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ON THE EXISTENCE OF A GENERALIZED SOLUTION TO A THREE-DIMENSIONAL ELLIPTIC EQUATION WITH RADIATION BOUNDARY CONDITION*

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Abstract. For a second order elliptic equation with a nonlinear radiation-type boundary condition on the surface of a three-dimensional domain, we prove existence of generalized solutions without explicit conditions (like $u\big|_{\Gamma} \in L_5(\Gamma)$) on the trace of solutions. In the boundary condition, we admit polynomial growth of any fixed degree in the unknown solution, and the heat exchange and emissivity coefficients may vary along the radiating surface.

Our generalized solution is contained in a Sobolev space with an exponent q which is greater than $9/4$ for the fourth power law.

Keywords: radiation boundary condition, generalized solution, existence

MSC 2000: 35D05, 35J65

1. Introduction

Second order elliptic equations with a nonlinear radiation boundary condition have been considered in quite a number of papers. Classical solutions for the multidimensional case have been investigated in [4], and, under certain conditions, existence, unicity and stability of positive solutions have been shown. [6] considers mainly the parabolic case (but derives also results for elliptic equations) for a much broader class of functions figuring in the boundary condition than [4]. For more literature on classical solutions see the references in both the papers.

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For the foundation of finite element methods, weak solutions have been considered for the same problem, e.g. in [15], [12] for the two-dimensional case. Research on weak solutions for the three-dimensional case was restricting the Sobolev space in which the solution was searched by requiring the trace of the solution on the boundary to be in the space $L_5(\Gamma)$, see [5]. The point is that—as revealed by the imbedding theorems—in three dimensions (as opposed to the case of two dimensions) the trace of an H^1-function is in general not contained in an L_q space convenient for the investigation. The line of [5] has been continued in the recent paper [11] in which the authors investigate also finite element solutions under such assumptions.

For the research presented in this paper, the motivation was [7] where the boundary condition for temperature would have been more realistic if it were of radiation type.

We will prove existence of a generalized solution without explicit conditions on the trace of the solution. To do so, we add, however, to the radiation boundary condition a linear term in order to be able to guarantee the unicity of the corresponding linear problem. Physically, this addition corresponds to the admission of a convective heat transfer from the surface which anyway takes place but may be negligible for sufficiently high temperatures.

Our main result is the existence of a generalized solution in a Sobolev space $W^{2,q}(\Omega)$, for q sufficiently large, the space being defined on a general bounded and smooth three-dimensional domain Ω, for a general nonlinear Neumann-type boundary condition (with polynomial growth of any fixed degree in the unknown solution, and admitting the heat exchange and emissivity coefficients to vary along the radiating surface), and for sufficiently small σT_0^3, where σ is the Stefan-Boltzmann constant and T_0 the temperature of the surrounding medium.

This result is apparently unknown and therefore of theoretical interest. Moreover, we think that our result is interesting also from the numerical point of view (in spite of the exponent of the Sobolev space being different from 2 and in fact greater than 9/4 for the fourth power law), since under such circumstances estimates for numerical solutions are known in literature, see, e.g., [2], Chapter 7, and (for the interpolation error) [3], Chapter 3.

Consider a bounded open domain $\Omega \subset \mathbb{R}^3$ with a boundary $\partial\Omega = \Gamma$ which is supposed to be of class $C^{1,1}$ (i.e. continuously differentiable, and its derivatives are Lipschitz continuous, see Def. 1.2.1.1 in [8]).

In Ω the following boundary value problem is given in which the derivatives are understood in the generalized sense:

$$Au = -\operatorname{div}(a \operatorname{grad} u) = 0, \quad x = (x_1, x_2, x_3) \in \Omega, \tag{1}$$

$$a(x)\frac{\partial u}{\partial n} + h(x)u = \varphi(x) - bd(x,u), \quad x \in \Gamma. \tag{2}$$

Here d is a continuously differentiable function of its arguments with polynomial growth in u (more specific conditions will be formulated below). Moreover, b is a small positive constant, a is a Lipschitz continuous function of $x \in \overline{\Omega}$ and bounded away from 0: $a(x) \geqslant a_0 = \mathrm{const} > 0$.

Further, n is the unit outward normal vector to Γ, h is Lipschitz continuous on Γ, nonnegative but not identically zero, and finally, φ is sufficiently smooth (see below) on Γ.

