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On the existence of a new class of semi-Riemannian manifolds

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Abstract

The present paper deals with the existence of a new class of semi-Riemannian manifolds which are weakly generalized recurrent, pseudo quasi-Einstein and fulfill the condition $R \cdot R = Q(S, R)$. For this purpose, we presented a metric, computed its curvature properties, and finally checked various geometric structures arising out from the different curvatures by means of their covariant derivatives of first and second order.

Keywords: Recurrent; Weakly generalized recurrent; Pseudo quasi-Einstein; Pseudosymmetric; Ricci-generalized pseudosymmetric; Conformal; Projective; Conharmonic curvature tensor

MSC: 53C15; 53C25; 53C35

Introduction

A topological manifold is a connected, second-countable Hausdorff topological space which is locally Euclidean. A smooth manifold is a topological manifold endowed with a smooth structure, which is a maximal collection of coordinate charts whose transition functions are smooth. A semi-Riemannian manifold is a smooth manifold endowed with a semi-Riemannian metric. Let M, dim $M = n \geq 3$, be a connected semi-Riemannian smooth manifold endowed with a semi-Riemannian metric g of signature $(s, n - s)$, $0 \leq s \leq n$. If $s = 0$ or $s = n$, then M is a Riemannian manifold, and if $s = 1$ or $s = n - 1$, then M is a Lorentzian manifold. A semi-Riemannian manifold has mainly three notions of curvature tensors, namely, Riemann-Christoffel curvature tensor R (simply called curvature tensor), the Ricci tensor S, and the scalar curvature r. The curvature tensor R consists of second-order partial derivatives of the metric g with respect to the coordinate functions. Hence, the curvature tensor R provides the complete information of the manifold at the curvature level on g. Therefore, the metric plays the key role in the study of differential geometry of manifolds and their applications to the general theory of relativity and cosmology. The Ricci tensor S is the trace of R with respect to g while r is the trace of S with respect to g. We mention that any one-dimensional semi-Riemannian manifold is a void field because a change of metric can be translated as a mere change of parametrization for the curve. However, any two one-dimensional semi-Riemannian manifolds are locally isometric. Also for any two-dimensional semi-Riemannian manifold, the notions of three curvatures are equivalent. Hence, throughout the paper we will confine ourselves with a semi-Riemannian manifold M of dimension $n \geq 3$, and all the manifolds to be considered throughout the paper are assumed to be smooth and connected. Let ∇ be the Levi-Civita connection on M, which is a unique torsion-free metric connection on M. In terms of local coordinates, the components of R and S are respectively given by R_{hijk} and S_{ij}.

Symmetry plays an important role in the natural life of all living beings of our universe. Symmetry means something that is well proportioned and well balanced, and symmetry denotes the sort of concordance of several parts by which they integrate into a whole. Every spatial object in nature bears a symmetry and the beauty is bound up with symmetry. The geometric concept of symmetry has various forms such as bilateral, translatory, rotational, ornamental, crystallographic symmetry, etc. The main idea behind all these special forms is the invariance of a configuration of elements under a group of automorphic transformations. The principle of symmetry has wide applications in arts and in inorganic and organic nature for harmonious perfection of any spatial objects.

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Symmetry is a relative term which is significant in art and nature. Symmetry plays a great role in ordering the atomic and molecular spectra, for the understanding of which the principles of quantum physics provide the key. The physical occurrences are happening not only in space but also in space and time, and the world is spread out not as a three- but a four-dimensional continuum. The symmetry, the relativity, or homogenity of the four-dimensional medium was first correctly described by Albert Einstein. If nature were all lawful, then every phenomenon would share the full symmetry of the universal laws of nature as formulated by the theory of relativity. For details about the various forms of symmetry, we refer to the book of Weyl [1]. For physical significance of natural symmetries and Deszcz pseudosymmetries, we refer the reader to the work of Deszcz et al. [2].

It is well known that M is called locally symmetric [3,4] if $\nabla R = 0$ (i.e., if $R_{hijk,l} = 0$, ',' denotes the covariant derivative with respect to coordinate functions), which can be stated that the local geodesic symmetry at each point of M is an isometry. The study on generalization of locally symmetric manifolds started in 1946 and continued to date in different directions such as κ-space by Ruse [5-7] (which is called recurrent space by Walker in 1950 [8]), two-recurrent manifolds by Lichnerowicz [9], weakly symmetric manifolds by Selberg [10], generalized recurrent manifolds by Dubey [11], quasi-generalized recurrent manifolds by Shaikh and Roy [12], hyper-generalized recurrent manifolds by Shaikh and Patra [13], weakly generalized recurrent manifolds by Shaikh and Roy [14], pseudosymmetric manifolds by Chaki [15], semisymmetric manifolds by Cartan [16] (which was classified by Szabó [17-19]), pseudosymmetric manifolds by Deszcz [20,21], weakly symmetric manifolds by Tamássy and Binh [22], conformally recurrent manifolds by Adati and Miyazawa [23], and projectively recurrent manifolds by Adati and Miyazawa [24]. It may be mentioned that the notion of weakly symmetric manifold by Selberg is different from that by Tamássy and Binh, and the pseudosymmetric manifold by Chaki is also different from the pseudosymmetric manifold by Deszcz.

The manifold M is said to be Ricci symmetric if $\nabla S = 0$ (i.e., if $S_{ij,k} = 0$). The notion of Ricci symmetry was also weakend by various ways such as Ricci recurrent by Patterson [25], Ricci pseudosymmetric by Deszcz [26], Ricci semisymmetric by Szabó [17-19], pseudo Ricci symmetric by Chaki [27], and weakly Ricci symmetric by Tamássy and Binh [28]. Weakly symmetric and weakly Ricci symmetric spaces by Tamássy and Binh were studied by Shaikh and his coauthors in various papers (see [29-42] and also references therein).

Again, a semi-Riemannian manifold is Einstein if its Ricci tensor is constant multiple to the metric tensor. As a generalization of Einstein manifold, the notion of quasi-Einstein manifold arose during the study of exact solutions of Einstein's field equation as well as during the study of quasi-umbilical hypersurfaces. The process of quasi-Einstein manifold generalization was started in different ways by various authors such as generalized quasi-Einstein manifold by Chaki [43] and also by De and Ghosh [44], pseudo quasi-Einstein manifold by Shaikh [45], pseudo generalized quasi-Einstein manifold by Shaikh and Jana [46], hyper-generalized quasi-Einstein manifold by Shaikh et al. [47], and generalized pseudo quasi-Einstein manifold by Shaikh and Patra [48]. The definitions of all the notions described above are given in the section 'Preliminaries'.

By the decomposition of the covariant derivative ∇S, Gray [49] obtained two classes, $\mathcal{A}$ and $\mathcal{B}$, of Riemannian manifolds which lie between the class of Ricci symmetric manifolds and the manifolds of constant scalar curvature. The class $\mathcal{A}$ (respectively $\mathcal{B}$) is the class of Riemannian manifolds whose Ricci tensor is cyclic parallel (respectively Codazzi tensor). Every Ricci symmetric manifold is of class $\mathcal{B}$ but not conversely. We note that every manifold of constant curvature and hence Einstein manifold are of class $\mathcal{A}$ as well as $\mathcal{B}$. The existence of both classes is given in [50].

Hence, a natural question arises:

Q.1 Does there exist a weakly generalized recurrent manifold which is not any one of the following? (i) Einstein, (ii) quasi-Einstein, (iii) locally symmetric, (iv) Ricci symmetric, (v) recurrent, (vi) Ricci recurrent, (vii) generalized recurrent, (viii) hyper-generalized recurrent, (ix) quasi-generalized recurrent, (x) Codazzi-type Ricci tensor, (xi) cyclic Ricci parallel, (xii) semisymmetric, (xiii) weakly symmetric, (xiv) weakly Ricci symmetric, (xv) Chaki pseudosymmetric, and (xvi) Chaki pseudo Ricci symmetric.

The geometric structures stated in (iii) to (xvi) of *Q.1* involve the first-order covariant differentials of curvature tensor and Ricci tensor. A semi-Riemannian manifold is said to be semisymmetric [16-19] if $R \cdot R = 0$ (locally, $R_{hijk,lm} - R_{hijk,ml} = 0$). We mention that every locally symmetric space is semisymmetric but the converse is not true, in general. However, the converse is true for $n = 3$. As a proper generalization of semisymmetric manifold, the notion of pseudosymmetric manifolds arose during the study of semisymmetric totally umbilical submanifolds in manifolds admitting semisymmetric generalized curvature tensors [51]. A semi-Riemannian manifold $(M, g), n \geq 3$, is said to be pseudosymmetric by Deszcz [20,21] (respectively Ricci pseudosymmetric [26,52]) if $R \cdot R$ (respectively $R \cdot S$) and the Tachibina tensor $Q(g, R)$ (respectively $Q(g, S)$) are linearly dependent at every point of M.

