z},\mathbb{X})$, we need to define a suitable norm. We recall that the space of N-periodic discrete functions equipped with the norm

$$\|u\| := \max_{n\in[0,N]} \|u(n)\|_{\mathbb{X}} \tag{2.5}$$

is a Banach space.

Proposition 2.7 $\mathbb{P}_{N\lambda}(\mathbb{Z},\mathbb{X})$ *is a Banach space with the norm*

$$\|f\|_{N\lambda} := \max_{n\in[0,N]} \|\lambda^\wedge(-n)f(n)\|_{\mathbb{X}}. \tag{2.6}$$

Proof The proof follows from Proposition 2.2 and the fact that $\mathbb{P}_N(\mathbb{Z},\mathbb{X})$ is a Banach space with the norm (2.5). □

Next, we present a convolution theorem. This result is a useful tool in order to study the existence and uniqueness of (N,λ)-periodic discrete solutions of abstract Volterra difference equations.

Theorem 2.8 *Let* $f \in \mathbb{P}_{N\lambda}(\mathbb{Z},\mathbb{X})$ *and assume that* $b : \mathbb{N}_0 \to \mathbb{C}$ *is such that the sequence* $b^\backsim(n) := \lambda^\wedge(-n)b(n)$ *is summable. Then* $b*f$ *defined by*

$$(b*f)(n) = \sum_{j=-\infty}^{n} b(n-j)f(j), \quad n\in\mathbb{Z},$$

is well defined with respect to the norm $\|\cdot\|_{N\lambda}$ *and belongs to* $\mathbb{P}_{N\lambda}(\mathbb{Z},\mathbb{X})$.

Proof Let $p(n) := (b*f)(n)$, $n\in\mathbb{Z}$. First, note that p is well defined with respect to the norm $\|\cdot\|_{N\lambda}$. Indeed,

$$\begin{aligned}\|\lambda^\wedge(-n)p(n)\|_{\mathbb{X}} &\le \sum_{j=-\infty}^{n} \left|\lambda^\wedge\big(-(n-j)\big)b(n-j)\right| \|\lambda^\wedge(-j)f(j)\|_{\mathbb{X}} \\ &\le \|f\|_{N\lambda}\sum_{j=-\infty}^{n} \left|\lambda^\wedge\big(-(n-j)\big)b(n-j)\right| = \|f\|_{N\lambda}\sum_{j=0}^{\infty}\left|b^\backsim(j)\right|.\end{aligned}$$

Therefore $\|p\|_{N\lambda} \le \|f\|_{N\lambda}\|b^\backsim\|_{\ell_1}$. Next, we prove that p is (N,λ)-periodic discrete. In fact,

$$\begin{aligned}p(n+N) &= \sum_{j=-\infty}^{n+N} b(n+N-j)f(j) = \sum_{j=-\infty}^{n+N} b\big(n-(j-N)\big)f(j) \\ &= \sum_{r=-\infty}^{n} b(n-r)f(r+N) = \lambda\sum_{r=-\infty}^{n} b(n-r)f(r) = \lambda p(n).\end{aligned}$$

Hence $p\in\mathbb{P}_{N\lambda}(\mathbb{Z},\mathbb{X})$. □

In order to prove the next composition result, we need the following useful lemma.

Lemma 2.9 *For every* $(m,x)\in\mathbb{Z}\times\mathbb{X}$, *there exists* $\phi\in\mathbb{P}_{N\lambda}(\mathbb{Z},\mathbb{X})$ *such that*

$$\phi(m) = x.$$

Proof It is enough to consider $\phi(n) := \lambda\hat{}(n-m)x$. □

Let $g : \mathbb{Z} \times \mathbb{X} \to \mathbb{X}$ and $\phi \in \mathbb{P}_{N\lambda}(\mathbb{Z}, \mathbb{X})$. We recall that the operator $\mathcal{N}(\phi)(\cdot) := g(\cdot, \phi(\cdot))$ is called the Nemytskii discrete composition operator. We study the invariance of $\mathcal{N}$ on $\mathbb{P}_{N\lambda}(\mathbb{Z}, \mathbb{X})$.