In (1), we could have added a source term f (anticipating the definition of the solution u below: $f \in L_q(\Omega)$ for some q), but we will concentrate here on considering the nonlinear boundary condition (2). Concerning the physical sense of this boundary condition, we remark that Γ is assumed to be a surface on which the convective and radiational heat exchange takes place. The continuity of the heat flux on Γ means

$$a\frac{\partial T}{\partial n} = q_\Gamma(T),$$

where T is the absolute temperature of the body considered and q_Γ is the heat flux on its surface, i.e. the sum of the convective and radiational heat flux (per unit surface area). Hence, we have (see, e.g., [9], p. 546)

$$q_\Gamma(T) = \alpha(T_0 - T) + \varepsilon\sigma(T_0^4 - T^4),$$

with α, ε and σ being, respectively, the heat exchange coefficient, the emissivity coefficient and the Stefan-Boltzmann constant (the last one being approximately $5.67 \cdot 10^{-8}$ if measured in $\mathrm{W}/(\mathrm{m}^2\mathrm{K}^4)$). Norming the temperature T to the (constant) temperature T_0 outside the radiating body, we obtain the function $u := T/T_0$ sought in (1), and for the second kind boundary condition we get

$$\begin{aligned} a\frac{\partial u}{\partial n} &= q_\Gamma(uT_0)/T_0 = \alpha(1-u) + \varepsilon\sigma T_0^3(1-u^4) \\ &= (1-u)(\alpha + \varepsilon\sigma T_0^3(1+u+u^2+u^3)). \end{aligned} \tag{3}$$

From here it is seen that in (2) we allow the heat transfer and emissivity coefficients to vary along the surface, having

$$h(x) = \alpha(x),\;\; \varphi(x) = \alpha(x) + b\varepsilon(x),\;\; b = \sigma T_0^3,\;\; d(x,u) = \varepsilon(x)u^4. \tag{4}$$

The specific forms (3) and (4) of the boundary condition illustrate the more general form (2) on which we focus our attention.

The generalized solution of (1)–(2) will be defined as an element of a Sobolev space $W^{2,q}(\Omega)$ with an exponent $\frac{3}{2} \leqslant q < \infty$ which will be chosen sufficiently large. For details on Sobolev function spaces see [1] and [8]. As is well known, see [8], Theorem 1.5.1.2, the trace mapping

$$W^{1,q}(\Omega) \to W^{1-1/q,q}(\Gamma)$$

is continuous. Hence

$$a\frac{\partial}{\partial n}\colon\; W^{2,q}(\Omega) \to W^{1-1/q,q}(\Gamma)$$

is well defined and continuous with respect to $u \in W^{2,q}(\Omega)$.

3. Existence in $W^{2,q}(\Omega)$ of a solution to the linear problem

Introducing the notation $\psi(x) := \varphi(x) - bd(x,u(x))$, instead of (1)–(2) we will consider the linear problem (1), (5), where

$$a(x)\frac{\partial u}{\partial n} + h(x)u = \psi(x), \quad x \in \Gamma. \tag{5}$$

Our aim in what follows is the selection of $\psi \in W^{1-1/q,q}(\Gamma)$ such that the solution of (1), (5) exists uniquely and, moreover, is a solution of (1), (2) as well.

For this we remark, first of all, that (1), (5) cannot possess more than one solution u in $W^{2,q}$. Indeed, if this linear problem admitted two solutions u_1 and u_2 then $v := u_1 - u_2$ would be a solution of the corresponding homogeneous problem. Thus, multiplying the obtained equation for v in the L_2 scalar product by v we would obtain (observe that for $q \geqslant \frac{3}{2}$ and in our case of three dimensions, for $u \in W^{2,q}(\Omega)$ we have $\operatorname{grad} u \in (L_2(\Omega))^3$)

$$\begin{aligned}0 = \int_\Omega (Av)v\,\mathrm{d}x &= \int_\Omega a(x)|\operatorname{grad} v|^2\,\mathrm{d}x - \int_\Gamma a(s)\frac{\partial v}{\partial n}v\,\mathrm{d}s\\ &= \int_\Omega a(x)|\operatorname{grad} v|^2\,\mathrm{d}x + \int_\Gamma h(s)v^2(s)\,\mathrm{d}s.\end{aligned}$$

From here we get $v = 0$ in $W^{2,q}$ since v must be constant in any case due to $h \geqslant 0$; but $h \not\equiv 0$ shows that this constant must be zero.