It is well known that the conformal transformation is an angle-preserving mapping, the projective transformation is a geodesic-preserving mapping whereas concircular transformation is the geodesic circle-preserving mapping, and conharmonic transformation is a harmonic function-preserving mapping. Again, a semi-Riemannian manifold (M, g), $n \geq 4$, is said to be conformally pseudosymmetric [53,54] if the tensor $R \cdot C$ and the Tachibana tensor $Q(g, C)$ are linearly dependent at every point of M. The conditions of pseudosymmetric, Ricci pseudosymmetric, and conformally pseudosymmetric or other conditions of this kind are called conditions of pseudosymmetry type. The explicit local expressions of various pseudosymmetric type conditions and their systematic developments are given in the section 'Preliminaries'.

Now, another question arises:

Q.2 Does there exist a semi-Riemannian manifold realizing the condition $R \cdot R = Q(S, R)$ which is not any one of the following?
(i) Pseudosymmetric, (ii) Ricci pseudosymmetric,
(iii) $R \cdot W = LQ(g, W)$, (iv) $R \cdot W_1 = LQ(S, W_1)$,
(v) $C \cdot W = LQ(g, W)$, (vi) $C \cdot W = LQ(S, W)$,
(vii) $P \cdot W = LQ(g, W)$, (viii) $P \cdot W_1 = LQ(S, W_1)$,
(ix) $K \cdot W = LQ(g, W)$, (x) $K \cdot W_1 = LQ(S, W_1)$,
(xi) $Z \cdot W = LQ(g, W)$, (xii) $Z \cdot W = LQ(S, W)$,
(xiii) $Z \cdot S = LQ(g, S)$,
where L is any smooth function; W is any one of R, C, P, K, and Z; and W_1 is any one of C, P, and Z. Here C, P, K, and Z respectively denote the conformal, projective, concircular, and conharmonic curvature tensor.

All the notions of *Q.2* involve the second-order covariant differentials of different curvature tensors. The answer to question *Q.2* is given partially by Deszcz and his coauthors in ([21,55-57] and also references therein). However, combining questions *Q.1* and *Q.2*, it is natural to ask the following question:

Q.3 Does there exist a semi-Riemannian manifold which is a weakly generalized recurrent, pseudo quasi-Einstein and fulfills the condition $R \cdot R = Q(S, R)$ but not realizing any one of (i) to (xvi) of *Q.1* and (i) to (xiii) of *Q.2*?

This paper provides the answer to this question as affirmative by an explicit example which induces a new class of semi-Riemannian manifolds. The paper is organized as follows. The definitions of all the notions and their interrelations in questions *Q.1* and *Q.2* are given in the section 'Preliminaries'. In the last section, we compute the curvature properties of the metric given by

$$ds^2 = g_{ij}dx^i dx^j = x^1x^2x^3(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (dx^4)^2, \quad i, j = 1, 2, 3, 4,$$

which produces the answer to *Q.3* as affirmative, and hence, we obtain a new class of semi-Riemannian manifolds realizing the conditions of *Q.3*. The applications of pseudo quasi-Einstein manifolds are presented in [45]. Since the metric described above is a model of pseudo quasi-Einstein manifold, it is significant geometrically and relevant physically.

Preliminaries

It is well known that in a semi-Riemannian manifold (M, g), the local expression of the curvature tensor R is given by

$$R^h_{kji} = \frac{\partial \Gamma^h_{ji}}{\partial x^k} - \frac{\partial \Gamma^h_{ki}}{\partial x^j} + \Gamma^l_{ji}\Gamma^h_{kl} - \Gamma^l_{ki}\Gamma^h_{jl}$$

where $R^h_{kji} = g^{hl}R_{lkji}$, and Γ denotes the Christoffel symbol of second kind and is given by

$$\Gamma^h_{ij} = \frac{1}{2}g^{hk}\left[\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right].$$

Here g^{ij} are the components of g^{-1}. The local expression of the Ricci tensor S_{ij} and the scalar curvature r are respectively given by

$$S_{ij} = \frac{\partial \Gamma^k_{ki}}{\partial x^j} - \frac{\partial \Gamma^k_{ij}}{\partial x^k} + \Gamma^q_{ki}\Gamma^k_{qj} - \Gamma^q_{ji}\Gamma^k_{qk},$$

$$r = g^{hk}S_{hk},$$

where $S_{ij} = g^{hk}R_{hijk}$.

A connected semi-Riemannian manifold (M, g), $n \geq 3$, is Einstein if its Ricci tensor S of type (0, 2) is given by $S = \frac{r}{n}g$. We mention that any two-dimensional manifold is always Einstein. Let (M, g), $n \geq 3$, be a semi-Riemannian manifold and $U_S = \{x \in M : \left(S - \frac{r}{n}g\right)_x \neq 0\}$. Then the manifold M is said to be quasi-Einstein [58-67] if on $U_S \subset M$ we have

$$S - \alpha g = \beta A \otimes A$$

for some 1-form A on U_S and for some functions α, β on U_S, where $\otimes$ is the tensor product. It is clear that the 1-form A as well as the function β are non-zero at every point of U_S.

Again, let $U_{S_1} = \{x \in M : \left(S - \nu_1 g - \nu_2 A_1 \otimes A_1\right)_x \neq 0$ for any scalars ν_1, ν_2 and any 1-form $A_1\}$. Then the manifold M is said to be pseudo quasi-Einstein [45] if on $U_{S_1} \subset M$ we have

$$S - \alpha g - \beta A \otimes A = \gamma D,$$

where α, β, and γ are some functions on U_{S_1}, and A is any non-zero 1-form such that $g(X, U) = A(X)$ for all vector fields X with U being a unit vector field called the generator of the manifold. It is clear that the 1-form A as well as the functions β and γ are non-zero at every point on U_{S_1}. D is a symmetric tensor of type (0, 2) on U_{S_1} such that $tr\, D = 0$ and satisfying $D(X, U) = 0$ for all vector fields X, and D is called the structure tensor of the manifold. Such a manifold is denoted by PQE_n.

It follows that every quasi-Einstein manifold is pseudo quasi-Einstein but not conversely, which follows from the metric presented in the present paper. It is known that the outer product of two covariant vectors is a tensor of type (0, 2), but the converse is not true in general [68]. Consequently, the tensor D cannot be decomposed into product of two 1-forms. In particular, if $D = B \otimes B$, B being a non-zero 1-form, then a pseudo quasi-Einstein manifold reduces to a generalized quasi-Einstein manifold by De and Ghosh [44]. Again, if $D = A \otimes B + B \otimes A$, then a pseudo quasi-Einstein manifold turns into a generalized quasi-Einstein manifold by Chaki [43].

Again in 2008, Shaikh and Jana [46] introduced a generalized class of quasi-Einstein manifolds called pseudo generalized quasi-Einstein manifold, defined as follows:

Let M be a semi-Riemannian manifold. Let $U_{S_2} = \{x \in M : S - \nu_3 g - \nu_4 A_2 \otimes A_2 - \nu_5 B_2 \otimes B_2 \neq 0$ at x, for any scalars ν_3, ν_4, ν_5 and any 1-forms $A_2, B_2\}$. Then the manifold M is said to be a pseudo generalized quasi-Einstein manifold [46] if on $U_{S_2} \subset M$ the relation

$$S = \alpha g + \beta A \otimes A + \gamma B \otimes B + \delta D$$

holds for some 1-forms A, B, and some functions α, β, γ, δ on U_{S_2}, where D is any symmetric (0, 2) tensor with zero trace, which satisfies the condition $D(X, U) = 0$ for all vector fields X. It is obvious that the 1-forms A, B, and D, as well as the functions $\alpha, \beta, \gamma, \delta$, are non-zero at every point of U_{S_2}. Also α, β, γ, δ are called the associated scalars; A, B are the associated 1-forms of the manifold and D is called the structure tensor of the manifold. Such an n-dimensional manifold is denoted by $PGQE_n$. If $\gamma = 0$, then a $PGQE_n$ turns into a PQE_n.

Recently, Catino [69] introduced the notion of generalized quasi-Einstein manifolds which are different from that of Chaki [43] and also from De and Ghosh [44]. A complete semi-Riemannian manifold M is said to be a generalized quasi-Einstein manifold [69] if there exist three smooth functions f, μ and λ on M such that

$$S + \nabla^2 f - \mu df \otimes df = \lambda g.$$

Extending the notion of generalized quasi-Einstein manifolds by Chaki [43], recently Shaikh et al. [47] introduced the notion of hyper-generalized quasi-Einstein manifold.

