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Research Article

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On the existence of a weak solution for some singular $p(x)$-biharmonic equation with Navier boundary conditions

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Abstract: In the present paper, we investigate the existence of solutions for the following inhomogeneous singular equation involving the $p(x)$-biharmonic operator:

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) = g(x)u^{-\gamma(x)} \mp \lambda f(x,u) & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with C^2 boundary, λ is a positive parameter, $\gamma : \overline{\Omega} \to (0,1)$ is a continuous function, $p \in C(\overline{\Omega})$ with $1 < p^- := \inf_{x\in\Omega} p(x) \leq p^+ := \sup_{x\in\Omega} p(x) < \frac{N}{2}$, as usual, $p^*(x) = \frac{Np(x)}{N-2p(x)}$,

$$g \in L^{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}}(\Omega),$$

and $f(x,u)$ is assumed to satisfy assumptions (f1)–(f6) in Section 3. In the proofs of our results, we use variational techniques and monotonicity arguments combined with the theory of the generalized Lebesgue Sobolev spaces. In addition, an example to illustrate our result is given.

Keywords: Navier boundary condition, singular problem, $p(x)$-biharmonic operator, variational methods, existence results, generalized Lebesgue Sobolev spaces

MSC 2010: 35J20, 35J60, 35G30, 35J35

1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with C^2 boundary condition. In this paper, we are dealing with the following problem:

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) = g(x)u^{-\gamma(x)} \mp \lambda f(x,u) & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathrm{P}_{\mp\lambda}$}$$

where λ is a positive parameter, $\gamma(x) \in C(\overline{\Omega})$ satisfying $0 < \gamma^- = \inf_{x\in\Omega} \gamma(x) \leq \gamma^+ = \sup_{x\in\Omega} \gamma(x) < 1$, $p \in C(\overline{\Omega})$ with $1 < p^- := \inf_{x\in\Omega} p(x) \leq p^+ := \sup_{x\in\Omega} p(x) < \frac{N}{2}$, as usual, $p^*(x) = \frac{Np(x)}{N-2p(x)}$, and

$$g \in L^{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}}(\Omega)$$

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and is almost everywhere positive in Ω. In the sequel, X will denote the Sobolev space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$. Associated to problem ($P_{\mp\lambda}$), we have the singular functional $I_{\mp\lambda} : X \to \mathbb{R}$ given by

$$I_{\mp\lambda}(u) = J(u) - \Phi_{\mp\lambda}(u),$$

where

$$J(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx$$

and

$$\Phi_{\mp\lambda}(u) = \int_\Omega \frac{g(x)}{1-\gamma(x)} |u|^{1-\gamma(x)}\, dx \mp \lambda \int_\Omega F(x, u(x))\, dx,$$

where $F(x, t) = \int_0^t f(x, s)\, ds$.

Definition 1.1. If for all $v \in X$,

$$\int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v\, dx = \int_\Omega g(x)|u|^{-\gamma(x)} v\, dx \mp \lambda \int_\Omega f(x, u) v\, dx,$$

then $u \in X$ is called a weak solution of ($P_{\mp\lambda}$).

The operator $\Delta^2_{p(x)} u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is called the $p(x)$-biharmonic operator of fourth order, where p is a continuous non-constant function. This differential operator is a natural generalization of the p-biharmonic operator $\Delta^2_p u := \Delta(|\Delta u|^{p-2}\Delta u)$, where $p > 1$ is a real constant. However, the $p(x)$-biharmonic operator possesses more complicated non-linearity than the p-biharmonic operator, due to the fact that $\Delta^2_{p(x)}$ is not homogeneous. This fact implies some difficulties; for example, we can not use the Lagrange multiplier theorem in many problems involving this operator.

The study of this kind of operators occurs in interesting areas such as electrorheological fluids (see [19]), elastic mechanics (see [25]), stationary thermo-rheological viscous flows of non-Newtonian fluids, image processing (see [6]) and the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium (see [1]).

Problem ($P_{\mp\lambda}$) is a new variant of $p(x)$-biharmonic equations due to the singular term and the indefinite one. Note that results for $p(x)$-Laplace equations with singular non-linearity are rare. Meanwhile, elliptic and singular elliptic equations involving the $p(x)$-Laplace and the $p(x)$-biharmonic operators can be found in [1, 2, 5, 8–10, 13, 14, 17, 18, 20–23].