Next observe that (1), (5) is selfadjoint [10], that is, the adjoint problem also has at most one solution. Then we may refer to [8], Lemma 2.4.2.1, for the fact that our linear problem possesses a solution in $W^{2,q}(\Omega)$ for boundary data $\psi \in W^{1-1/q,q}(\Gamma)$, and this solution satisfies the estimate

$$\|u\|_{W^{2,q}(\Omega)} \leqslant c_0 \|\psi\|_{W^{1-1/q,q}(\Gamma)}, \tag{6}$$

where for the $W^{2,q}$ norm we employ the expression

$$\|u\|_{W^{2,q}(\Omega)} = \|u\|_{q,\Omega} + \sum_{k=1}^{3} \left\| \frac{\partial u}{\partial x_k} \right\|_{q,\Omega} + \sum_{k,\ell=1}^{3} \left\| \frac{\partial^2 u}{\partial x_k \partial x_\ell} \right\|_{q,\Omega}.$$

Here and below, the standard $L_q(\Omega)$ norms are denoted shortly by $\|\cdot\|_{q,\Omega}$.

4. The fixed point problem

We denote the solution u of (1), (5) by $u = F(\psi)$. By construction, $u = F(\psi) \in W^{2,q}(\Omega)$ will solve the original problem (1)–(2) iff

$$\psi = \varphi - bd(\cdot, F(\psi))\big|_{\Gamma}. \tag{7}$$

In other terms, ψ must be a fixed point of the mapping

$$\Phi\colon\ W^{1-1/q,q}(\Gamma) \to W^{1-1/q,q}(\Gamma) \tag{8}$$

defined by

$$\Phi(\psi) := \varphi - bd(\cdot, F(\psi))\big|_{\Gamma}. \tag{9}$$

Using similar arguments as in [13]–[14], we will show that the Schauder fixed point theorem can be applied to (8), (9).

Theorem 1. *For q sufficiently large and for any $\varphi \in W^{1-1/q,q}(\Gamma)$, the mapping* (8)–(9) *is continuous and compact.*

P r o o f. First, from (6) and the linearity of (1), (5) we see that F is a linear continuous mapping for $\psi \in W^{1-1/q,q}(\Gamma)$:

$$F\colon\ W^{1-1/q,q}(\Gamma) \to W^{2,q}(\Omega).$$

Next we clarify under which conditions the mapping

$$u \to d(\cdot, u) \tag{10}$$

is a continuous and compact mapping from $W^{2,q}(\Omega)$ into $W^{1,q}(\Omega)$, see (12)–(15) below. Taking this for granted for a moment, we may conclude—since the trace mapping

$$W^{1,q}(\Omega) \to W^{1-1/q,q}(\Gamma)$$

is continuous—that

$$u \to d(\cdot, u)\big|_{\Gamma}$$

is a continuous and compact mapping from $W^{2,q}(\Omega)$ into $W^{1-1/q,q}(\Gamma)$. Hence, for $\varphi \in W^{1-1/q,q}(\Gamma)$, (9) is also continuous and compact.

Therefore we now consider $u \in W^{2,q}(\Omega)$. We remark first that the Rellich-Kondrashov theorem implies that the imbedding $W^{2,q}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ is compact for r appropriately chosen; specifically, from [1], p. 144, we have this property under the following conditions:

$$\text{if } 3 \leqslant q \text{ then for all finite } r, \quad \text{if } 3 > q \text{ then for } r < \frac{3q}{3-q}. \tag{11}$$

In the following lemma we answer the question under which conditions (10) is continuous and bounded as a mapping from $W^{1,r}(\Omega)$ into $W^{1,q}(\Omega)$. Then (10) as a mapping $W^{2,q}(\Omega) \hookrightarrow W^{1,r}(\Omega) \to W^{1,q}(\Omega)$ is continuous and compact, and Theorem 1 will be proved. □

Lemma 1. *Assume that, in* (2) *and* (9), $d = d(x,v)$ *satisfies the conditions* (12)–(16) *below (where c_d is a positive constant and $\gamma \geqslant 1$) for all $x \in \Omega$, for all real v, v_1, v_2 and for $1 \leqslant k \leqslant 3$. Then* (10) *is a continuous and bounded mapping $W^{1,r}(\Omega) \to W^{1,q}(\Omega)$ for $r := q(\gamma+1)$. We have*

$$|d(x,v)|, \left|\frac{\partial d(x,v)}{\partial x_k}\right| \leqslant c_d(1+|v|)^{\gamma+1}, \tag{12}$$

$$|d(x,v_1) - d(x,v_2)| \leqslant c_d|v_1 - v_2|(1+|v_1|+|v_2|)^{\gamma}, \tag{13}$$

$$\left|\frac{\partial d(x,v_1)}{\partial x_k} - \frac{\partial d(x,v_2)}{\partial x_k}\right| \leqslant c_d|v_1 - v_2|(1+|v_1|+|v_2|)^{\gamma}, \tag{14}$$

$$\left|\frac{\partial d(x,v)}{\partial v}\right| \leqslant c_d(1+|v|)^{\gamma}, \tag{15}$$

$$\left|\frac{\partial d(x,v_1)}{\partial v} - \frac{\partial d(x,v_2)}{\partial v}\right| \leqslant c_d|v_1 - v_2|(1+|v_1|+|v_2|)^{\gamma-1}. \tag{16}$$

