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On the existence of a fundamental total and bounded biorthogonal sequence in every separable Banach space, and related constructions of uniformly bounded orthonormal systems in L^2

by

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Abstract. (1) In every separable Banach space X a biorthogonal sequence (x_n, x_n^*) is constructed such that $\sup_n \|x_n\|\|x_n^*\| < \infty$, the linear combinations of the x_n's are dense in X and, for every x in X, if $x_n^*(x) = 0$, for all n, then $x = 0$.

(2) Linear subspaces of $L^2[0, 1]$ which admit an orthonormal basis consisting of uniformly bounded functions are characterized.

The present paper consists of three sections. In the first one, using a trick invented by Olevskiĭ ([9], Lemmas 3 and 4), we prove

THEOREM 1. *In every separable Banach space X there exists a fundamental and total biorthogonal sequence (x_n, x_n^*) such that*

$$\sup_n \|x_n\| \|x_n^*\| < \infty.$$

Recall that a sequence (x_n, x_n^*) of pairs consisting of elements of a Banach space X and bounded linear functionals on X, i.e. elements of X^* – the dual of X, is said to be *biorthogonal* if $x_n^*(x_m) = \delta_n^m$ for $n, m = 1, 2, \ldots$ A biorthogonal sequence (x_n, x_n^*) is *fundamental* if linear combinations of the x_n's are dense in X, and is *total* if the condition $x_n^*(x) = 0$ for $n = 1, 2, \ldots$ implies that $x = 0$.

Theorem 1 answers a question of Banach ([1], p. 238). A slightly weaker result has previously been obtained by Davis and Johnson [4].

The main result of the second section is

THEOREM 2. *Let E be a separable linear subspace of a Hilbert space $L^2(\mu)$ where μ is a probability measure on a sigma field of subsets of a set S. Then E admits an orthonormal basis consisting of uniformly bounded functions if and only if*

(i) *$E \cap L^\infty(\mu)$ is dense in E in the $L^2(\mu)$ norm,*

(ii) *$E \cap \{f \in L^\infty(\mu): \|f\|_\infty \leq 1\}$ is not a totally bounded subset of $L^2(\mu)$.*

Moreover, if $E\cap L^\infty(\mu)$ is a separable subspace of $L^\infty(\mu)$, then the orthonormal basis can be constructed so that it spans a linear subspace which is dense in the norm $\|\cdot\|_\infty$ in $E\cap L^\infty(\mu)$.

As a corollary we obtain that every subspace of $L^2[0,1]$ of finite codimension admits a uniformly bounded orthonormal basis consisting of trigonometric polynomials. This answers a question of H. Shapiro [14].

In the third section we consider Banach spaces X with the following property

(*) *there exist a compact Hausdorff space S, an isometrically isomorphic embedding $j: X\to C(S)$ and a Borel probability measure μ on S such that the unit ball of $j(X)$ regarded as a subset of $L^2(\mu)$ is not totally bounded.*

Using a recent profound result of Rosenthal [13] we show that a Banach space X has the property (*) if and only if it contains a closed linear subspace isomorphic to the space l^1 of all absolutely convergent series of scalars.

1. Proof of Theorem 1. If A is a non-empty subset of a Banach space X, then $[A]$ denotes the closed linear subspace of X generated by A, and $\operatorname{lin} A$ the linear subspace of X generated by A.

We begin with a lemma which is a modification of Olevskiĭ's Lemma 3 of [9].

LEMMA 1. *Let X be a Banach space and let n be a positive integer. Let $x_0, x_1, \ldots, x_{2^n-1}$ be elements of X and let $x_0^*, x_1^*, \ldots, x_{2^n-1}^*$ be elements of X^* such that $x_p^*(x_q)=\delta_p^q$ for $p, q = 0, 1, \ldots, 2^n-1$.*

Then there exists a unitary real matrix $(a_{k,j}^n)_{0\leqslant k,j<2^n}$ such that if

$$e_k=\sum_{j=0}^{2^n-1} a_{k,j}^n x_j \quad \text{and} \quad e_k^*=\sum_{j=0}^{2^n-1} a_{k,j}^n x_j^* \quad \text{for} \quad k=0,1,\ldots,2^n-1,$$

then

(1) $$\max_{0\leqslant p<2^n}\|e_p\| < (1+\sqrt{2})\max_{1\leqslant j<2^n}\|x_j\| + 2^{-n/2}\|x_0\|,$$