Recently, Ayoujil and Amrouss [4] studied the following problem:

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda |u|^{q(x)-2} u & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P}$$

In the case when $\max_{x\in\Omega} q(x) < \min_{x\in\Omega} p(x)$, they proved that the energy functional associated to problem (P) has a nontrivial minimum for any positive λ; see [4, Theorem 3.1]. In the case when $\min_{x\in\Omega} q(x) < \min_{x\in\Omega} p(x)$ and $q(x)$ has a subcritical growth, they used Ekeland's variational principle in order to prove the existence of a continuous family of eigenvalues which lies in a neighborhood of the origin. Finally, when

$$\max_{x\in\Omega} p(x) < \min_{x\in\Omega} q(x) \le \max_{x\in\Omega} q(x) < \frac{Np(x)}{N-2p(x)},$$

they showed (see [4, Theorem 3.8]) that for every $\Lambda > 0$ the energy functional Φ_λ corresponding to (P) has a Mountain Pass-type critical point which is nontrivial and nonnegative, and hence $\Lambda = (0, +\infty)$, where Λ is the set of the eigenvalues. The same problem, for $p(x) = q(x)$, is studied by Ayoujil and Amrouss in [3]. They established the existence of infinitely many eigenvalues for problem (P) by using an argument based on the Ljusternik–Schnirelmann critical point theory. They showed that $\sup \Lambda = +\infty$, and they pointed out that only under special conditions, which are somehow connected with a kind of monotony of the function $p(x)$, one has $\inf \Lambda > 0$ (this is in contrast with the case when $p(x)$ is a constant, where one always has $\inf \Lambda > 0$).

Later, Ge, Zhou and Wu [10] considered the following problem:

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda V(x)|u|^{q(x)-2}u & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P1}$$

where V is an indefinite weight and λ is a positive real number. They considered different situations concerning the growth rates, and they proved, using the mountain pass lemma and Ekeland's principle, the existence of a continuous family of eigenvalues. A recent paper concerning this type of problems is [12].

Inspired by the above-mentioned papers, we study problem ($\mathrm{P}_{\mp\lambda}$), which contains a singular term and indefinite many more general terms than the one studied in [10]. In this new situation, we will show the existence of a weak solution for problem ($\mathrm{P}_{\mp\lambda}$). The paper is organized as follows: In Section 2, we recall some definitions concerning variable exponent Lebesgue spaces, $L^{p(x)}(\Omega)$, as well as Sobolev spaces, $W^{k,p(x)}(\Omega)$. Moreover, some properties of these spaces will also be exhibited to be used later. Our main results are stated in Section 3. The proofs of our results will be presented in Section 4 and Section 5.

2 Notations and preliminaries

To study $p(x)$-biharmonic problems, we need some results on the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ (for details, see [9, 18]) and some properties of the $p(x)$-biharmonic operator, which will be needed later. Set

$$C_+(\overline{\Omega}) := \{h : h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

Let p be a Lipschitz continuous function on $\overline{\Omega}$. We set $1 < p^- := \min_{x\in\overline{\Omega}} p(x) \le p^+ = \max_{x\in\overline{\Omega}} p(x) < \infty$ and

$$L^{p(x)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ measurable such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\Big\}.$$

We recall the following so-called Luxemburg norm on this space defined by the formula

$$|u|_{p(x)} = \inf\Big\{\mu > 0 : \int_\Omega \Big|\frac{u(x)}{\mu}\Big|^{p(x)}\,dx \le 1\Big\}.$$

Clearly, when $p(x) = p$, a positive constant, the space $L^{p(x)}(\Omega)$ reduces to the classical Lebesgue space $L^p(\Omega)$, and the norm $|u|_{p(x)}$ reduces to the standard norm

$$\|u\|_{L^p} = \Big(\int_\Omega |u|^p\,dx\Big)^{\frac{1}{p}} \quad \text{in } L^p(\Omega).$$

For any positive integer k, as in the constant exponent case, let

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega),\ |\alpha| \le k\},$$

where $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index, $|\alpha| = \sum_{i=1}^N \alpha_i$ and

$$D^\alpha u = \frac{\partial^{|\alpha|}u}{\partial^{\alpha_1}x_1 \cdots \partial^{\alpha_N}x_N}.$$

Then $W^{k,p(x)}(\Omega)$ is a separable and reflexive Banach space equipped with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha|\le k} |D^\alpha u|_{p(x)}.$$

Furthermore, $W_0^{k,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$. Let $L^{p'(x)}(\Omega)$ be the conjugate space of $L^{p(x)}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Then the Hölder-type inequality

$$\Big|\int_\Omega uv\,dx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)|u|_{p(x)}|v|_{p'(x)}, \quad u \in L^{p(x)}(\Omega),\ v \in L^{p'(x)}(\Omega) \tag{2.1}$$

holds.