Let M be a semi-Riemannian manifold. Let $U_{S_3} = \{x \in M : S - \nu_6 g - \nu_7 A_3 \otimes A_3 - \nu_8 [A_3 \otimes B_3 + B_3 \otimes A_3] \neq 0$ at x for any scalars ν_6, ν_7, ν_8 and any 1-forms $A_3, B_3\}$. Then the manifold M is said to be a hyper-generalized quasi-Einstein manifold [47] if on $U_{S_3} \subset M$ the relation

$$S - \alpha g - \beta A \otimes A - \gamma [A \otimes B + B \otimes A] = \delta [A \otimes I + I \otimes A]$$

holds for some 1-forms A, B, I and some functions $\alpha, \beta, \gamma, \delta$ on U_{S_3}. It is clear that the 1-forms A, B, I, as well as the functions $\alpha, \beta, \gamma, \delta$, are non-zero at every point of U_{S_3}. The scalars $\alpha, \beta, \gamma, \delta$ are known as the associated scalars of the manifold, and A, B, D are called the associated 1-forms of the manifold. Such an n-dimensional manifold is denoted by $HGQE_n$.

Extending the notion of PQE_n of [45], recently Shaikh and Patra [48] introduced the notion of generalized pseudo quasi-Einstein manifold, defined as follows:

Let M be a semi-Riemannian manifold. Let $U_{S_4} = \{x \in M : S - \nu_9 g - \nu_{10} A_4 \otimes A_4 - \nu_{11} D_4 \neq 0$ at x for any scalars $\nu_9, \nu_{10}, \nu_{11}$, any 1-form A_4, and any tensor D_4 of type (0, 2)$\}$. Then the manifold M is said to be a generalized pseudo quasi-Einstein manifold [48] if on $U_{S_4} \subset M$ the relation

$$S - \alpha g - \beta A \otimes A - \gamma D = \delta J$$

holds for any 1-form A on U_{S_4}, some smooth functions α, β, γ, δ on U_{S_4}, and for some trace-free symmetric tensors D, J of type (0, 2) such that $D(X, U) = 0$ and $J(X, U) = 0$ for any vector field X. It is obvious that the 1-form A and the tensors D and J, as well as the functions $\alpha, \beta, \gamma, \delta$, are non-zero at every point of U_{S_4}. Such an n-dimensional manifold is denoted by $GPQE_n$. It follows that every quasi-Einstein manifold as well as PQE_n is a $GPQE_n$ but not conversely. Especially, if $\delta = 0$, then a $GPQE_n$ turns into a PQE_n, and if $\delta = \gamma = 0$, then a $GPQE_n$ reduces to a quasi-Einstein manifold. We note that if $D = B \otimes B$, B being a non-zero 1-form, then a $GPQE_n$ turns into a pseudo generalized quasi-Einstein manifold by Shaikh and Jana [46]. Also, if $D = A \otimes B + B \otimes A$ and $J = A \otimes N + N \otimes A$, N being a non-zero 1-form, then a $GPQE_n$ turns into a hyper generalized quasi-Einstein manifold [47].

In a semi-Riemannian manifold (M, g), $n \geq 3$, the Ricci tensor S is said to be a Codazzi tensor [70,71] (respectively cyclic Ricci parallel [49]) if it satisfies

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z) \ \ (\text{locally, } S_{ij,k} = S_{ik,j})$$
$$(\text{respectively} \ \ (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0,$$
$$\text{locally, } S_{ij,k} + S_{jk,i} + S_{ki,j} = 0)$$

for all vector fields $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of all smooth vector fields on M.

Let $U_L = \{x \in M : R \neq 0$ and $\nabla R \neq 0$ at $x\}$. A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be recurrent [8] if on $U_L \subset M$ the curvature tensor R satisfies the following:

$$\nabla R = A \otimes R,$$

where A is any non-zero 1-form. The defining condition of a recurrent manifold in local form is given by

$$R_{hijk,l} = A_l R_{hijk}.$$

Let $U_{L_1} = \{x \in M : S \neq 0 \text{ and } \nabla S \neq 0 \text{ at } x\}$. Then the manifold (M, g), $n \geq 3$, is said to be Ricci recurrent [25] if on $U_{L_1} \subset M$ the Ricci tensor S satisfies the following:

$$\nabla S = A \otimes S,$$

where A is any non-zero 1-form. Also the defining condition of a Ricci recurrent manifold in local form is given by

$$S_{ij,k} = A_k S_{ij}.$$

Generalizing the notion of recurrent manifold and extending the notion of generalized recurrent manifolds by Dubey [11], recently Shaikh and his coauthors [12-14] introduced three classes of generalized recurrent manifolds, namely, quasi-generalized recurrent manifolds, hyper-generalized recurrent manifolds, and weakly generalized recurrent manifolds.

Let $U_Q = \{x \in M : (R)_x \neq 0 \text{ and } (\nabla R - \Theta \otimes R)_x \neq 0 \text{ for any 1-forms } \Theta\}$. A non-flat semi-Riemannian manifold (M, g), $n \geq 3$, is said to be a quasi-generalized recurrent manifold [12] (briefly QGK_n) if on $U_Q \subset M$ the condition

$$\nabla R = A \otimes R + B \otimes [G + g \wedge H],$$

holds for some non-zero 1-forms A, B, where $H = \eta \otimes \eta$, η being a non-zero 1-form, and the Kulkarni-Nomizu product $E_1 \wedge E_2$ of two (0,2)-tensors, E_1 and E_2, is defined by (see, e.g., [64,72,73])

$$\begin{aligned}(E_1 \wedge E_2)(X_1, X_2; X, Y) = & E_1(X_1, Y)E_2(X_2, X) \\ & + E_1(X_2, X)E_2(X_1, Y) \\ & - E_1(X_1, X)E_2(X_2, Y) \\ & - E_1(X_2, Y)E_2(X_1, X),\end{aligned}$$

$X_1, X_2, X_3, X_4 \in \chi(M)$. Especially if $\eta = 0$, then a QGK_n turns out to be a generalized recurrent manifold (i.e., GK_n) by Dubey [11]. In terms of local coordinates, the defining condition of a QGK_n is given by

$$\begin{aligned}R_{hijk,l} = A_l R_{hijk} + B_l \big[& G_{hijk} + g_{hk}\eta_i\eta_j + g_{ij}\eta_h\eta_k - g_{hj}\eta_i\eta_k \\ & - g_{ik}\eta_h\eta_j \big].\end{aligned}$$

The manifold (M, g), $n \geq 3$, is said to be a hyper-generalized recurrent manifold [13] (briefly HGK_n) if on $U_Q \subset M$, the condition

$$\nabla R = A \otimes R + B \otimes (S \wedge g)$$

$$(\text{locally, } R_{hijk,l} = A_l R_{hijk} + B_l [S_{hk}g_{ij} + S_{ij}g_{hk} - S_{hj}g_{ik} - S_{ik}g_{hj}])$$

holds for some non-zero 1-forms A, B.

Again the manifold (M, g), $n \geq 3$, is said to be a weakly generalized recurrent manifold [14] (briefly WGK_n) if on $U_Q \subset M$, the condition

$$\nabla R = A \otimes R + B \otimes \frac{1}{2}(S \wedge S)$$

holds for some non-zero 1-forms A, B such that $A(X) = g(\sigma, X)$ and $B(X) = g(\rho, X)$. In terms of local components, the defining condition of a WGK_n can be written as

$$R_{hijk,l} = A_l R_{hijk} + B_l [S_{hk}S_{ij} - S_{ik}S_{hj}]. \tag{1}$$

We note that for $\alpha = \beta$, a quasi-Einstein manifold is WGK_n if and only if it is QGK_n. Again, we also note that for $2\alpha = \beta$, a quasi-Einstein manifold is HGK_n if and only if it is QGK_n.