Theorem 2.10 *Let* $g : \mathbb{Z} \times \mathbb{X} \to \mathbb{X}$. *Then the following assertions are equivalent*:

(i) *for every* $\phi \in \mathbb{P}_{N\lambda}(\mathbb{Z}, \mathbb{X})$ *we have that* $\mathcal{N}(\phi)$ *is* (N, λ)*-periodic discrete.*

(ii) g *is* N*-periodic in the first variable and homogeneous in the second variable, that is,* $g(n+N, \lambda x) = \lambda g(n, x)$ *for all* $(n, x) \in \mathbb{Z} \times \mathbb{X}$.

Proof Assume (ii), then for $\phi \in \mathbb{P}_{N\lambda}(\mathbb{Z}, \mathbb{X})$ and all $n \in \mathbb{Z}$ we have

$$\mathcal{N}(\phi)(n+N) = g\bigl(n+N, \phi(n+N)\bigr) = g\bigl(n+N, \lambda\phi(n)\bigr) = \lambda\mathcal{N}(\phi)(n).$$

Thus, we conclude that $\mathcal{N}(\phi) \in \mathbb{P}_{N\lambda}(\mathbb{Z}, \mathbb{X})$. Suppose (i) and let $(n, x) \in \mathbb{Z} \times \mathbb{X}$ be arbitrary. By Lemma 2.9 there exists $\phi \in \mathbb{P}_{N\lambda}(\mathbb{Z}, \mathbb{X})$, such that $\phi(n) = x$. Therefore, for such ϕ we have

$$\begin{aligned}\lambda g(n, x) &= \lambda g\bigl(n, \phi(n)\bigr) = \lambda\mathcal{N}(\phi)(n) = \mathcal{N}(\phi)(n+N) = g\bigl(n+N, \phi(n+N)\bigr) \\ &= g\bigl(n+N, \lambda\phi(n)\bigr) = g(n+N, \lambda x),\end{aligned}$$

which gives the claim. □

3 (N, λ)-Periodic discrete solutions of abstract Volterra difference equations

3.1 The linear case

In this part, we establish the existence of (N, λ)-periodic discrete solutions for the following class of linear Volterra difference equations defined on a Banach space $\mathbb{X}$ (see [10]):

$$u(n+1) = \alpha \sum_{j=-\infty}^{n} a(n-j)u(j) + f(n), \quad n \in \mathbb{Z}, \tag{3.1}$$

where α is a given complex number, a is summable, and $f \in \mathbb{P}_{N\lambda}(\mathbb{Z}, \mathbb{X})$ for N, λ fixed. Let $s(\alpha, k)$ be the solution of the difference equation

$$\begin{aligned}&s(\alpha, n+1) = \alpha \sum_{j=0}^{n} a(n-j)s(\alpha, j), \quad n \in \mathbb{N}_0, \\ &s(\alpha, 0) = 1,\end{aligned} \tag{3.2}$$

and define the set

$$\Omega_{\lambda s}^{N} := \left\{ \alpha \in \mathbb{C} : \sum_{j=0}^{\infty} \bigl| s^{\smile}(\alpha, j) \bigr| < \infty \right\},$$

where $s^{\smile}(\alpha, j) = \lambda\hat{}(-j)s(\alpha, j)$. Note that $0 \in \Omega_{\lambda s}^{N}$.