The modular on the space $L^{p(x)}(\Omega)$ is the map $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) := \int_\Omega |u|^{p(x)}\,dx,$$

and it satisfies the following propositions.

Proposition 2.1 (see [16]). *For all $u \in L^{p(x)}(\Omega)$, we have the following assertions:*
(i) $|u|_{p(x)} < 1$ *(resp.* $= 1, > 1$*) if and only if* $\rho_{p(x)}(u) < 1$ *(resp.* $= 1, > 1$*).*
(ii) $\min(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}) \le \rho_{p(x)}(u) \le \max(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+})$.
(iii) $\rho_{p(x)}(u_n - u) \to 0$ *if and only if* $|u_n - u|_{p(x)} \to 0$.

Proposition 2.2 (see [7]). *Let p and q be two measurable functions such that $p \in L^\infty(\Omega)$ and $1 \le p(x)q(x) \le \infty$ for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then*

$$\min(|u|_{p(x)q(x)}^{p^+}, |u|_{p(x)q(x)}^{p^-}) \le ||u|^{p(x)}|_{q(x)} \le \max(|u|_{p(x)q(x)}^{p^-}, |u|_{p(x)q(x)}^{p^+}).$$

For more details concerning the modular, see [9, 16].

Definition 2.3. Assuming that E and F are Banach spaces, we define the norm on the space $X := E \cap F$ as $\|u\|_X = \|u\|_E + \|u\|_F$.

In order to discuss problems ($P_{\mp\lambda}$), we need some theories on the space $X := W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$. From Definition 2.3 we know that for any $u \in X$,

$$\|u\| = \|u\|_{1,p(x)} + \|u\|_{2,p(x)},$$

and thus

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)} + \sum_{|\alpha|=2} |D^\alpha u|_{p(x)}.$$

Zang and Fu [24], proved the equivalence of the norms, and they even proved that the norm $|\Delta u|_{p(x)}$ is equivalent to the norm $\|u\|$ (see [24, Theorem 4.4]). Let us choose on X the norm defined by $\|u\| = |\Delta u|_{p(x)}$. Note that $(X, \|\cdot\|)$ is also a separable and reflexive Banach space and that the modular is defined as $\rho_{p(x)} : X \to \mathbb{R}$ by $\rho_{p(x)}(\Delta u) = \int_\Omega |\Delta u|\,dx$ and satisfies the same properties as in Proposition 2.1. Hereafter, let

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - 2p(x)}, & p(x) < \dfrac{N}{2}, \\ +\infty, & p(x) \ge \dfrac{N}{2}. \end{cases}$$

Remark 2.4. If $q \in C^+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \Omega$, by [4, Theorem 3.2] we deduce that X is continuously and compactly embedded in $L^{q(x)}(\Omega)$.

Throughout this paper, the letters $k, c, C, C_i, i = 1, 2, \dots,$ denote positive constants which may change from line to line.

3 Hypotheses and main results

Let us impose the following hypotheses on the non-linearity $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$:
(f1) f is a C^1 function such that $f(x, 0) = 0$.
(f2) There exists $\Omega_1 \Subset \Omega$ with $|\Omega_1| > 0$, and a nonnegative function h_1 on Ω_1 such that $h_1 \in L^{s_1(x)}(\Omega)$ with

$$\lim_{|t|\to 0} \frac{f(x, t)}{h_1(x)|t|^{r_1(x)-1}} = 0 \quad \text{for } x \in \Omega \text{ uniformly.}$$

(f3) There exists a positive function h on Ω such that $h \in L^{s(x)}(\Omega)$ and

$$\lim_{|t|\to+\infty} \frac{f(x,t)}{h(x)|t|^{r(x)-1}} = 0 \quad \text{for } x \in \Omega \text{ uniformly},$$

where s, s_1, r and $r_1 \in C(\overline{\Omega})$ are such that $1 < \max\{r(x), r_1(x)\} < p(x) < \frac{N}{2} < \min(s(x), s_1(x))$ for all $x \in \Omega$.