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be weakly symmetric by Tamássy and Binh [22] if on $U_Q \subset M$, $\nabla R = \mathcal{L}$, where $\mathcal{L}$ is a tensor of type $(0, 5)$ defined by

$$\begin{aligned}\mathcal{L}(X, X_1, X_2, X_3, X_4) = & A(X)R(X_1, X_2, X_3, X_4) \\ & + B(X_1)R(X, X_2, X_3, X_4) + B(X_2)R(X_1, X, X_3, X_4) \\ & + D(X_3)R(X_1, X_2, X, X_4) + D(X_4)R(X_1, X_2, X_3, X)\end{aligned}$$

for all vector fields $X, X_i \in \chi(M)$ $(i = 1, 2, 3, 4)$ and for some 1-forms A, B, D on M. In terms of local coordinates, the above expression can be written as

$$R_{hijk,l} = A_l R_{hijk} + B_h R_{lijk} + B_i R_{hljk} + D_j R_{hilk} + D_k R_{hijl}.$$

Again, in 1993 Tamássy and Binh [28] introduced the notion of weakly Ricci symmetric manifold defined as follows:

Let $U_T = \{x \in M : (\nabla S - \Theta_1 \otimes S)_x \neq 0 \text{ for any 1-forms } \Theta_1\}$. Then the semi-Riemannian manifold (M, g), $n \geq 3$, is called weakly Ricci symmetric if on $U_T \subset M$ its Ricci tensor S is not identically zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(Y)S(X, Z) + D(Z)S(Y, X) \tag{2}$$

for some 1-forms A, B, D (not simultaneously zero), where $X, Y, Z \in \chi(M)$. The local form of the above condition is given as

$$S_{ij,k} = A_k S_{ij} + B_i S_{kj} + D_j S_{ik}.$$

A non-flat semi-Riemannian manifold (M, g), $n \geq 3$, is said to be pseudosymmetric by Chaki [15] if on $U_L \subset M$ its curvature tensor R satisfies the relation

$$\begin{aligned}(\nabla_X R)(Y, Z, U, V) = & 2A(X)R(Y, Z, U, V) \\ & + A(Y)R(X, Z, U, V) \\ & + A(Z)R(Y, X, U, V) \\ & + A(U)R(Y, Z, X, V) \\ & + A(V)R(Y, Z, U, X)\end{aligned}$$

$$\text{(or locally, } R_{hijk,l} = 2A_l R_{hijk} + A_h R_{lijk} + A_i R_{hljk} + A_j R_{hilk} + A_k R_{hijl})$$

for any 1-forms A, where $X, Y, Z, U, V \in \chi(M)$.

In 1988, Chaki [27] introduced the notion of pseudo Ricci symmetric manifolds defined as follows:

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be pseudo Ricci symmetric if on $U_{L_1} \subset M$ its Ricci tensor S is not identically zero and satisfies the relation

$$(\nabla_X S)(Y, Z) = 2A(X)S(Y, Z) + A(Y)S(X, Z) + A(Z)S(X, Y) \tag{3}$$

$$\text{(or locally,} S_{ij,k} = 2A_k S_{ij} + A_i S_{kj} + A_j S_{ih})$$

for any non-zero 1-form A, where $X, Y, Z \in \chi(M)$.

We define on a semi-Riemannian manifold (M, g), $n \geq 3$, the endomorphisms $X \wedge_E Y$ and $\mathcal{R}(X, Y)$ by [58,64,72]

$$(X \wedge_E Y)Z = E(Y, Z)X - E(X, Z)Y,$$
$$\mathcal{R}(X, Y)Z = [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z,$$

respectively, where E is a $(0, 2)$-tensor on M, $X, Y, Z \in \chi(M)$.

Now we define the Gaussian curvature tensor G, the Riemann-Christoffel curvature tensor R, the Weyl conformal curvature tensor C, projective curvature tensor P, concircular curvature tensor K, and conharmonic curvature tensor Z of (M, g) respectively by [58,64,72,74-76])

$$\begin{aligned} G(X_1, X_2, X_3, X_4) &= g((X_1 \wedge_g X_2)X_3, X_4), \\ R(X_1, X_2, X_3, X_4) &= g(\mathcal{R}(X_1, X_2)X_3, X_4), \\ C(X_1, X_2, X_3, X_4) &= R(X_1, X_2, X_3, X_4) \\ &\quad - \frac{1}{n-2}(g \wedge S)(X_1, X_2, X_3, X_4) \\ &\quad + \frac{r}{(n-2)(n-1)} G(X_1, X_2, X_3, X_4), \end{aligned}$$

$$\begin{aligned} P(X_1, X_2, X_3, X_4) &= R(X_1, X_2, X_3, X_4) \\ &\quad - \frac{1}{n-1}\big[g(X_1, X_4)S(X_2, X_3) \\ &\qquad - g(X_2, X_4)S(X_1, X_3)\big], \\ K(X_1, X_2, X_3, X_4) &= R(X_1, X_2, X_3, X_4) \\ &\quad - \frac{r}{n(n-1)} G(X_1, X_2, X_3, X_4), \\ Z(X_1, X_2, X_3, X_4) &= R(X_1, X_2, X_3, X_4) \\ &\quad - \frac{1}{n-2}(g \wedge S)(X_1, X_2, X_3, X_4). \end{aligned}$$

In terms of local coordinates, the expressions of C, P, K, and Z are respectively given by

$$\begin{aligned} C_{hijk} &= R_{hijk} - \frac{1}{n-2}\left[S_{hk}g_{ij} + S_{ij}g_{hk} - S_{hj}g_{ik} - S_{ik}g_{hj}\right] \\ &\quad + \frac{r}{(n-1)(n-2)}\left[g_{hk}g_{ij} - g_{ik}g_{hj}\right], \end{aligned}$$

$$P_{hijk} = R_{hijk} - \frac{1}{n-1}\left[S_{hk}g_{ij} - S_{hj}g_{ik}\right],$$

$$K_{hijk} = R_{hijk} - \frac{r}{n(n-1)}\left[g_{hk}g_{ij} - g_{ik}g_{hj}\right],$$

$$Z_{hijk} = R_{hijk} - \frac{1}{n-2}\left[S_{hk}g_{ij} + S_{ij}g_{hk} - S_{hj}g_{ik} - S_{ik}g_{hj}\right].$$

For a $(0, k)$-tensor T, $k \geq 1$, and a symmetric $(0, 2)$-tensor E, we define the $(0, k)$-tensor $E \cdot T$ and the $(0, k+2)$-tensors $R \cdot T$, $C \cdot T$, and $Q(E, T)$ by [58,64,72,77]

$$\begin{aligned} (E \cdot T)(X_1, \cdots, X_k) &= -T(\mathcal{E}X_1, X_2, \cdots, X_k) - \cdots \\ &\quad - T(X_1, X_2, \cdots, \mathcal{E}X_k), \\ (R \cdot T)(X_1, \cdots, X_k; X, Y) &= (\mathcal{R}(X, Y) \cdot T)(X_1, \cdots, X_k) \\ &= -T(\mathcal{R}(X, Y)X_1, X_2, \cdots, X_k) - \cdots \\ &\quad - T(X_1, \cdots, X_{k-1}, \mathcal{R}(X, Y)X_k), \\ (C \cdot T)(X_1, \cdots, X_k; X, Y) &= (\mathcal{C}(X, Y) \cdot T)(X_1, \cdots, X_k) \\ &= -T(\mathcal{C}(X, Y)X_1, X_2, \cdots, X_k) - \cdots \\ &\quad - T(X_1, \cdots, X_{k-1}, \mathcal{C}(X, Y)X_k), \\ Q(E, T)(X_1, \cdots, X_k; X, Y) &= ((X \wedge_E Y) \cdot T)(X_1, \cdots, X_k) \\ &= -T((X \wedge_E Y)X_1, X_2, \cdots, X_k) - \cdots \\ &\quad - T(X_1, \cdots, X_{k-1}, (X \wedge_E Y)X_k), \end{aligned}$$

where $\mathcal{E}$ is the endomorphism of $\chi(M)$ defined by $g(\mathcal{E}X, Y) = E(X, Y)$. In terms of local components, these tensors can be written as

$$\begin{aligned} (E \cdot T)_{i_1 i_2 \cdots i_k} &= -g^{pq}(E_{qi_1} T_{pi_2 \cdots i_k} + E_{qi_2} T_{i_1 p \cdots i_k} + \cdots \\ &\quad + E_{qi_k} T_{i_1 i_2 \cdots p}), \end{aligned}$$

$$\begin{aligned} (R \cdot T)_{i_1 i_2 \cdots i_k uv} &= -g^{pq}(R_{uvqi_1} T_{pi_2 \cdots i_k} + R_{uvqi_2} T_{i_1 p \cdots i_k} \\ &\quad + \cdots + R_{uvqi_k} T_{i_1 i_2 \cdots p}), \end{aligned}$$

$$\begin{aligned} (C \cdot T)_{i_1 i_2 \cdots i_k uv} &= -g^{pq}(C_{uvqi_1} T_{pi_2 \cdots i_k} + C_{uvqi_2} T_{i_1 p \cdots i_k} \\ &\quad + \cdots + C_{uvqi_k} T_{i_1 i_2 \cdots p}), \end{aligned}$$

$$\begin{aligned} Q(E, T)_{i_1 i_2 \ldots i_k uv} &= E_{i_1 u} T_{vi_2 \ldots i_k} + E_{i_2 u} T_{i_1 v \cdots i_k} + \cdots \\ &\quad + E_{i_k u} T_{i_1 i_2 \cdots v} - E_{i_1 v} T_{ui_2 \cdots i_k} \\ &\quad - E_{i_2 v} T_{i_1 u \cdots i_k} - \cdots - E_{i_k v} T_{i_1 i_2 \cdots u}. \end{aligned}$$

Putting in the above formulas $T = R, T = S, T = C$, $T = K$, $T = Z$ or $T = P$, $E = g$ or $E = S$, we obtain the following tensors: $R \cdot R$, $R \cdot S$, $R \cdot C$, $R \cdot K$, $C \cdot R$, $C \cdot S$, $C \cdot C$, $C \cdot K$, $Q(g,R)$, $Q(g,S)$, $Q(g,C)$, $Q(g,K)$, $Q(S,R)$, $Q(S,C)$, $Q(g,K)$, $S \cdot R$, $S \cdot C$, $S \cdot K$, $Z \cdot Z$, $Z \cdot R$, $R \cdot Z$, $Z \cdot S$, $P \cdot R$, $P \cdot P$, $R \cdot P$, $P \cdot S$, etc. The tensor $Q(E,T)$ is called the Tachibana tensor of the tensors E and T, or the Tachibana tensor for short [77].