Theorem 3.1 *Let $a:\mathbb{N}_0\to\mathbb{C}$ and $f\in\mathbb{P}_{N\lambda}(\mathbb{Z},\mathbb{X})$ be given. Suppose that a is summable and $\alpha\in\Omega^N_{\lambda s}$. Then there is an (N,λ)-periodic discrete solution of (3.1) given by*

$$u(n+1)=\sum_{j=-\infty}^{n}s(\alpha,n-j)f(j). \tag{3.3}$$

Proof Since $f\in\mathbb{P}_{N\lambda}(\mathbb{Z},\mathbb{X})$ and $\alpha\in\Omega^N_{\lambda s}$, applying Theorem 2.8 we obtain that (3.3) is a well defined (N,λ)-periodic discrete function. Moreover, since a is summable, following the same lines as in the proof of [10, Theorem 3.1], we find that u satisfies (3.1). Indeed,

$$\begin{aligned}
&\alpha\sum_{j=-\infty}^{n}a(n-j)u(j)+f(n)\\
&\quad=\alpha\sum_{j=-\infty}^{n}a(n-j)\left(\sum_{i=-\infty}^{j-1}s(\alpha,j-1-i)f(i)\right)+f(n)\\
&\quad=\alpha\sum_{j=-\infty}^{n-1}\sum_{i=-\infty}^{j}a(n-1-j)s(\alpha,j-i)f(i)+f(n)\\
&\quad=\alpha\sum_{i=-\infty}^{n-1}\sum_{j=i}^{n-1}a(n-1-j)s(\alpha,j-i)f(i)+f(n)\\
&\quad=\alpha\sum_{i=-\infty}^{n-1}\left(\sum_{j=0}^{n-1-i}a(n-1-i-j)s(\alpha,j)\right)f(i)+f(n)\\
&\quad=\sum_{i=-\infty}^{n-1}s(\alpha,n-i)f(i)+s(\alpha,0)f(n)=\sum_{i=-\infty}^{n}s(\alpha,n-i)f(i)=u(n+1).
\end{aligned}$$

□

Remark 3.2 Uniqueness of solutions to the linear case follows directly from [3, Remark 2.4].

3.2 The semilinear case

In this subsection, we consider the problem of existence and uniqueness of (N,λ)-periodic discrete solutions for the class of semilinear Volterra difference equations on a Banach space $\mathbb{X}$ given by

$$u(n+1)=\alpha\sum_{j=-\infty}^{n}a(n-j)u(j)+f\bigl(n,u(n)\bigr),\quad n\in\mathbb{Z}, \tag{3.4}$$

where $\alpha\in\mathbb{C}$ and f satisfies suitable conditions. Here, we assume that $a^{\smallsmile}(k):=\lambda^{\wedge}(-k)a(k)$ $(k\in\mathbb{N}_0)$ is such that $\|a^{\smallsmile}\|_{\ell_1}<\infty$. For example, if a is summable and $|\lambda|\geq 1$ then $a^{\smallsmile}\in\ell_1(\mathbb{N}_0)$.

Theorem 3.3 *Let $f:\mathbb{Z}\times\mathbb{X}\to\mathbb{X}$ be given. Assume the following conditions:*

(i) *There exists $(N,\lambda)\in\mathbb{Z}\times(\mathbb{C}\setminus\{0\})$ such that $f(n+N,\lambda x)=\lambda f(n,x)$ for all $(n,x)\in\mathbb{Z}\times\mathbb{X}$;*

(ii) *There exists a constant $L > 0$ such that*

$$\big\|f(n,x) - f(n,y)\big\|_{\mathbb{X}} \leq L\|x - y\|_{\mathbb{X}}$$

for all $x, y \in \mathbb{X}$ and $n \in \mathbb{Z}$;

(iii) $\alpha \in \Omega^N_{\lambda s}$;

(iv) $L\sum_{k=0}^{\infty} |s^{\frown}(\alpha,k)| < 1$.

Then equation (3.4) *has a unique solution in $\mathbb{P}_{N\lambda}(\mathbb{Z},\mathbb{X})$ satisfying*

$$u(n+1) = \sum_{j=-\infty}^{n} s(\alpha, n-j) f\big(j, u(j)\big).$$

Proof We define the operator $G : \mathbb{P}_{N\lambda}(\mathbb{Z},\mathbb{X}) \to \mathbb{P}_{N\lambda}(\mathbb{Z},\mathbb{X})$ by