(f4) There exists $A > 0$ such that

$$\int_\Omega F(x,t)\,\mathrm{d}x > 0 \quad \text{for all } t > A.$$

(f5) $f(x,t) \le Ch(x)|t|^{r(x)-2}t$ for all $t \in \mathbb{R}$ and all $x \in \overline{\Omega}$, where C is a positive constant, $h \in L^{s(x)}(\Omega)$ and $s, r \in C(\overline{\Omega})$ are such that for all $x \in \overline{\Omega}$ we have $1 < r(x) < p(x) < \frac{N}{2} < s(x)$.

(f6) There exists $\Omega_1 \Subset \Omega$ with $|\Omega_1| > 0$ such that $f(x,t), h(x) > 0$ in Ω_1.

Some remarks regarding the hypotheses are in order.

Remark 3.1. Under assumptions (f3) and (f4), we have the following assertions.

(i) $(1-\gamma(x))p(x) < p(x) < p^*(x)$ for any $x \in \overline{\Omega}$, so the injection of $X \hookrightarrow L^{(1-\gamma(x))p(x)}(\Omega)$ is compact and continuous.

(ii) $s'(x)r(x) < p^*(x)$ for any $x \in \overline{\Omega}$, where $\frac{1}{s(x)} + \frac{1}{s'(x)} = 1$, so $X \hookrightarrow L^{s'(x)r(x)}(\Omega)$ is compact and continuous.

(iii) $s_1'(x)r_1(x) < p^*(x)$ for any $x \in \overline{\Omega}$, where $\frac{1}{s_1(x)} + \frac{1}{s_1'(x)} = 1$, so $X \hookrightarrow L^{s_1'(x)r_1(x)}(\Omega)$ is compact and continuous.

(iv) $X \hookrightarrow L^{p^*(x)}(\Omega)$ is continuous.

Moreover, under conditions (f5) and (f6), we remark the following.

Remark 3.2. (i) There exists $K > 0$ such that $\eta f(x,\eta) \le Kr(x)F(x,\eta)$ for all $x \in \Omega_1$, $\eta \in \mathbb{R}_+$.

(ii) Due to condition (f5), there exists $C' > 0$ such that

$$F(x,\eta) \le C'h(x)|\eta|^{r(x)-1}\eta \quad \text{in } \Omega \times \mathbb{R}.$$

(iii) Conditions (f5) and (f6) assure that $F(x,\eta) > 0$ for all $x \in \Omega_1$, $\eta \in \mathbb{R}$.

(iv) Put $f(x,t) = Ch(x)|t|^{r(x)-2}t$, $x \in \Omega$, $t \in \mathbb{R}$. Then the first condition in the remark is satisfied.

Here we state our main results asserted in the following two theorems.

Theorem 3.3. *Assume that hypotheses* (f1), (f2), (f3) *and* (f4) *are fulfilled. Then for all* $\lambda > 0$ *problem* $(\mathrm{P}_{-\lambda})$ *has at least one nontrivial weak solution with negative energy.*

Theorem 3.4. *Assume that hypotheses* (f5) *and* (f6) *are fulfilled. Then for all* $\lambda > 0$ *problem* $(\mathrm{P}_{+\lambda})$ *has at least one nontrivial weak solution with negative energy.*

4 Proof of Theorem 3.3

The study of the existence of solutions to problem $(\mathrm{P}_{-\lambda})$ is done by looking for critical points to the functional $I_{-\lambda}\colon X \to \mathbb{R}$ defined by

$$I_{-\lambda}(u_\lambda) = \int_\Omega \frac{|\Delta u_\lambda|^{p(x)}}{p(x)}\,\mathrm{d}x - \int_\Omega \frac{g(x)u_\lambda^{1-\gamma(x)}}{1-\gamma(x)}\,\mathrm{d}x + \lambda \int_\Omega F(x,u_\lambda)\,\mathrm{d}x$$

in the Sobolev space X. The proof of the Theorem 3.3 is organized in several lemmas. Firstly, under Remark 3.1 one has

$$|u_\lambda|_{(1-\gamma(x))p(x)} \le C_2\|u_\lambda\| \quad \text{for all } u_\lambda \in X \tag{4.1}$$

and

$$|u_\lambda|_{s'(x)r(x)} \le C_3\|u_\lambda\| \quad \text{for all } u_\lambda \in X.$$

Now, we are in a position to show that $I_{-\lambda}$ possesses a nontrivial global minimum point in X.

Lemma 4.1. *Under assumptions* (f2), (f3) *and* (f4), *the functional* $I_{-\lambda}$ *is coercive on* X.