A semi-Riemannian manifold (M,g), $n \geq 3$, is said to be pseudosymmetric [20,21] if the condition

$$R \cdot R = L_R\, Q(g,R) \tag{4}$$

holds on $U_R = \{x \in M : \left(R - \frac{r}{n(n-1)}G\right)_x \neq 0\}$, where L_R is some function on this set.

A semi-Riemannian manifold (M,g), $n \geq 3$, is Ricci pseudosymmetric [26,52] if and only if

$$R \cdot S = L_S\, Q(g,S) \tag{5}$$

holds on U_S, where L_S is some function on this set. We note that $U_S \subset U_R$.

A semi-Riemannian manifold (M,g), $n \geq 4$, is said to be a manifold with pseudosymmetric Weyl conformal curvature tensor [53,54] if the tensor $C \cdot C$ and the Tachibana tensor $Q(g,C)$ are linearly dependent at every point of M, that is,

$$C \cdot C = L_C\, Q(g,C) \tag{6}$$

on $U_C = \{x \in M : C \neq 0 \text{ at } x\}$, where L_C is some function on this set. We note that $U_C \subset U_R$.

A semi-Riemannian manifold (M,g), $n \geq 3$, is said to be Ricci-generalized pseudosymmetric [55,56] if at every point of M, the tensor $R \cdot R$ and the Tachibana tensor $Q(S,R)$ are linearly dependent. Hence, (M,g) is Ricci-generalized pseudosymmetric if and only if

$$R \cdot R = L\, Q(S,R) \tag{7}$$

holds on $U = \{x \in M : Q(S,R) \neq 0 \text{ at } x\}$, where L is some function on this set. An important subclass of Ricci-generalized pseudosymmetric manifolds is formed by the manifolds realizing the condition [54,55]

$$R \cdot R = Q(S,R). \tag{8}$$

We note that from [14] (see Theorem 2.1(v) and equation (2.17) of [14]), it follows that a WGK_n is semisymmetric if σ, ρ are codirectional. But in general, a WGK_n is not semisymmetric. Also, a WGK_n is not, in general, pseudosymmetric and Ricci-generalized pseudosymmetric, but we presented a metric in the present paper which fulfills the condition $R \cdot R = Q(S,R)$.

Examples of WGK_n and $R \cdot R = Q(S,R)$

Let M be an open connected subset of $\mathbb{R}^4$ endowed with the product metric

$$\begin{aligned} ds^2 = g_{ij}dx^i dx^j =& x^1x^2x^3(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 \\ &+ (dx^4)^2, \quad i,j = 1,2,3,4, \end{aligned} \tag{9}$$

$x^1 > 0, x^2 > 0, x^3 > 0$.

Then the non-zero components of the Christoffel symbols of second kind are given by

$$\Gamma^1_{11} = -\frac{x^1x^3}{2}, \Gamma^2_{11} = \frac{1}{2}x^2x^3\left((x^1)^2x^3 + 1\right),$$
$$\Gamma^3_{11} = -\frac{x^1x^2}{2}, \Gamma^2_{12} = \frac{x^1x^3}{2}, \Gamma^2_{13} = \frac{x^1x^2}{2}.$$

The non-zero components of curvature tensor and Ricci tensor (up to symmetry) are given by

$$R_{1213} = -\frac{x^1}{2}, \quad S_{13} = -\frac{x^1}{2}. \tag{10}$$

The scalar curvature of this metric is given by $r = 0$. Again the non-zero components $R_{hijk,l}$ and $S_{ij,l}$ of the covariant derivatives of curvature tensor and Ricci tensor (up to symmetry) are given by

$$R_{1213,1} = -\frac{1}{4}\left((x^1)^2x^{3+2}\right), \quad R_{1313,1} = \frac{(x^1)^2x^2}{2}, \tag{11}$$

$$S_{11,1} = -\frac{(x^1)^2x^2}{2}, \; S_{13,1} = -\frac{1}{4}\left((x^1)^2x^3 + 2\right).$$

The non-zero components of $(R \cdot R)_{hijklm}$, $(R \cdot S)_{ijlm}$, $(R \cdot C)_{hijklm}$, $(R \cdot P)_{hijklm}$, $(R \cdot K)_{hijklm}$, $(R \cdot Z)_{hijklm}$, $(C \cdot R)_{hijklm}$, $(C \cdot S)_{ijlm}$, $(C \cdot C)_{hijklm}$, $(C \cdot P)_{hijklm}$, $(C \cdot K)_{hijklm}$, $(C \cdot Z)_{hijklm}$, $(P \cdot R)_{hijklm}$, $(P \cdot S)_{ijlm}$, $(P \cdot C)_{hijklm}$, $(P \cdot P)_{hijklm}$, $(P \cdot K)_{hijklm}$, $(P \cdot Z)_{hijklm}$, $(K \cdot R)_{hijklm}$, $(K \cdot S)_{ijlm}$, $(K \cdot C)_{hijklm}$, $(K \cdot P)_{hijklm}$, $(K \cdot K)_{hijklm}$, $(K \cdot Z)_{hijklm}$, $(Z \cdot R)_{hijklm}$, $(Z \cdot S)_{ijlm}$, $(Z \cdot C)_{hijklm}$, $(Z \cdot P)_{hijklm}$, $(Z \cdot K)_{hijklm}$, $(Z \cdot Z)_{hijklm}$, $Q(g,R)_{hijklm}$, $Q(g,S)_{ijlm}$, $Q(S,R)_{hijklm}$, $Q(g,C)_{hijklm}$, $Q(S,C)_{hijklm}$, $Q(g,P)_{hijklm}$, $Q(g,K)_{hijklm}$, $Q(S,K)_{hijklm}$, $Q(g,Z)_{hijklm}$, $Q(S,Z)_{hijklm}$, and $Q(S,P)_{hijklm}$ are respectively given below:

$$\begin{aligned} (R \cdot R)_{131312} = -2(R \cdot R)_{121313} = \frac{(x^1)^2}{2}, \\ (R \cdot S)_{1112} = 2(R \cdot S)_{1313} = -\frac{(x^1)^2}{2}. \end{aligned} \tag{12}$$

$$\begin{aligned} (R \cdot C)_{131312} =& -(R \cdot C)_{141412} = -2(R \cdot C)_{121313} \\ =& -2(R \cdot C)_{143413} = \frac{(x^1)^2}{4}. \end{aligned} \tag{13}$$

$$\left.\begin{array}{c}(R\cdot P)_{133112}=2(R\cdot P)_{121313}=-2(R\cdot P)_{132113}=-\frac{(x^1)^2}{2},\quad (R\cdot P)_{131312}\\ =2(R\cdot P)_{123113}=-2(R\cdot P)_{121112}=-2(R\cdot P)_{141412}=-2(R\cdot P)_{131213},\\ =4(R\cdot P)_{231113}=4(R\cdot P)_{341413}=-4(R\cdot P)_{133313}=-4(R\cdot P)_{143413}=\frac{(x^1)^2}{3},\\ (R\cdot P)_{131113}=\frac{1}{12}(x^1)^3x^2x^3,\end{array}\right\} \tag{14}$$

$$(R\cdot K)_{131312}=-2(R\cdot K)_{121313}=\frac{(x^1)^2}{2}. \tag{15}$$

$$(R\cdot Z)_{131312}=-(R\cdot Z)_{141412}=-2(R\cdot Z)_{121313}=-2(R\cdot Z)_{143413}=\frac{(x^1)^2}{4}. \tag{16}$$

$$(C\cdot R)_{131312}=2(C\cdot R)_{121414}=2(C\cdot R)_{131434}=-2(C\cdot R)_{121313}=\frac{(x^1)^2}{4}. \tag{17}$$