(2) $$\max_{0\leqslant p<2^n}\|e_p^*\| < (1+\sqrt{2})\max_{1\leqslant j<2^n}\|x_j^*\| + 2^{-n/2}\|x_0^*\|,$$

(3) $$e_p^*(e_q)=\delta_p^q \quad \text{for} \quad p, q = 0, 1, \ldots, 2^n-1,$$

(4) $$[\{e_p\}_{0\leqslant p<2^n}] = [\{x_p\}_{0\leqslant p<2^n}]; \quad [\{e_p^*\}_{0\leqslant p<2^n}] = [\{x_p^*\}_{0\leqslant p<2^n}].$$

Proof. Conditions (3) and (4) are satisfied for every unitary $2^n\times 2^n$-matrix. The specific unitary matrix for which (1) and (2) hold is defined to be the matrix which transforms the unit vector basis of the 2^n-dimensional

Hilbert space $l_{2^n}^2$ onto the Haar basis of this space. We put

$$a_{k,0}^n = 2^{-n/2} \quad \text{for} \quad 0\leqslant k<2^n,$$

$$a_{k,2^s+r}^n = \begin{cases} 2^{(s-n)/2} & \text{for} \quad 2^{n-s-1}2r\leqslant k<2^{n-s-1}(2r+1), \\ -2^{(s-n)/2} & \text{for} \quad 2^{n-s-1}(2r+1)\leqslant k<2^{n-s-1}(2r+2), \\ 0 & \text{for} \quad k<2^{n-s-1}2r \text{ and for } k\geqslant 2^{n-s-1}(2r+2) \end{cases}$$

$$(s=0,1,\ldots,n-1;\ r=0,1,\ldots,2^s-1).$$

We have

(5) $$\sum_{j=1}^{2^n-1}|a_{k,j}^n| = \sum_{s=0}^{n-1} 2^{-(n-s)/2} < 1+\sqrt{2} \quad \text{for} \quad 0\leqslant k<2^n.$$

Clearly, (5) implies (1) and (2).

PROPOSITION 1. *Let (x_n, x_n^*) be a fundamental and total biorthogonal sequence in a Banach space X such that there exists an increasing infinite sequence (n_k) such that $\sup_k \|x_{n_k}\|\|x_{n_k}^*\| = M<\infty$.*

Then there exists a fundamental and total biorthogonal sequence (e_n, e_n^*) in X such that

$$\sup_n \|e_n\|\|e_n^*\| \leqslant M(1+\sqrt{2})^2+1$$

and

$$\operatorname{lin}\{e_n\}_{n=1}^\infty = \operatorname{lin}\{x_n\}_{n=1}^\infty \quad \text{and} \quad \operatorname{lin}\{e_n^*\}_{n=1}^\infty = \operatorname{lin}\{x_n^*\}_{n=1}^\infty.$$

Proof. Without loss of generality one may assume that $\|x_n\|=1$ for all n. Pick a permutation $p(\cdot)$ of the indices and an increasing sequence (m_r) of the indices so that if $\tilde{x}_n = x_{p(n)}$ and $\tilde{x}_n^* = x_{p(n)}^*$ for all n and $q_r = \sum_{p=0}^{r} 2^{m_p}$ for all r, then

if $n\neq q_r$ for all r, then $\|\tilde{x}_n\|\|\tilde{x}_n^*\|\leqslant M$,

if $n=q_r$ for some $r=0,1,\ldots$, then

$$(1+\sqrt{2})^2M+1 > [(1+\sqrt{2})M+\|\tilde{x}_n^*\|2^{-m_r/2}][(1+\sqrt{2})+\|\tilde{x}_n\|2^{-m_r/2}].$$