Proof. First, we recall that in view of assumptions (f3), (f4), inequality (4.1), Remark 3.1 and Proposition 2.1 one has for any $u_\lambda \in X$ with $\|u_\lambda\| > \max(1, A)$,

$$\begin{aligned} I_{-\lambda}(u_\lambda) &\geq \int_\Omega \frac{1}{p(x)}|\Delta u_\lambda|^{p(x)}\,dx - C_4|g|_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}}\,||u_\lambda|^{1-\gamma(x)}|_{p^*(x)} + \lambda\int_\Omega F(x,u_\lambda)\,dx \\ &\geq \int_\Omega \frac{1}{p(x)}|\Delta u_\lambda|^{p(x)}\,dx - C_4|g|_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}}\,||u_\lambda|^{1-\gamma(x)}|_{p(x)} \\ &\geq \int_\Omega \frac{1}{p(x)}|\Delta u_\lambda|^{p(x)}\,dx - C_4|g|_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}}\,\|u_\lambda\|^{1-\gamma^-} \\ &\geq \frac{1}{p^+}\rho_{p(x)}(\Delta u_\lambda) - C_4|g|_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}}\,\|u_\lambda\|^{1-\gamma^-} \\ &\geq \frac{1}{p^+}\|u_\lambda\|^{p^-} - C_4|g|_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}}\,\|u_\lambda\|^{1-\gamma^-}. \end{aligned}$$

Since $1-\gamma^- < p^-$, we infer that $I_{-\lambda}(u_\lambda) \to \infty$ as $\|u_\lambda\| \to \infty$; in other words, $I_{-\lambda}$ is coercive on X. The proof of Lemma 4.1 is now completed. □

Lemma 4.2. *Suppose assumptions* (f2) *and* (f3) *are fulfilled. Then there exists* $\varphi \in X$ *such that* $\varphi \geq 0$, $\varphi \neq 0$ *and* $I_{-\lambda}(t\varphi) < 0$ *for* $t > 0$ *small enough.*

Proof. Let $\varphi \in C_0^\infty(\Omega)$ such that $\operatorname{supp}(\varphi) \subset \Omega_1 \Subset \Omega$, $\varphi = 1$ in a subset $\Omega' \subset \operatorname{supp}(\varphi)$ and $0 \leq \varphi \leq 1$ in Ω_1. Using assertions on the functions g and F and assumption (f2), we have

$$\begin{aligned} I_{-\lambda}(t\varphi) &= \int_\Omega \frac{1}{p(x)}|\Delta t\varphi|^{p(x)}\,dx - \int_\Omega \frac{1}{1-\gamma(x)}g(x)|t\varphi|^{1-\gamma(x)}\,dx + \lambda\int_\Omega F(x,t\varphi)\,dx \\ &\leq \frac{t^{p^-}}{p^-}\rho_{p(x)}(\Delta\varphi) - \frac{t^{1-\gamma^-}}{1-\gamma^-}\int_\Omega g(x)|\varphi|^{1-\gamma(x)}\,dx + \lambda C_1 t^{r_1^-}\int_{\Omega_1} h_1(x)|\varphi|^{r_1(x)}\,dx \\ &\leq t^{r_1^-}\bigg[\frac{1}{p^-}\rho_{p(x)}(\Delta\varphi) + \lambda C_1\int_{\Omega_1} h_1(x)|\varphi|^{r_1(x)}\,dx\bigg] - \frac{t^{1-\gamma^-}}{1-\gamma^-}\int_\Omega g(x)|\varphi|^{1-\gamma(x)}\,dx, \end{aligned}$$

so

$$I_{-\lambda}(t\varphi) < 0 \quad \text{for } t < \psi^{\frac{1}{r_1^- -(1-\gamma^-)}}$$

with

$$0 < \psi < \min\bigg\{1, \frac{\frac{1}{1-\gamma^-}\int_\Omega g(x)|\varphi|^{1-\gamma(x)}\,dx}{\frac{1}{p^-}\rho_{p(x)}(\Delta\varphi) + \lambda C_1\int_{\Omega_1} h_1(x)|\varphi|^{r_1(x)}\,dx}\bigg\}.$$

Finally, we point out, using the hypothesis on φ and the definition of the modular on X, that

$$\frac{1}{p^-}\rho_{p(x)}(\Delta\varphi) + \lambda C\int_{\Omega_1} h_1(x)|\varphi|^{r_1(x)}\,dx > 0.$$

In fact, if

$$\frac{1}{p^-}\rho_{p(x)}(\Delta\varphi) + \lambda C_1\int_{\Omega_1} h_1(x)|\varphi|^{r_1(x)}\,dx = 0,$$

then $\rho_{p(x)}(\Delta\varphi) = 0$, and consequently $\|\varphi\| = 0$, which contradicts the choice of φ and gives the proof of Lemma 4.2. □

In the sequel, we put $m_\lambda = \inf_{u_\lambda \in X} I_{-\lambda}(u_\lambda)$. Then we have the following lemma.