$$G(u)(n) := \sum_{j=-\infty}^{n} s(\alpha, n-j) f\big(j, u(j)\big).$$

By hypothesis (i) and Theorems 2.10 and 2.8, we have that and $G(u)$ is an (N,λ)-periodic discrete function and therefore G is well defined. Now, for $u, v \in \mathbb{P}_{N\lambda}(\mathbb{Z},\mathbb{X})$ we have by hypothesis (ii)

$$\begin{aligned}
&\left\| \lambda^\wedge(-n) \sum_{j=-\infty}^{n} s(\alpha, n-j)\big[f(j,u(j)) - f(j,v(j))\big]\right\|_{\mathbb{X}} \\
&\quad = \left\| \sum_{j=-\infty}^{n} \lambda^\wedge\big(-(n-j)\big) s(\alpha,n-j)\lambda^\wedge(-j)\big[f(j,u(j)) - f(j,v(j))\big]\right\|_{\mathbb{X}} \\
&\quad \leq \sum_{j=-\infty}^{n} \big|\lambda^\wedge\big(-(n-j)\big)s(\alpha,n-j)\big| \big\|\lambda^\wedge(-j)\big[f(j,u(j)) - f(j,v(j))\big]\big\|_{\mathbb{X}} \\
&\quad \leq \sum_{j=-\infty}^{n} \big|\lambda^\wedge\big(-(n-j)\big)s(\alpha,n-j)\big| |\lambda|^{-j/N} \big\|\big[f(j,u(j)) - f(j,v(j))\big]\big\|_{\mathbb{X}} \\
&\quad \leq L \sum_{j=-\infty}^{n} \big|\lambda^\wedge\big(-(n-j)\big)s(\alpha,n-j)\big| |\lambda|^{-j/N} \big\|\big[u(j) - v(j)\big]\big\|_{\mathbb{X}} \\
&\quad = L \sum_{j=-\infty}^{n} \big|\lambda^\wedge\big(-(n-j)\big)s(\alpha,n-j)\big| |\lambda|^{-j/N} \big\|\lambda^\wedge(j)\lambda^\wedge(-j)\big[u(j) - v(j)\big]\big\|_{\mathbb{X}} \\
&\quad = L \sum_{j=-\infty}^{n} \big|\lambda^\wedge\big(-(n-j)\big)s(\alpha,n-j)\big| \big\|\lambda^\wedge(-j)\big[u(j) - v(j)\big]\big\|_{\mathbb{X}} \\
&\quad \leq \|u - v\|_{N\lambda} L \sum_{k=0}^{\infty} \big|s^{\frown}(\alpha,k)\big|.
\end{aligned}$$

By (iii) and (iv), we obtain

$$\|G(u)-G(v)\|_{N\lambda} = \max_{n\in[0,N]} \left\| \lambda^{\wedge}(-n) \sum_{j=-\infty}^{n} s(\alpha, n-j)\big[f(j,u(j)) - f(j,v(j))\big] \right\|_{\mathbb{X}}$$
$$\leq \|u-v\|_{N\lambda} L \sum_{k=0}^{\infty} \big|s^{\smile}(\alpha,k)\big| < \|u-v\|_{N\lambda}.$$

It follows that G is a contraction. Then there exists a unique function $u \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$ such that $Gu = u$. Hence u is the unique solution of equation (3.4). □

Example 3.4 We consider the following difference equation in the Banach space $\mathbb{X} = \mathbb{R}$,

$$u(n+1) = \alpha \sum_{j=-\infty}^{n} p^{n-j} u(j) + \nu g(n) \cos\big(h(n)u(n)\big), \quad n \in \mathbb{Z}, \tag{3.5}$$

where $g \in \mathbb{P}_{N\lambda}(\mathbb{Z}, \mathbb{R})$, $h \in \mathbb{P}_{N\frac{1}{\lambda}}(\mathbb{Z}, \mathbb{R})$, $p \in \mathbb{C}$ is such that $|p| < 1$ and

$$\alpha \in \mathbb{D} := \big\{ z \in \mathbb{C} : |z+p| < |\lambda|^{1/N} \big\}.$$