Next put

$$e_n=\tilde{x}_n \quad \text{and} \quad e_n^*=\tilde{x}_n^* \quad \text{for} \quad n<2^{m_0},$$

$$e_{k+q_{r-1}} = \sum_{j=0}^{2^{m_r}-1} a_{k,j}^{m_r}\tilde{x}_{j+q_{r-1}}; \quad e_{k+q_{r-1}}^* = \sum_{j=0}^{2^{m_r}-1} a_{k,j}^{m_r}\tilde{x}_{j+q_{r-1}}^*$$

$$\text{for} \quad 0\leqslant k<2^{m_r};\ r=1,2,\ldots$$

where $a_{k,j}^{m_r}$ are defined as in Lemma 1 for $n=m_r$. Using Lemma 1, we easily verify that the sequence (e_n, e_n^*) has the desired properties.

Proof of Theorem 1. We shall assume that $\dim X = \infty$. Then the separability of X implies that there exist sequences $E_1 \subset E_2 \subset \ldots$ of subspaces of X and $F_1 \subset F_2 \subset \ldots$ of subspaces of X^* such that $\dim E_i = \dim F_i = i$ for $i = 1, 2, \ldots, \bigcup_{i=1}^{\infty} E_i$ is dense in X and if $f^*(x) = 0$, for all $f^* \in \bigcup_{i=1}^{\infty} F_i$, then $x = 0$. In view of Proposition 1, it is enough to construct a biorthogonal sequence (x_n, x_n^*) in X such that if $G_n = [x_1, x_2, \ldots, x_n]$ and $H_n = [x_1^*, x_2^*, \ldots, x_n^*]$ then for all s

$$G_{3s-2} \supset E_s; \quad H_{3s-1} \supset F_s; \quad \|x_{3s}\|\,\|x_{3s}^*\| \leqslant 3. \tag{6}$$

Pick $x_1 \in X$ and $x_1^* \in X^*$ so that $0 \neq x_1 \in E_1$ and $x_1^*(x_1) = 1$. Assume that, for some $n-1 \geqslant 1$, the elements $x_1, x_2, \ldots, x_{n-1}$ in X and the functionals $x_1^*, x_2^*, \ldots, x_{n-1}^*$ in X^* have been defined to satisfy (6) and so that $x_p^*(x_q) = \delta_p^q$ for $p, q = 1, 2, \ldots, n-1$. We consider separately three cases.

Case 1: $n = 3s-2$. If $G_{n-1} \supset E_s$ we define $x_n \in X$ and $x_n^* \in X^*$ arbitrarily, so that

$$x_n^*(x_q) = \delta_n^q \quad \text{and} \quad x_p^*(x_n) = \delta_p^n \quad \text{for} \quad p, q = 1, 2, \ldots, n.$$

If $E_s \setminus G_{n-1}$ is non-empty, say $e \in E_s \setminus G_{n-1}$, then we put

$$x_n = e - \sum_{p=1}^{n-1} x_p^*(e) x_p \quad \text{and} \quad G_n = [G_{n-1} \cup \{x_n\}].$$

Clearly, $x_n \neq 0$. Since $\dim E_s = \dim E_{s-1} + 1$ and $e \in G_n \setminus E_{s-1}$ and since the inductive hypothesis implies that $E_{s-1} \subset G_{n-1}$, we infer that $G_n \supset E_s$. Since $x_n \in G_n \setminus G_{n-1}$, there exists a bounded linear functional on G_n, say g^*, such that $g^*(x_n) = 1$ and $g^*(g) = 0$ for $g \in G_{n-1}$. We define x_n^* to be any extension of g^* to a bounded linear functional on X.

Case 2: $n = 3s-1$. If $H_{n-1} \supset F_s$ we define $x_n \in X$ and $x_n^* \in X^*$ arbitrarily so that $x_n^*(x_q) = \delta_n^q$ and $x_p^*(x_n) = \delta_p^n$ for $p, q = 1, 2, \ldots, n$. If $F_s \setminus H_{n-1}$ is non-empty, say $f^* \in F_s \setminus H_{n-1}$, then we put