R e m a r k . Before proving Lemma 1, we observe that the conditions (2), (12)–(16) not only are more general than (4) but also better correspond to radiation boundary conditions in practical use, since often a polynomial expression in T with rational exponents is taken where the highest exponent (say $\gamma+1$) satisfies $\gamma+1 \in (3,4]$.

P r o o f . To show the boundedness, we obtain for $d(x,u(x))$ from (12) and from $(\gamma+1)q = r$ that

$$\|d\|_{q,\Omega}^q = \int_\Omega |d(x,u(x))|^q \,\mathrm{d}x \leqslant c_d^q \int_\Omega (1+|u|)^{(\gamma+1)q} \,\mathrm{d}x = (c_d \|1+|u|\,\|_{r,\Omega}^{\gamma+1})^q. \tag{17}$$

Next we have, for every k,

$$\left\| \frac{\partial[d(\cdot,u(\cdot))]}{\partial x_k} \right\|_{q,\Omega} \leqslant \left\| \frac{\partial d(\cdot,u)}{\partial x_k} \right\|_{q,\Omega} + \left\| \frac{\partial d}{\partial v} \frac{\partial u}{\partial x_k} \right\|_{q,\Omega},$$

where the first term on the right-hand side is further estimated analogously to (17). Using the Hölder inequality, for the second term we find

$$\begin{aligned} \int_\Omega \left| \frac{\partial d}{\partial v} \frac{\partial u}{\partial x_k} \right|^q \mathrm{d}x &\leqslant c_d^q \int_\Omega (1+|u|)^{\gamma q} \left| \frac{\partial u}{\partial x_k} \right|^q \mathrm{d}x \\ &\leqslant c_d^q \Big(\int_\Omega (1+|u|)^r \,\mathrm{d}x \Big)^{\frac{\gamma}{\gamma+1}} \Big(\int_\Omega \left| \frac{\partial u}{\partial x_k} \right|^{q(\gamma+1)} \mathrm{d}x \Big)^{\frac{1}{\gamma+1}} \\ &= c_d^q \|1+|u|\,\|_{r,\Omega}^{q\gamma} \left\| \frac{\partial u}{\partial x_k} \right\|_{r,\Omega}^q \end{aligned}$$

for all k.

To show the continuity take the q-th power of (13), integrate over Ω and apply the Hölder inequality to get

$$\begin{aligned} \int_\Omega |d(x,u_1(x)) - d(x,u_2(x))|^q \,\mathrm{d}x \leqslant c_d^q &\Big\{ \int_\Omega |u_1-u_2|^r \,\mathrm{d}x \Big\}^{\frac{q}{r}} \\ &\times \Big\{ \int_\Omega (1+|u_1|+|u_2|)^{\frac{\gamma q r}{r-q}} \,\mathrm{d}x \Big\}^{\frac{r-q}{r}}. \end{aligned} \tag{18}$$

Using $\gamma q/(r-q) = 1$ and taking the qth root we find

$$\|d(\cdot,u_1) - d(\cdot,u_2)\|_{q,\Omega} \leqslant c_d \|u_1-u_2\|_{r,\Omega} \|1+|u_1|+|u_2|\,\|_{r,\Omega}^\gamma.$$

To estimate the L_q norm of $\frac{\partial}{\partial x_k}[d(x,u_1(x))-d(x,u_2(x))]$, we start from

$$\tag{19} \begin{aligned} &\frac{\partial}{\partial x_k}[d(x,u_1(x))-d(x,u_2(x))] \\ &\quad= \Big(\frac{\partial d}{\partial x_k}(x,u_1)-\frac{\partial d}{\partial x_k}(x,u_2)\Big)+\Big(\frac{\partial d}{\partial v}(x,u_1)-\frac{\partial d}{\partial v}(x,u_2)\Big)\frac{\partial u_1}{\partial x_k} \\ &\qquad+\frac{\partial d}{\partial v}(x,u_2)\Big(\frac{\partial u_1}{\partial x_k}-\frac{\partial u_2}{\partial x_k}\Big). \end{aligned}$$