Let $\varphi(n) := g(n)h(n)$. Note that φ is a periodic function with period N. Then, there exists a constant τ such that $\tau := \max_{n\in[0,N]} |\varphi(n)|$. We claim that if

$$|\nu| < \frac{|\lambda|^{1/N} - |\alpha+p|}{(|\lambda|^{1/N} - |\alpha+p| + |\alpha|)|\tau|}, \tag{3.6}$$

then (3.5) has a unique (N, λ)-periodic discrete solution. In order to show this, first, let us determine the solution $s(\alpha, n)$ of the problem

$$s(\alpha, n+1) = \alpha \sum_{j=0}^{n} p^{k-j} s(\alpha, j), \quad n \in \mathbb{N}_0,$$
$$s(\alpha, 0) = 1,$$

using the Z-transform. Indeed, we have $Z[s(\alpha, n+1)] = \alpha Z[\sum_{j=0}^{n} p^{k-j} s(\alpha, j)]$, that is, $z\tilde{s}(z) - zs(\alpha, 0) = \alpha\tilde{p}(z)\tilde{s}(z)$ or, equivalently, $z\tilde{s}(z) - z = \alpha(\frac{z}{z-p})\tilde{s}(z)$. Then,

$$\tilde{s}(z) = \frac{z}{z - \alpha(\frac{z}{z-p})} = \frac{z-p}{z-p-\alpha}.$$

Hence,

$$s(\alpha, n) = (\alpha+p)^n - p(p+\alpha)^{n-1} = \alpha(\alpha+p)^{n-1}, \quad n \geq 1. \tag{3.7}$$

It follows that $\alpha \in \mathbb{D} \subset \Omega_{\lambda s}^{N}$, which proves condition (iii) of Theorem 3.3.

On the other hand, note that $f(n, x) := \nu g(n) \cos(h(n)x)$ satisfies hypotheses (i) and (ii) of Theorem 3.3:

(i)

$$f(n+N,\lambda x) = \nu g(n+N)\cos\bigl(h(n+N)\lambda x\bigr) = \nu\lambda g(n)\cos\left(\frac{1}{\lambda}h(n)\lambda x\right)$$
$$= \lambda\nu g(n)\cos\bigl(h(n)x\bigr) = \lambda f(n,x);$$

(ii)

$$\bigl|f(n,x)-f(n,y)\bigr| \le \bigl|\nu g(n)h(n)\bigr||x-y| \le |\nu\tau||x-y| := L|x-y|.$$

Next, we show part (iv) of Theorem 3.3. Indeed, using (3.6) and (3.7) we have that

$$L\sum_{j=0}^{\infty}\bigl|s^{\frown}(\alpha,j)\bigr|$$
$$= |\nu\tau|\left(1+\sum_{j=1}^{\infty}\bigl|\lambda^{\wedge}(-j)\alpha(\alpha+p)^{n-1}\bigr|\right) = |\nu\tau|\left(1+\frac{|\alpha|}{|\lambda|^{1/N}}\sum_{j=1}^{\infty}\left(\frac{|\alpha+p|}{|\lambda|^{1/N}}\right)^{n-1}\right)$$
$$= |\nu\tau|\left(1+\frac{|\alpha|}{|\lambda|^{1/N}-|\alpha+p|}\right) = |\nu\tau|\frac{|\lambda|^{1/N}-|\alpha+p|+|\alpha|}{|\lambda|^{1/N}-|\alpha+p|} < 1.$$

Thus, we have checked all the hypotheses of Theorem 3.3. Hence there exists a unique (N,λ)-periodic discrete solution u of (3.5) satisfying

$$u(n+1) = \nu\sum_{j=-\infty}^{n} s(\alpha,n-j)g(j)\cos\bigl(h(j)u(j)\bigr).$$

Remark 3.5 As a particular case of the previous example, we can consider the functions $h(n) := (1/2)^{n/8}\sin(n\pi/4)$ and $g(n) := (2)^{n/8}\cos(n\pi/4)$.

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