$$(C\cdot S)_{1112}=2(C\cdot S)_{1313}=-2(C\cdot S)_{1414}=-\frac{(x^1)^2}{4}. \tag{18}$$

$$\left.\begin{array}{c}(C\cdot C)_{131312}=(C\cdot C)_{131434}=-(C\cdot C)_{141412}=2(C\cdot C)_{121414}=-2(C\cdot C)_{121313}\\ =-2(C\cdot C)_{143413}=-2(C\cdot C)_{133414}=\frac{(x^1)^2}{8}.\end{array}\right\} \tag{19}$$

$$\left.\begin{array}{c}(C\cdot P)_{131113}=-(C\cdot P)_{141114}=\frac{1}{24}(x^1)^3x^2x^3,\\ (C\cdot P)_{121414}=(C\cdot P)_{131434}=(C\cdot P)_{132113}=(C\cdot P)_{141334}=-(C\cdot P)_{121313}\\ =-(C\cdot P)_{134134}=-(C\cdot P)_{142114}=-(C\cdot P)_{143134}=-\frac{1}{2}(C\cdot P)_{133112}=\frac{(x^1)^2}{8},\\ (C\cdot P)_{123113}=(C\cdot P)_{141214}=-(C\cdot P)_{121112}=-(C\cdot P)_{124114}=-(C\cdot P)_{131213}\\ =-(C\cdot P)_{141412}=\frac{1}{2}(C\cdot P)_{131312}=2(C\cdot P)_{134314}=2(C\cdot P)_{144414}\\ =2(C\cdot P)_{231113}=-2(C\cdot P)_{133313}=-2(C\cdot P)_{143413}=-2(C\cdot P)_{241114}\\ =-2(C\cdot P)_{341314}=-2(C\cdot P)_{341413}=\frac{(x^1)^2}{12}.\end{array}\right\} \tag{20}$$

$$(C\cdot K)_{131312}=2(C\cdot K)_{121414}=2(C\cdot K)_{131434}=-2(C\cdot K)_{121313}=\frac{(x^1)^2}{4}. \tag{21}$$

$$\left.\begin{array}{c}(C\cdot Z)_{131312}=(C\cdot Z)_{131434}=-(C\cdot Z)_{141412}=2(C\cdot Z)_{121414}\\ =-2(C\cdot Z)_{121313}=-2(C\cdot Z)_{133414}=-2(C\cdot Z)_{143413}=\frac{(x^1)^2}{8}.\end{array}\right\} \tag{22}$$

$$(P\cdot R)_{131312}=-(P\cdot R)_{131321}=2(P\cdot R)_{121331}=-2(P\cdot R)_{121313}=\frac{(x^1)^2}{3}. \tag{23}$$

$$(P\cdot S)_{1121}=-(P\cdot S)_{1112}=2(P\cdot S)_{1331}=-2(P\cdot S)_{1313}=\frac{(x^1)^2}{2}. \tag{24}$$

$$\left.\begin{array}{c}(P\cdot C)_{141421}=-(P\cdot C)_{141412}=2(P\cdot C)_{143431}=-2(P\cdot C)_{143413}=\frac{(x^1)^2}{4},\\ (P\cdot C)_{131312}=-(P\cdot C)_{131321}=2(P\cdot C)_{121331}=-2(P\cdot C)_{121313}=\frac{(x^1)^2}{6},\\ (P\cdot C)_{131434}=(P\cdot C)_{133441}=-(P\cdot C)_{131443}=-(P\cdot C)_{133414}=\frac{(x^1)^2}{24}.\end{array}\right\} \tag{25}$$

$$\left.\begin{array}{c}(P\cdot P)_{132113}=-(P\cdot P)_{121112}=-(P\cdot P)_{121313}=-(P\cdot P)_{141412}=-2(P\cdot P)_{143413}\\ =-2(P\cdot P)_{341413}=\frac{(x^1)^2}{6},\quad (P\cdot P)_{131113}=\frac{1}{36}(x^1)^3x^2x^3,\\ (P\cdot P)_{123113}=-(P\cdot P)_{131213}=2(P\cdot P)_{231113}=-2(P\cdot P)_{131123}=\frac{(x^1)^2}{9},\\ (P\cdot P)_{131434}=-(P\cdot P)_{133414}=-(P\cdot P)_{143134}=-(P\cdot P)_{143314}=-(P\cdot P)_{341134}\\ =-(P\cdot P)_{341314}=\frac{1}{7}(P\cdot P)_{131312}=-\frac{1}{5}(P\cdot P)_{133313}=-\frac{1}{13}(P\cdot P)_{133112}=\frac{(x^1)^2}{36}.\end{array}\right\} \tag{26}$$

$$(P\cdot K)_{131312}=-(P\cdot K)_{131321}=2(P\cdot K)_{121331}=-2(P\cdot K)_{121313}=\frac{(x^1)^2}{3}. \tag{27}$$

$$\left.\begin{array}{c}(P\cdot Z)_{144112}=-2(P\cdot Z)_{143413}=\frac{(x^1)^2}{4},\\ (P\cdot Z)_{131312}=-2(P\cdot Z)_{121313}=4(P\cdot Z)_{131434}=-4(P\cdot Z)_{133414}=\frac{(x^1)^2}{6}.\end{array}\right\} \tag{28}$$

$$(K\cdot R)_{131312}=-2(K\cdot R)_{121313}=\frac{(x^1)^2}{2},\quad (K\cdot S)_{1112}=2(K\cdot S)_{1313}=-\frac{(x^1)^2}{2}. \tag{29}$$

$$(K\cdot C)_{131312}=-(K\cdot C)_{141412}=-2(K\cdot C)_{121313}=-2(K\cdot C)_{143413}=\frac{(x^1)^2}{4}. \tag{30}$$

$$\left.\begin{array}{c}(K\cdot P)_{133112}=2(K\cdot P)_{121313}=-2(K\cdot P)_{132113}=-\frac{(x^1)^2}{2},\\ (K\cdot P)_{131312}=2(K\cdot P)_{123113}=-2(K\cdot P)_{141412}=-2(K\cdot P)_{121112}\\ =-2(K\cdot P)_{131213}=\frac{(x^1)^2}{3},\quad (K\cdot P)_{131113}=\frac{1}{12}(x^1)^3x^2x^3,\\ (K\cdot P)_{341413}=(K\cdot P)_{133313}=(K\cdot P)_{143413}=-(K\cdot P)_{231113}=-\frac{(x^1)^2}{12}.\end{array}\right\} \tag{31}$$

$$(K\cdot K)_{131312}=-2(K\cdot K)_{121313}=\frac{(x^1)^2}{2}. \tag{32}$$

$$(K\cdot Z)_{131312}=-(K\cdot Z)_{141412}=-2(K\cdot Z)_{121313}=-2(K\cdot Z)_{143413}=\frac{(x^1)^2}{4}. \tag{33}$$

$$(Z\cdot R)_{131312}=2(Z\cdot R)_{131434}=2(Z\cdot R)_{121414}=-2(Z\cdot R)_{121313}=\frac{(x^1)^2}{4}. \tag{34}$$

$$(Z\cdot S)_{1112}=2(Z\cdot S)_{1313}=-2(Z\cdot S)_{1414}=-\frac{(x^1)^2}{4}. \tag{35}$$

$$\left.\begin{array}{c}(Z\cdot C)_{131312}=(Z\cdot C)_{131434}=-(Z\cdot C)_{141412}=2(Z\cdot C)_{121414}\\ =-2(Z\cdot C)_{121313}=-2(Z\cdot C)_{133414}=-2(Z\cdot C)_{143413}=\frac{(x^1)^2}{8}.\end{array}\right\} \tag{36}$$

$$\left.\begin{array}{c}(Z\cdot P)_{121414}=(Z\cdot P)_{132113}=(Z\cdot P)_{141334}=(Z\cdot P)_{131434}=-(Z\cdot P)_{121313}\\ =-(Z\cdot P)_{134134}=-(Z\cdot P)_{142114}=-(Z\cdot P)_{143134}=-\frac{1}{2}(Z\cdot P)_{133112}=\frac{(x^1)^2}{8},\\ (Z\cdot P)_{123113}=(Z\cdot P)_{141214}=-(Z\cdot P)_{121112}=-(Z\cdot P)_{124114}=-(Z\cdot P)_{131213}\\ =-(Z\cdot P)_{141412}=\frac{1}{2}(Z\cdot P)_{131312}=2(Z\cdot P)_{134314}=2(Z\cdot P)_{144414}=2(Z\cdot P)_{231113}\\ =-2(Z\cdot P)_{133313}=-2(Z\cdot P)_{143413}=-2(Z\cdot P)_{241114}=-2(Z\cdot P)_{341314}\\ =-2(Z\cdot P)_{341413}=\frac{(x^1)^2}{12},\quad (Z\cdot P)_{131113}=-(Z\cdot P)_{141114}=\frac{1}{24}(x^1)^3x^2x^3.\end{array}\right\} \tag{37}$$