Lemma 4.3. *Let* $\lambda \geq 0$, $\gamma \in C(\bar\Omega, (0,1))$,

$$g \in L^{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}}(\Omega)$$

with $g(x) > 0$ *for almost every* $x \in \Omega$, *and assume that hypothesis* (f1), (f2), (f3) *and* (f4) *are fulfilled. Then* $I_{-\lambda}$ *reaches its global minimizer in* X, *that is, there exists* $u_\lambda \in X$ *such that* $I_{-\lambda}(u_\lambda) = m_\lambda < 0$.

Proof. Let $\{u_n\}$ be a minimizing sequence, that is to say $I_{-\lambda}(u_n) \to m_\lambda$. Suppose $\{u_n\}$ is not bounded, so $\|u_n\| \to +\infty$ as $n \to +\infty$. Since I_λ is coercive, we have

$$I_{-\lambda}(u_n) \to +\infty \quad \text{as } \|u_n\| \to +\infty.$$

This contradicts the fact that $\{u_n\}$ is a minimizing sequence, so $\{u_n\}$ is bounded in X, and therefore, up to a subsequence, there exists $u_\lambda \in X$ such that

$$\begin{aligned} u_n &\rightharpoonup u_\lambda && \text{weakly in } X, \\ u_n &\to u_\lambda && \text{strongly in } L^{s(x)}(\Omega),\ 1 \le s(x) < p^*(x), \\ u_n(x) &\to u_\lambda(x) && \text{a.e. in } \Omega. \end{aligned}$$

Since $J : X \to \mathbb{R}$ is sequentially weakly lower semi-continuous (see [12]), we have

$$\int_\Omega \frac{1}{p(x)} |\Delta u_\lambda|^{p(x)} \le \liminf_{n\to+\infty} \int_\Omega \frac{1}{p(x)} |\Delta u_n|^{p(x)}. \tag{4.2}$$

On the other hand, by Vital's theorem (see [16, p. 113]), we can claim that

$$\lim_{n\to\infty} \int_\Omega g(x)|u_n|^{1-\gamma(x)}\,dx = \int_\Omega g(x)|u_\lambda|^{1-\gamma(x)}\,dx. \tag{4.3}$$

Indeed, we only need to prove that

$$\Big\{ \int_\Omega g(x)|u_n|^{1-\gamma(x)}\,dx : n \in \mathbb{N} \Big\}$$

is equi-absolutely-continuous. Note that $\{u_n\}$ is bounded in X, so Remark 3.1 implies that $\{u_n\}$ is bounded in $L^{p^*(x)}(\Omega)$. For every $\varepsilon > 0$, using Proposition 2.1 and the absolutely-continuity of

$$\int_\Omega |g(x)|^{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}}\,dx,$$

there exist $\zeta, \xi > 0$ such that

$$|g|^\zeta_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}} \le \int_\Omega |g(x)|^{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}}\,dx \le \varepsilon^\zeta \quad \text{for any } \Omega_2 \subset \Omega \text{ with } |\Omega_2| < \xi.$$

Consequently, by the Hölder inequality and Proposition 2.1 one has

$$\int_\Omega |g(x)|\,|u_n|^{1-\gamma(x)}\,dx \le |g|_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}} \big||u_n|^{1-\gamma(x)}\big|_{p^*(x)} \le |g|_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}} |u_n|^k_{(1-\gamma(x))p^*(x)}.$$

Since $(1-\gamma(x))p^*(x) < p^*(x)$, we have

$$|u_n|_{(1-\gamma(x))p^*(x)} \le C_7 |u_n|_{p^*(x)},$$

so

$$\int_\Omega |g(x)|\,|u_n|^{1-\gamma(x)}\,dx \le |g|_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}} \big||u_n|^{1-\gamma(x)}\big|_{p^*(x)} \le |g|_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}} |u_n|^k_{(1-\gamma(x))p^*(x)} < \varepsilon C_7^k |u_n|^k_{p^*(x)}.$$

Since $|u_n|_{p^*(x)}$ is bounded, claim (4.3) is valid.