$$x_n^* = f^* - \sum_{q=1}^{n-1} f^*(x_q) x_q^*.$$

Since $f^* \notin H_{n-1}$, there exists an $x \in X$ such that

$$1 = f^*(x) \neq \sum_{q=1}^{n-1} f^*(x_q) x_q^*(x).$$

We put $x_n = x - \sum_{p=1}^{n-1} x_p^*(x) x_p$. It is easy to check that $x_n^*(x_q) = \delta_n^q$ and $x_p^*(x_n) = \delta_p^n$ for $p, q = 1, 2, \ldots, n$. Let $H_n = [H_{n-1} \cup \{x_n^*\}]$. Since the inductive hypothesis implies that $F_{s-1} \subset H_{n-1}$ and since $\dim F_s = \dim F_{s-1} + 1$ and $f^* \in F_s \setminus F_{s-1}$, we infer that $H_n \supset F_s$.

Case 3: $n = 3s$. Using Mazur's technique (cf. [10], Lemma) we pick an $x_n \in X$ with $\|x_n\| = 1$ so that $x^*(x_n) = 0$ for every $x^* \in H_{n-1}$ and, for all g in G_{n-1} and for all scalars t, $\|g + tx_n\| \geqslant (1 - \frac{1}{3})\|g\|$. Define g^* on G_n by $g^*(g + tx_n) = t$. Then

$$|t| = \|tx_n\| \leqslant \|g + tx_n\| + \|g\| \leqslant (1 + \tfrac{3}{2})\|g + tx_n\|.$$

Thus $\|g^*\| \leqslant 3$. We define x_n^* to be any norm preserving extension of g^* to a linear functional on X.

Remark 1. Using in Case 3 Day's technique (cf. [3]) which bases on the Borsuk antipodal mapping theorem one can choose (both in the case of real and of complex scalars) x_{3s} and x_{3s}^* so that

$$\|x_{3s}\| = \|x_{3s}^*\| = x_{3s}^*(x_{3s}) = 1 \quad \text{for} \quad s = 1, 2, \ldots$$

Now the inspection of the proof of Theorem 1 yields that in every separable Banach space for every $\varepsilon > 0$ there exists a fundamental total and bounded biorthogonal sequence (e_n, e_n^*) such that $\|e_n\|\,\|e_n^*\| < (1 + \sqrt{2})^2 + \varepsilon$ for all n. However, as it was observed by C. Bessaga, we have

Corollary 1. *Every separable Banach space X admits an equivalent norm $|||\cdot|||$ such that there exists in X a fundamental and total biorthogonal sequence (e_n, e_n^*) with $|||e_n|||\cdot|||e_n^*||| = 1$.*

Proof. We admit $|||x||| = \max(\|x\|, \sup_n |e_n^*(x)|)$ for $x \in X$ where (e_n, e_n^*) is any fundamental and total biorthogonal sequence in X such that $\|e_n\| = 1$ for all n and $\sup_n \|e_n^*\| < \infty$.

Remark 2. A similar argument to that which was used in the proof of Theorem 1 allows us to prove the following

Theorem 1'. *Let X and Y be Banach spaces and let $T\colon X \to Y$ be a one-to-one bounded linear operator. If X is separable, $T(X)$ is dense in Y and T is not compact, then there exist fundamental and total biorthogonal sequences (x_n, x_n^*) in X and (y_n, y_n^*) in Y such that*

$$\sup_n \max(\|x_n\|\,\|x_n^*\|, \|y_n\|\,\|y_n^*\|) < \infty \quad \textit{and} \quad T(x_n) = y_n \quad \textit{for all } n.$$

2. Constructions of uniformly bounded orthonormal sequences. We employ the following notation. If μ is a probability measure (= a non-negative normalized measure) on a sigma field of subsets of a set S then

$$\langle x, y\rangle = \int_S x(s)\overline{y(s)}\mu(ds),$$

$$\|x\|_2 = \langle x, x\rangle^{1/2} \quad \text{and} \quad \|x\|_\infty = \inf_{\mu(B)=1} \sup_{s \in B} |x(s)|$$

for any μ-absolutely square summable scalar valued functions x and y on S. $L^\infty(\mu)$ and $L^2(\mu)$ denote as usually the Banach spaces of those x that $\|x\|_\infty < \infty$ and $\|x\|_2 < \infty$, respectively.