$$(Z\cdot K)_{131312}=2(Z\cdot K)_{121414}=2(Z\cdot K)_{131434}=-2(Z\cdot K)_{121313}=\frac{(x^1)^2}{4}. \tag{38}$$

$$\left.\begin{array}{c}(Z\cdot Z)_{131312}=(Z\cdot Z)_{131434}=-(Z\cdot Z)_{141412}=2(Z\cdot Z)_{121414}\\ =-2(Z\cdot Z)_{121313}=-2(Z\cdot Z)_{133414}=-2(Z\cdot Z)_{143413}=\frac{(x^1)^2}{8}.\end{array}\right\} \tag{39}$$

$$\left.\begin{array}{c}Q(g,R)_{121312}=Q(g,R)_{121434}=Q(g,R)_{131424}=-Q(g,R)_{132313}=-Q(g,R)_{123414}\\ =-Q(g,R)_{132414}=\frac{1}{2}Q(g,R)_{131323}=-2Q(g,R)_{121213}=\frac{x^1}{2}.\end{array}\right\} \tag{40}$$

$$\left.\begin{array}{c}Q(g,S)_{1312}=-Q(g,S)_{1213}=Q(g,S)_{3414}=Q(g,S)_{1434}=-\frac{1}{2}Q(g,S)_{1123}\\ =\frac{1}{2}Q(g,S)_{3313}=\frac{x^1}{2},\quad Q(g,S)_{1113}=-(x^1)^2x^2x^3.\end{array}\right\} \tag{41}$$

$$Q(S,R)_{131312} = -2Q(S,R)_{121313} = \frac{(x^1)^2}{2}. \tag{42}$$

$$\left.\begin{array}{c} Q(g,C)_{343413} = Q(g,C)_{131323} = Q(g,C)_{131424} = -Q(g,C)_{123414} = -Q(g,C)_{141423} \\ = -Q(g,C)_{121213} = 2Q(g,C)_{121312} = 2Q(g,C)_{143412} = -2Q(g,C)_{132313} \\ = -2Q(g,C)_{142413} = -2Q(g,C)_{132414} = 2Q(g,C)_{142314} = 2Q(g,C)_{121434} \\ = -2Q(g,C)_{133434} = \frac{x^1}{2}, \quad Q(g,C)_{141413} = -2Q(g,C)_{131414} = -\frac{1}{2}(x^1)^2x^2x^3. \end{array}\right\} \tag{43}$$

$$Q(S,C)_{131312} = 2Q(S,C)_{133414} = -2Q(S,C)_{121313} = -2Q(S,C)_{131434} = \frac{(x^1)^2}{4}. \tag{44}$$

$$\left.\begin{array}{c} Q(g,P)_{141413} = Q(g,P)_{131313} = Q(g,P)_{133113} = 2Q(g,P)_{121113} = 2Q(g,P)_{131112} \\ = 2Q(g,P)_{134114} = 2Q(g,P)_{141134} = 2Q(g,P)_{143114} = -\frac{1}{3}(x^1)^2x^2x^3, \\ Q(g,P)_{121312} = Q(g,P)_{121434} = Q(g,P)_{124314} = Q(g,P)_{131424} = Q(g,P)_{141324} \\ = Q(g,P)_{342114} = -Q(g,P)_{123414} = -Q(g,P)_{132112} = -Q(g,P)_{132414} \\ = -Q(g,P)_{134124} = -Q(g,P)_{142134} = -Q(g,P)_{143124} = -Q(g,P)_{241314} \\ = -Q(g,P)_{341214} = -\frac{1}{2}Q(g,P)_{133123} = \frac{x^1}{2}, \\ Q(g,P)_{131212} = Q(g,P)_{141234} = Q(g,P)_{134214} = Q(g,P)_{343413} = Q(g,P)_{243114} \\ = -Q(g,P)_{123112} = -Q(g,P)_{124134} = -Q(g,P)_{121123} = -Q(g,P)_{141423} \\ = 2Q(g,P)_{133213} = 2Q(g,P)_{134334} = 2Q(g,P)_{133312} = 2Q(g,P)_{144434} \\ = 2Q(g,P)_{143412} = 2Q(g,P)_{233113} = Q(g,P)_{341412} = Q(g,P)_{344414} \\ = -Q(g,P)_{142413} = -Q(g,P)_{231112} = -Q(g,P)_{143214} = -Q(g,P)_{234114} \\ = -Q(g,P)_{241134} = -Q(g,P)_{241413} = -Q(g,P)_{341334} = -Q(g,P)_{343314} \\ = \frac{2}{5}Q(g,P)_{122113} = -\frac{2}{5}Q(g,P)_{121213} = \frac{1}{2}Q(g,P)_{131323} \\ = -\frac{1}{2}Q(g,P)_{132313} = -\frac{1}{2}Q(g,P)_{231313} = \frac{x^1}{3}. \end{array}\right\} \tag{45}$$

$$\left.\begin{array}{c} Q(g,K)_{121312} = Q(g,K)_{131424} = Q(g,K)_{121434} = -Q(g,K)_{132313} = -Q(g,K)_{123414} \\ = -Q(g,K)_{132414} = \frac{1}{2}Q(g,K)_{131323} = -\frac{1}{2}Q(g,K)_{121213} = \frac{x^1}{2}. \end{array}\right\} \tag{46}$$

$$Q(S,K)_{131312} = -2Q(S,K)_{121313} = \frac{(x^1)^2}{2}. \tag{47}$$

$$\left.\begin{array}{c} Q(g,Z)_{131323} = Q(g,Z)_{131424} = Q(g,Z)_{343413} = -Q(g,Z)_{121213} = -Q(g,Z)_{123414} \\ = -Q(g,Z)_{141423} = 2Q(g,Z)_{121312} = 2Q(g,Z)_{142314} = 2Q(g,Z)_{121434} \\ = 2Q(g,Z)_{143412} = -2Q(g,Z)_{132313} = -2Q(g,Z)_{132414} = -2Q(g,Z)_{133434} \\ = -2Q(g,Z)_{142413} = \frac{x^1}{2}, \quad Q(g,Z)_{141413} = -2Q(g,Z)_{131414} = -\frac{1}{2}(x^1)^2x^2x^3. \end{array}\right\} \tag{48}$$

$$Q(S,Z)_{131312} = 2Q(S,Z)_{133414} = -2Q(S,Z)_{121313} = -2Q(S,Z)_{131434} = \frac{(x^1)^2}{4}. \tag{49}$$

$$\left.\begin{array}{c} Q(S,P)_{132113} = -Q(S,P)_{121313} = \frac{(x^1)^2}{4}, \quad Q(S,P)_{123113} \\ = Q(S,P)_{131123} = Q(S,P)_{133313} = -Q(S,P)_{131213} = 2Q(S,P)_{133414} \\ = 2Q(S,P)_{143314} = 2Q(S,P)_{143134} = 2Q(S,P)_{341314} = 2Q(S,P)_{231113} \\ = 2Q(S,P)_{341134} = -2Q(S,P)_{131434} = \frac{2}{5}Q(S,P)_{131312} \\ = -\frac{2}{5}Q(S,P)_{133112} = \frac{(x^1)^2}{6}, \quad Q(S,P)_{131113} = \frac{1}{6}(x^1)^3x^2x^3. \end{array}\right\} \tag{50}$$

In terms of local coordinates, if we consider the components of the 1-forms A and B as

$$\left.\begin{array}{l} A_i(x) = \begin{cases} 2x^1 & \text{for } \; i = 1 \\ 0 & \text{otherwise,} \end{cases} \\ \\ B_i(x) = \begin{cases} -\frac{2+(x^1)^2x^3}{2x^1} & \text{for } \; i = 1 \\ 0 & \text{otherwise,} \end{cases} \end{array}\right\} \tag{51}$$

at any point $x \in M$, then by virtue of (10), (11), and (51), it follows that (1) holds for all $i, j, h, k, l = 1, 2, 3, 4$.