In what follows, we remark, using assumptions (f2) and (f3), that for all $\varepsilon > 0$ there exists C_ε such that

$$|F(x, u_\lambda(x))| \le \varepsilon \frac{C_1}{r_1^-} |h_1(x)|\,|u_\lambda|^{r_1(x)} + C_\varepsilon \frac{C}{r^-} |h(x)|\,|u_\lambda|^{r(x)}.$$

Then by the Hölder inequality one has

$$\int_\Omega |F(x, u_\lambda(x)| \le \varepsilon \frac{C_1}{r_1^-} |h_1|_{s_1(x)} \big||u_\lambda|^{r_1(x)}\big|_{s_1'(x)} + C_\varepsilon \frac{C}{r^-} |h|_{s(x)} \big||u_\lambda|^{r(x)}\big|_{s'(x)}.$$

Besides, if $u_n \rightharpoonup u_\lambda$ in X, then we have strong convergence in $L^{s'(x)r(x)}(\Omega)$ and $L^{s_1'(x)r_1(x)}(\Omega)$. So the Lebesgue dominated convergence theorem and Proposition 2.2 enable us to state the following assertion: If

$$u \mapsto \lambda \int_\Omega F(x, u_\lambda(x))\,dx$$

is weakly continuous, then

$$\lim_{n\to+\infty} \Phi_{-\lambda}(u_n) = \Phi_{-\lambda}(u_\lambda). \tag{4.4}$$

Using (4.2), (4.3) and (4.4), we deduce that $I_{-\lambda}$ is weakly lower semi-continuous, and consequently

$$m_\lambda \le I_{-\lambda}(u_\lambda) \le \liminf_{n\to+\infty} I_{-\lambda}(u_n) = m_\lambda.$$

The proof of Lemma 4.3 is now completed. □

Proof of Theorem 3.3. Now, let us show that the weak limit u_λ is a weak solution of problem $(\mathrm{P}_{-\lambda})$ if $\lambda > 0$ is sufficiently large. Firstly, observe that $I_{-\lambda}(0) = 0$. So, in order to prove that the solution is nontrivial, it suffices to prove that there exists $\lambda_* > 0$ such that

$$\inf_{u_\lambda\in X} I_{-\lambda}(u_\lambda) < 0 \quad \text{for all } \lambda > 0.$$

For this purpose, we consider the variational problem with constraints

$$\lambda_* := \inf\Big\{\int_\Omega \frac{1}{p(x)}|\Delta w|^{p(x)}\,dx + \int_\Omega \frac{g(x)}{1-\gamma(x)}|w|^{1-\gamma(x)}\,dx : w \in X \text{ and } \int_\Omega F(x, w(x))\,dx = 1\Big\}, \tag{4.5}$$

and define

$$\Lambda_* := \inf\{\lambda > 0 \colon (\mathrm{P}_{-\lambda}) \text{ admits a nontrivial weak solution}\}.$$

From above we have

$$I_{-\lambda}(u_\lambda) = \lambda_* - \lambda < 0 \quad \text{for any } \lambda > \lambda_*.$$

Therefore, the above remarks show that $\lambda_* \ge \Lambda_*$ and that problem $(\mathrm{P}_{-\lambda})$ has a solution for all $\lambda > \lambda_*$.

We now argue that problem $(\mathrm{P}_{-\lambda})$ has a solution for all $\lambda > \Lambda_*$. Fixing $\lambda > \Lambda_*$, by the definition of Λ_* we can take $\mu \in (\Lambda_*, \lambda)$ such that $I_{-\mu}$ has a nontrivial critical point $u_\mu \in X$. Since $\mu < \lambda$, we obtain that u_μ is a sub-solution of problem $(\mathrm{P}_{-\lambda})$. We now want to construct a super-solution of problem $(\mathrm{P}_{-\lambda})$ which dominates u_μ. For this purpose, we introduce the constrained minimization problem

$$\inf\{I_{-\lambda}(w) : w \in X \text{ and } w \ge u_\mu\}.$$

By using the previous arguments to treat (4.5), follows that the above minimization problem has a solution $u_\lambda > u_\mu$. Moreover, u_λ is also a weak solution of problem $(\mathrm{P}_{-\lambda})$ for all $\lambda > \Lambda_*$. With the arguments developed in [15], we deduce that problem $(\mathrm{P}_{-\lambda})$ has a solution if $\lambda = \Lambda_*$.