The proof of Theorem 2 is similar to the proof of Theorem 1. Instead of Proposition 1 we apply the following result due to Olevskiĭ ([9], Lemma 4).

PROPOSITION 2. *Let μ be a probability measure on a sigma field of subsets of a set S. Let (x_n) be an infinite orthonormal (with respect to the inner product $\langle , \rangle$) sequence of functions in $L^\infty(\mu)$ such that $\liminf_n \|x_n\|_\infty < \infty$. Then there exists an orthonormal sequence (e_n) such that*

$$\operatorname{lin}\{x_n\}_{n=1}^\infty = \operatorname{lin}\{e_n\}_{n=1}^\infty \quad \text{and} \quad \sup_n \|e_n\|_\infty < \infty.$$

The proof of Proposition 2 can be obtained by a non-essential modification of the proofs of Lemma 1 and Proposition 1.

To prove Theorem 2 it is convenient to use the following simple fact.

LEMMA 2. *Let (g_n) be a normalized sequence in $L^2(\mu)$ which weakly in $L^2(\mu)$ converges to zero and let $\sup_n \|g_n\|_\infty = M < \infty$. Then for every finite dimensional subspace of $L^\infty(\mu)$, say F, and for $k > 0$ there exist an index $n_0 > k$ and a function h in the orthogonal complement of F such that*

$$[F \cup \{g_{n_0}\}] = [F \cup \{h\}], \quad \|h\|_2 = 1 \quad \text{and} \quad \|h\|_\infty < M + 2^{-k}.$$

Proof. Let $p = \dim F$. Let $e_1, e_2, \ldots, e_p$ be any orthonormal basis for F. Pick $\varepsilon > 0$ so that

$$\frac{M + \varepsilon \sum_{j=1}^{p} \|e_j\|_\infty}{1 - \varepsilon p} < M + 2^{-k}.$$

Since (g_n) converges weakly to 0 in $L^2(\mu)$, there exists an index $n_0 > k$ such that $|\langle g_{n_0}, e_j\rangle| < \varepsilon$ for $1 \leqslant j \leqslant p$. Put

$$h = \Big(g_{n_0} - \sum_{j=1}^{p} \langle g_{n_0}, e_j\rangle e_j\Big) \Big\|g_{n_0} - \sum_{j=1}^{p} \langle g_{n_0}, e_j\rangle e_j\Big\|_2^{-1}.$$

Clearly, h belongs to the orthogonal complement of F, $\|h\|_2 = 1$ and $[F \cup \{g_{n_0}\}] = [F \cup \{h\}]$. We have

$$\Big\|g_{n_0} - \sum_{j=1}^{p} \langle g_{n_0}, e_j\rangle e_j\Big\|_\infty \leqslant \|g_{n_0}\|_\infty + \Big\|\sum_{j=1}^{p} \langle g_{n_0}, e_j\rangle e_j\Big\|_\infty \leqslant M + \varepsilon \sum_{j=1}^{p} \|e_j\|_\infty$$

and

$$\Big\|g_{n_0} - \sum_{j=1}^{p} \langle g_{n_0}, e_j\rangle e_j\Big\|_2 \geqslant \|g_{n_0}\|_2 - \Big\|\sum_{j=1}^{p} \langle g_{n_0}, e_j\rangle e_j\Big\|_2 \geqslant 1 - \varepsilon p.$$

Thus

$$\|h\|_\infty \leqslant \Big(M + \varepsilon \sum_{j=1}^{p} \|e_j\|_\infty\Big)(1 - \varepsilon p)^{-1} < M + 2^{-k}.$$

Proof of Theorem 2. It follows from (i) that there exists in E an increasing sequence of finite dimensional subspaces $F_1 \subset F_2 \subset \ldots$ such that $\dim F_p = p$ and $\bigcup_{p=1}^\infty F_p$ is dense in E. Clearly, if $E \cap L^\infty(\mu)$ is a separable subset of $L^\infty(\mu)$ one can choose the sequence (F_p) so that the union $\bigcup_{p=1}^\infty F_p$ is dense in $E \cap L^\infty(\mu)$ in the $L^\infty(\mu)$ norm. Condition (ii) yields that there exists in E a sequence (g_n) satisfying the assumption of Lemma 2. In view of Proposition 2 it is enough to define inductively an orthonormal sequence (h_n) in $L^\infty(\mu) \cap E$ so that, for $s = 1, 2, \ldots$,

$$[\{h_1, h_2, \ldots, h_{2s-1}\}] \supset F_s, \tag{7}$$

$$\|h_{2s}\|_\infty < M + 2^{-s} \quad \text{where} \quad M = \sup_n \|g_n\|_\infty. \tag{8}$$