In terms of local coordinates, the defining condition of a pseudo quasi-Einstein manifold can be expressed as

$$S_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma D_{ij}, \quad 1 \le i,j \le n. \tag{52}$$

We consider the associated scalars α, β, and γ, the components of the associated 1-form A, and the structure tensor D respectively as follows:

$$\alpha = -\frac{x^1}{2(1+x^1x^2x^3)}, \quad \beta = \frac{x^1(2+x^1x^2x^3)}{2(1+x^1x^2x^3)} \tag{53}$$

and γ is any nowhere vanishing scalar function,

$$A_i(x) = \begin{cases} 1 & \text{for } \; i=3, \\ 0 & \text{otherwise}, \end{cases}, \tag{54}$$

$$D_{ij}(x) = \begin{cases} \frac{(x^1)^2x^2x^3}{2\gamma(1+x^1x^2x^3)} & \text{for } \; i=1, j=1, \\ \frac{x^1}{2\gamma(1+x^1x^2x^3)} & \text{for } \; i=1, j=2, \\ \frac{x^1}{2\gamma} & \text{for } \; i=1, j=3, \\ -\frac{x^1}{2\gamma} & \text{for } \; i=3, j=3, \\ \frac{x^1}{2\gamma(1+x^1x^2x^3)} & \text{for } \; i=4, j=4, \\ 0 & \text{otherwise}, \end{cases} \tag{55}$$

at any point $x \in M$. Then it is easy to check that the generator U is taken as $(0,0,1,0)$ and $A =< \cdot, U >$, the structure tensor D is symmetric such that $trD = 0$ and $D(X,U) = 0$ for all X. From (55) it follows that D cannot be expressed as the outer product of two covectors. Then from (9), (10), (53), (54), and (55), it follows that (M^4, g) endowed with the metric (9) is pseudo quasi-Einstein. Also, from the above results, it is easy to check that the metric (9) does not satisfy any one of (i) to (xvi) of *Q.1* and (i) to (xiii) of *Q.2*.

Hence from (10), (11), (12), and (42), we can state the following:

Theorem 3.1. *Let (M^4, g) be a semi-Riemannian manifold equipped with the metric (9). Then (M^4, g) is (i) weakly generalized recurrent, (ii) pseudo quasi-Einstein, and (iii) fulfills the condition $R \cdot R = Q(S,R)$.*

As a consequence of $R \cdot R = Q(S,R)$ and $r = 0$ and also from the above calculations, we can state the following:

Theorem 3.2. *Let (M^4, g) be a semi-Riemannian manifold equipped with the metric (9). Then (M^4, g) satisfies the following:*
(i) $Q(g,C) = Q(g,Z)$, (ii) $Q(g,R) = Q(g,K)$,
(iii) $Q(S,R) = Q(S,K)$, (iv) $Q(S,C) = Q(S,Z)$,
(v) $R \cdot S = P \cdot S$, (vi) $R \cdot K = R \cdot R$,
(vii) $R \cdot K = K \cdot R$, (viii) $R \cdot K = Q(S,K)$,
(ix) $K \cdot R = Q(S,R)$, (x) $K \cdot K = Q(S,K)$,
(xi) $P \cdot R = \frac{2}{3}Q(S,R)$, (xii) $P \cdot K = \frac{2}{3}Q(S,K)$,
(xiii) $C \cdot Z = Z \cdot C = Z \cdot Z$, (xiv) $Z \cdot R = Z \cdot K$.

Again, if we consider the signature of the metric (9) as semi-Riemannian or Lorentzian given by

$$ds^2 = g_{ij}dx^idx^j = -x^1x^2x^3(dx^1)^2 + 2dx^1dx^2 - (dx^3)^2 - (dx^4)^2,$$
$$ds^2 = g_{ij}dx^idx^j = -x^1x^2x^3(dx^1)^2 + 2dx^1dx^2 - (dx^3)^2 + (dx^4)^2,$$
$$ds^2 = g_{ij}dx^idx^j = -x^1x^2x^3(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 - (dx^4)^2,$$
$$ds^2 = g_{ij}dx^idx^j = x^1x^2x^3(dx^1)^2 + 2dx^1dx^2 - (dx^3)^2 - (dx^4)^2,$$
$$ds^2 = g_{ij}dx^idx^j = -x^1x^2x^3(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (dx^4)^2,$$
$$ds^2 = g_{ij}dx^idx^j = x^1x^2x^3(dx^1)^2 + 2dx^1dx^2 - (dx^3)^2 + (dx^4)^2,$$
$$ds^2 = g_{ij}dx^idx^j = x^1x^2x^3(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 - (dx^4)^2,$$

$i,j = 1,2,3,4$, then it can be easily shown that the results of Theorem 3.1 and Theorem 3.2 remain unchanged.

Again if we consider the metric as

$$ds^2 = g_{ij}dx^idx^j = x^2x^3(dx^1)^2 + 2dx^1dx^2 + 2dx^1dx^3 + x^1(dx^3)^2 + (dx^4)^2 \tag{56}$$

and

$$ds^2 = g_{ij}dx^idx^j = x^1x^2x^3(dx^1)^2 + 2dx^1dx^3 + (dx^2)^2 + (dx^4)^2, \tag{57}$$

$i,j = 1,2,3,4$, then the results of Theorem 3.1 and Theorem 3.2 also remain unchanged. It may be mentioned that if the signature of the metrics (56) and (57) are considered as semi-Riemannian or Lorentzian, then the results of Theorem 3.1 and Theorem 3.2 also hold.

By extending the dimension of the metrics (9), (56), and (57) given as

$$ds^2 = g_{ij}dx^idx^j = x^1x^2x^3(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (dx^4)^2 + \delta_{ab}dx^adx^b, \tag{58}$$

$$ds^2 = g_{ij}dx^idx^j = x^2x^3(dx^1)^2 + 2dx^1dx^2 + 2dx^1dx^3 + x^1(dx^3)^2 + (dx^4)^2 + \delta_{ab}dx^adx^b, \tag{59}$$

and

$$ds^2 = g_{ij}dx^idx^j = x^1x^2x^3(dx^1)^2 + 2dx^1dx^3 + (dx^2)^2 + (dx^4)^2 + \delta_{ab}dx^adx^b, \tag{60}$$

where δ_{ab} denotes the Kronecker delta, $5 \le a,b \le n$, and $i,j = 1,2,...,n$, it is easy to check that the metrics (58), (59), and (60) are also WGK$_n$, pseudo quasi-Einstein, and fulfills the condition $R \cdot R = Q(S,R)$. We note that if we consider the metrics as

$$ds^2 = g_{ij}dx^i dx^j = f_{13}(x^1,x^3)(c_1 + x^2c_2)(dx^1)^2 + 2f_1(x^1)dx^1dx^2 + f_{13}(x^1,x^3)(dx^3)^2 + f_4(x^4)(dx^4)^2, \tag{61}$$

$$ds^2 = g_{ij}dx^i dx^j = f_{14}(x^1,x^4)(c_1 + x^2c_2)(dx^1)^2 + 2f_1(x^1)dx^1dx^2 + f_{13}(x^1,x^3)(dx^3)^2 + f_{14}(x^1,x^4)(dx^4)^2, \tag{62}$$

and

$$ds^2 = g_{ij}dx^i dx^j = f_{13}(x^1,x^3)(c_1 + x^2c_2)(dx^1)^2 + 2f_1(x^1)dx^1dx^2 + 2f_{13}(x^1,x^3)dx^1dx^3 + f_{13}(x^1,x^3)(dx^3)^2 + f_4(x^4)(dx^4)^2, \tag{63}$$

$i,j = 1,2,3,4$, where c_1, c_2 are constants, f_1, f_{13}, f_{14}, and f_4 are respectively the functions of x^1, (x^1,x^3), (x^1,x^4) and x^4, then they are also WGK_n, pseudo quasi-Einstein and realizes the condition $R \cdot R = Q(S,R)$. This leads to the following:

Theorem 3.3. *Let (M^n,g), $n \geq 4$, be a semi-Riemannian manifold equipped with any one metric given in (58) to (63). Then (M^n,g) is (i) weakly generalized recurrent, (ii) pseudo quasi-Einstein, and (iii) fulfills the condition $R \cdot R = Q(S,R)$.*

Conclusions

From the above results and discussion, we conclude that we obtain a new class of semi-Riemannian manifolds which is WGK_n, pseudo quasi-Einstein, and fulfills $R \cdot R = Q(S,R)$ but does not satisfy any one of conditions (i) to (xvi) of *Q.1* and conditions (i) to (xiii) of *Q.2*. Also, metrics (9) and (56) to (63) presented in the paper do not realize the defining conditions of any one of the following: (a) generalized quasi-Einstein by Chaki as well as by De and Ghosh, (b) hyper-generalized quasi-Einstein, (c) pseudo generalized quasi-Einstein, and (d) generalized pseudo quasi-Einstein.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

AAS participated in posing the problems and describing the techniques of the solutions and how to find out the metrics of such geometric structures. IR mainly made the calculations using a programme in Wolfram mathematica. FRAS read the manuscript and made many suggestions for its improvement. All authors read and approved the final manuscript.

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