Here, the first term on the right-hand side can be estimated as (18) by using condition (14). For the second term, we proceed similarly on the basis of (16), using once more $r=(\gamma+1)q$:

$$\begin{aligned} &\int_\Omega\Big|\Big(\frac{\partial d}{\partial v}(x,u_1)-\frac{\partial d}{\partial v}(x,u_2)\Big)\frac{\partial u_1}{\partial x_k}\Big|^q\,\mathrm{d}x \\ &\qquad\leqslant c_d^q\|u_1-u_2\|_{r,\Omega}^q\Big\{\int_\Omega(1+|u_1|+|u_2|)^{(\gamma-1)\frac{r}{\gamma}}\Big|\frac{\partial u_1}{\partial x_k}\Big|^{\frac{r}{\gamma}}\,\mathrm{d}x\Big\}^{\frac{\gamma q}{r}}. \end{aligned}$$

By applying once more the Hölder inequality to the last integral on the right-hand side, we see that this expression is bounded by

$$\|1+|u_1|+|u_2|\,\|_{r,\Omega}^{q(\gamma-1)}\Big\|\frac{\partial u_1}{\partial x_k}\Big\|_{r,\Omega}^q.$$

We turn therefore to the third term in (19). Using (15) we obtain similarly as above

$$\Big\|\frac{\partial d}{\partial v}(\cdot,u_2)\Big(\frac{\partial u_1}{\partial x_k}-\frac{\partial u_2}{\partial x_k}\Big)\Big\|_{q,\Omega}\leqslant c_d\Big\|\frac{\partial u_1}{\partial x_k}-\frac{\partial u_2}{\partial x_k}\Big\|_{r,\Omega}\|1+|u_2|\,\|_{r,\Omega}^{\gamma}.$$

The above estimates show the lemma to be true. □

The following result delivers the remaining part for the application of the Schauder fixed point theorem to the mapping Φ defined in (8), (9). We shall use the notation $\|\cdot\|_\Gamma$ for the norm of the Banach space $W^{1-1/q,q}(\Gamma)$ since here this will be the basic space.

Lemma 2. *For a $\gamma\geqslant 1$, assume (12)–(16) hold. Then, for a sufficiently small b and for $q>3\gamma/(\gamma+1)$, in $W^{1-1/q,q}(\Gamma)$ there is a ball of some radius ϱ which is mapped by Φ into itself.*

P r o o f. From the proof of Theorem 1 and Lemma 1 we know that

$$\psi\to D(\psi),\ \text{ where } D(\psi):=d(\cdot,F(\psi))|_\Gamma,\quad F(\psi)=u,$$

as a mapping from $W^{1-1/q,q}(\Gamma)$ into itself is bounded, i.e., for all $\varrho > 0$ there is a $c_1 = c_1(\varrho) > 0$ such that

$$\text{if } \|\psi\|_\Gamma \leqslant \varrho \quad \text{then} \quad \|D(\psi)\|_\Gamma \leqslant c_1(\varrho). \tag{20}$$

For this to be true, in Lemma 1 we have supposed that $r = q(\gamma+1)$, and if $q \geqslant 3$ we have no restriction on r, whereas for $q < 3$ from (11) a restriction on $r = q(\gamma+1)$ arises which means just $q > 3\gamma/(\gamma+1)$.

Now choose a constant $\varrho \geqslant 2\|\varphi\|_\Gamma$ and let b satisfy $b \leqslant \frac{\varrho}{2c_1(\varrho)}$.

Then Φ maps the ball of radius ϱ in $W^{1-1/q,q}(\Gamma)$ into itself since $\|\psi\|_\Gamma \leqslant \varrho$, (9), and (20) yield

$$\|\Phi(\psi)\|_\Gamma \leqslant \|\varphi\|_\Gamma + bc_1(\varrho) \leqslant \frac{\varrho}{2} + \frac{\varrho}{2} = \varrho. \tag{21}$$

This completes the proof of the lemma. □

We remark that for $\gamma + 1 = 4$ the estimates of Lemma 1 correspond to the fourth power law, and then the restriction on q reads $q > 9/4$.

Summarizing the above results we get the following conclusion.

Theorem 2. *For a $\gamma \in [1,\infty)$, assume* (12)–(16) *holds and let $q > \frac{3\gamma}{\gamma+1}$. Then for any $\varphi \in W^{1-1/q,q}(\Gamma)$ there is a positive number $b_0(\varphi)$ such that for all* $0 < b < b_0(\varphi)$ *equation* (7) *has a solution in $W^{1-1/q,q}(\Gamma)$, and hence* (1)–(2) *has a solution in $W^{2,q}(\Omega)$.*

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