We define h_1 as any element of F_1 with $\|h_1\|_2 = 1$. Suppose that for some $n-1 \geqslant 1$ the functions $h_1, h_2, \ldots, h_{n-1}$ have been defined to satisfy the conditions (7) and (8) and so that $\langle h_p, h_q\rangle = \delta_p^q$ for $p, q = 1, 2, \ldots, n-1$. Let us consider separately two cases.

Case 1: $n = 2s$ for some $s = 1, 2, \ldots$ We put $h_n = h$ where h is that of Lemma 2 applied for $F = [\{h_1, h_2, \ldots, h_{n-1}\}]$ for (g_p) and for $k = s$.

Case 2: $n = 2s-1$ for some $s = 2, 3, \ldots$ If $F_s \subset [\{h_1, h_2, \ldots, h_{n-1}\}]$, we again define $h_n = h$ where h is that of Lemma 2 applied for $F = [\{h_1, h_2, \ldots, h_{n-1}\}]$ for (g_p) and for $k = 1$. If $F_m \not\subset [\{h_1, \ldots, h_{n-1}\}]$, then there exists an f which belongs to $F_s \setminus [\{h_1, h_2, \ldots, h_{n-1}\}]$. Let $\tilde{f}$ be the orthogonal projection of f onto $[\{h_1, h_2, \ldots, h_{n-1}\}]$. We put $h_n = (f - \tilde{f})\|f - \tilde{f}\|_2^{-1}$. Clearly, $\|h_n\|_2 = 1$ and h_n belongs to the orthogonal complement of $[\{h_1, h_2, \ldots, h_{n-1}\}]$. Obviously, we have $f \in [\{h_1, h_2, \ldots, h_n\}] \setminus \setminus [\{h_1, h_2, \ldots, h_{n-1}\}]$. By the inductive hypothesis, $F_{s-1} \subset [\{h_1, h_2, \ldots \ldots, h_{n-1}\}]$. Thus, $F_s \subset [\{h_1, h_2, \ldots, h_n\}]$ because $\dim F_s = \dim F_{s-1} + 1$.

This completes the induction and the proof of the sufficiency of conditions (i) and (ii). The necessity is trivial.

Remark 1. A similar argument gives

THEOREM 2'. *Let $T\colon X \to H$ be a one-to-one bounded linear operator from a Banach space X into a Hilbert space H. Let $E = T(X)$. If E is separable and T is not compact, then there exists a sequence (x_n) in X such that $\sup_n \|x_n\| < \infty$ and $(T(x_n))$ is an orthonormal basis for E.*

Moreover, if X is separable and $x_n^* \in X^*$ is defined by $x_n^*(x) = \langle T(x), T(x_n)\rangle_H$ for $x \in X$ and for $n = 1, 2, \ldots$, where $\langle \cdot, \cdot \rangle_H$ denotes the inner product of H, then (x_n) can be chosen so that (x_n, x_n^*) is a fundamental and total biorthogonal sequence in X and $\sup_n \|x_n\| \|x_n^*\| < \infty$.

Remark 2. There exists an orthonormal decomposition of $L^2[0, 1]$ onto subspaces E_1 and E_2 such that neither E_1 nor E_2 admit uniformly bounded orthonormal bases. It is enough to define $E_1 = [\{x_1\} \cup \{x_{2m}\}_{m=2}^{\infty}]$ and $E_2 = [\{x_2\} \cup \{x_{2m-1}\}]_{m=2}^{\infty}$ where (x_n) is any orthonormal basis for $L^2[0, 1]$ such that the functions x_1 and x_2 are unbounded, $x_{2m-1}(t) = 0$ for $0 \leqslant t < \frac{1}{2}$ and $x_{2m}(t) = 0$ for $\frac{1}{2} < t \leqslant 1$ $(m = 1, 2, \ldots)$. However, as was observed earlier by F. G. Arutunian (unpublished), we have

Corollary 2. *If E is a linear subspace of a separable space $L^2(\mu)$ where μ is a non-purely atomic probability measure and if the orthogonal complement of E is finite dimensional, then $[E]$ has a uniformly bounded orthonormal basis.*

Moreover, if $E \cap L^\infty(\mu)$ is dense in E then the basis can be chosen from elements of $E \cap L^\infty(\mu)$.

Proof. It is enough to show that $[E]$ satisfies conditions (i) and (ii) of Theorem 2. To check (i), first observe that the density of $L^\infty(\mu)$ regarded as a subspace of $L^2(\mu)$ in $L^2(\mu)$ implies that for every positive integer p and for every linearly independent $f_1, f_2, \ldots, f_{p+1}$ in $L^2(\mu)$ there exist $y_1, y_2, \ldots, y_{p+1}$ in $L^\infty(\mu)$ such that the matrix $(y_k, f_j)_{1 \leqslant k, j \leqslant p+1}$ is invertible. Let $(a_{i,k})_{1 \leqslant i,k \leqslant p+1}$ be the inverse matrix and let $z_i = \sum_{k=1}^{p+1} a_{i,k} y_k$ for $i = 1, 2, \ldots, p+1$. Then $z_i \in L^\infty(\mu)$ and $\langle z_i, f_j\rangle = \delta_i^j$ for $i, j = 1, 2, \ldots, p+1$. The above observation applied to any basis of the orthogonal complement of E and any non-zero element f of $[E]$ yields the existence of an y in $L^\infty(\mu)$ such that $\langle y, f\rangle = 1$ and $\langle y, g\rangle = 0$ for all g in the orthogonal complement of E. The last condition means that $y \in [E]$. Hence there is no $f \neq 0$ in $[E]$ which is orthogonal to all $y \in [E] \cap L^\infty(\mu)$, equivalently, $[E] \cap L^\infty(\mu)$ is dense in $[E]$. Hence $[E]$ satisfies (i).

Let P denote the orthogonal projection from $L^2(\mu)$ onto $[E]$, I the identity operator on $L^2(\mu)$, and I_μ: $L^\infty(\mu) \to L^2(\mu)$ the natural injection. I_μ is not compact because μ is not purely atomic, while $(I-P)I_\mu$ is compact because the orthogonal complement of E is finitely dimensional. Thus, PI_μ is not compact, equivalently, $[E]$ satisfies (ii).

The "moreover" part of the corollary follows from the observation that in this case if $[E]$ satisfies (ii) then E also satisfies (ii).

An immediate consequence of Corollary 2 is

Corollary 3. *Let f be any unbounded function in $L^2[0, 1]$. Then the orthogonal complement of f admits a uniformly bounded orthonormal basis consisting of trigonometric polynomials. This basis has no extension to any uniformly bounded orthonormal basis for $L^2[0, 1]$.*

Corollary 3 answers a question of Shapiro [14].

3. Fat subspaces of $C(S)$ spaces.

Definition. Let μ be a probability Borel measure on a compact Hausdorff space S. A closed linear subspace Z of $C(S)$ is said to be *fat with respect to* μ if the unit ball of Z regarded as a subset of the Hilbert space $L^2(\mu)$ is not totally bounded.

Let I_μ: $L^\infty(\mu) \to L^2(\mu)$ denote the natural injection. It is clear that Z is fat with respect to μ iff the restriction of I_μ to Z is not a compact operator or, equivalently, if $E = I_\mu(Z)$ satisfies condition (ii) of Theorem 2.

Our next result characterizes Banach spaces which admit fat isometric embeddings into $C(S)$ spaces. Some of the equivalent conditions are stated in terms of 2-absolutely summing operators, i.e. such bounded linear operators which admit a factorization through a natural injection I_μ for some measure μ (cf. [12] and [8]).

Proposition 3. *For every Banach space X the following conditions are equivalent:*

(a) *there exists a uniformly bounded sequence (x_n) of elements of X such that no subsequence of (x_n) is a weak Cauchy sequence,*

(b) *X contains a subspace isomorphic to l^1,*

(c) *there exists a 2-absolutely summing operator from X onto l^2,*

(d) *there exists a 2-absolutely summing non-compact operator from X into l^2,*

(e) *for every isometric embedding j of X into a $C(S)$ space there exists a probability Borel measure μ on S such that $j(X)$ is fat with respect to μ,*

(f) *for some isometric embedding j of X into a $C(S)$ space there exists a probability Borel measure μ on S such that $j(X)$ is fat with respect to μ.*

Proof. (a) $\Rightarrow$ (b). This is a profound recent result of Rosenthal [13].

(b) $\Rightarrow$ (c). Let T be a bounded linear operator from l^1 onto l^2 (cf. [2] for the existence of such operators). Then, by a result of Grothendieck [7] (cf. also [8]), T is 2-absolutely summing. Hence, by [12], T admits an extension to a 2-absolutely summing operator from X onto l^2.

(c) $\Rightarrow$ (d). Obvious.

(d) $\Rightarrow$ (e). Let T: $X \to l^2$ be a non-compact 2-absolutely summing operator and let S be a compact Hausdorff space. By a result of Persson and Pietsch [11], for every isometric embedding j: $X \to C(S)$ there exists a Borel probability measure μ on S such that $T = AI_\mu j$ for some bounded linear operator $A: L^2(\mu) \to l^2$. Since T is non-compact, the image of the unit ball of $j(X)$ under I_μ is not a totally bounded subset of $L^2(\mu)$. Thus, $j(X)$ is a fat subspace of $C(S)$ with respect to μ.

(e) $\Rightarrow$ (f). Obvious.

(f) $\Rightarrow$ (a). It follows from (f) that there exists a uniformly bounded sequence (x_n) in X such that $\|I_\mu j(x_n) - I_\mu j(x_m)\|_2 \geqslant 1$ for $n \neq m$ $(n, m = 1, 2, \ldots)$. Thus the sequence (x_n) does not contain weak Cauchy sequences because I_μ takes weak Cauchy sequences into strong Cauchy sequences.

A similar result to our Proposition 3 was recently independently discovered by Weis [16].

Our last result is related to Gaposhkin's [6] generalization of a result of Sidon [15].

Corollary 4. *Let μ be a probability measure on a sigma field of subsets of S. Let (g_n) be a uniformly bounded sequence in $L^\infty(\mu)$ such that (g_n) tends weakly to zero in $L^2(\mu)$ and $\limsup_n \|g_n\|_2 > 0$. Then there exists an infinite subsequence (g_{n_k}) and $c > 0$ such that*

$$\Big\|\sum_{k=1}^{p} c_k g_{n_k}\Big\|_\infty > c \sum_{k=1}^{p} |c_k|$$

for every finite sequence of scalars $c_1, c_2, \ldots, c_p$ $(p = 1, 2, \ldots)$.

Proof. Without loss of generality we may assume that $\inf_n \|g_n\|_2 > 0$. Then (g_n) does not have Cauchy (in $L^2(\mu)$) subsequences because (g_n) weakly converges in $L^2(\mu)$ to zero but no subsequence of (g_n) strongly converges to zero. Thus (g_n) regarded as a sequence of elements of $L^\infty(\mu)$ does not contain weak (in $L^\infty(\mu)$) Cauchy sequences because the natural injection I_μ: $L^\infty(\mu) \to L^2(\mu)$ takes weak Cauchy sequences in $L^\infty(\mu)$ into strong Cauchy sequences in $L^2(\mu)$. Since $\sup_n \|g_n\|_\infty < \infty$, to complete the proof it is enough to apply Rosenthal's criterion (cf. Rosenthal [13] for the real case, and Dor [5] for the complex case).

Added in proof. Since the completion of the present paper the second named author proved that in every separable Banach space, for every $\varepsilon > 0$, there exists a fundamental total and bounded by $1+\varepsilon$ biorthogonal sequence (cf. [17]).

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