Now, it remains to show that $\Delta u_\lambda = 0$ on $\partial\Omega$. Due to the above arguments, one has

$$\int_\Omega |\Delta u_\lambda|^{p(x)-2}\Delta u_\lambda \Delta v\,dx = \int_\Omega m(x) v\,dx \quad \text{for all } v \in X, \tag{4.6}$$

where

$$m(x) = g(x)u_\lambda^{-\gamma(x)} - \lambda f(x, u_\lambda).$$

Relation (4.6) implies that

$$\int_\Omega |\Delta u_\lambda|^{p(x)-2}\Delta u_\lambda \Delta v\,dx = \int_\Omega m(x) v\,dx \quad \text{for all } v \in C_0^\infty(\Omega). \tag{4.7}$$

Let ζ be the unique solution of the problem

$$\begin{cases} \Delta\zeta = m(x) & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases}$$

Relation (4.7) yields

$$\int_\Omega |\Delta u_\lambda|^{p(x)-2}\Delta u_\lambda \Delta v\,\mathrm{d}x = \int_\Omega (\Delta\zeta) v\,\mathrm{d}x \quad \text{for all } v \in C_0^\infty(\Omega).$$

Using the Green formula, we have

$$\int_\Omega (\Delta\zeta) v\,\mathrm{d}x = \int_\Omega \zeta\Delta v\,\mathrm{d}x.$$

Therefore,

$$\int_\Omega |\Delta u_\lambda|^{p(x)-2}\Delta u_\lambda \Delta v\,\mathrm{d}x = \int_\Omega \zeta\Delta v\,\mathrm{d}x \quad \text{for all } v \in C_0^\infty(\Omega). \tag{4.8}$$

On the other hand, for all $\tilde{u_\lambda} \in C_0^\infty(\Omega)$ there exists a unique $v \in C_0^\infty(\Omega)$ such that $\Delta v = \tilde{u_\lambda}$ in Ω. Thus, relation (4.8) can be rewritten as

$$\int_\Omega (|\Delta u_\lambda|^{p(x)-2}\Delta u_\lambda - \zeta)\tilde{u_\lambda}\,\mathrm{d}x = 0 \quad \text{for all } \tilde{u_\lambda} \in C_0^\infty(\Omega).$$

Applying the fundamental lemma of the calculus of variations, we deduce that

$$|\Delta u_\lambda|^{p(x)-2}\Delta u_\lambda - \zeta = 0 \quad \text{in } \Omega.$$

Since $\zeta = 0$ on $\partial\Omega$, we conclude that $\Delta u_\lambda = 0$ on $\partial\Omega$. Thus, u_λ is a nontrivial weak solution of problem $(\mathrm{P}_{-\lambda})$ such that $\Delta u_\lambda = 0$. This completes the proof of Theorem 3.3. □

5 Proof of Theorem 3.4

The proof of Theorem 3.4 is organized in several lemmas. Firstly, we show the existence of a local minimum for $I_{+\lambda}$ in a small neighborhood of the origin in X.

Lemma 5.1. *Under assumption* (f5), *the functional $I_{+\lambda}$ is coercive on X.*

Proof. Using Remark 3.1, inequality (2.1) and Proposition 2.1, we obtain that for any $v_\lambda \in X$ with $\|v_\lambda\| > 1$,

$$\begin{aligned}
I_{+\lambda}(v_\lambda) &= \int_\Omega \frac{1}{p(x)}|\Delta v_\lambda|^{p(x)}\,\mathrm{d}x - \frac{1}{1-\gamma(x)}\int_\Omega g(x)|v_\lambda|^{1-\gamma(x)}\,\mathrm{d}x - \lambda\int_\Omega F(x, v_\lambda(x))\,\mathrm{d}x \\
&\geq \frac{1}{p^+}\rho_{p(x)}(\Delta v_\lambda) - \frac{1}{1-\gamma^+}\int_\Omega g(x)|v_\lambda|^{1-\gamma(x)}\,\mathrm{d}x - \lambda\int_\Omega F(x, v_\lambda(x))\,\mathrm{d}x \\
&\geq \frac{1}{p^+}\|v_\lambda\|^{p^-} - \frac{1}{1-\gamma^+}|g|_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}}\,||v_\lambda|^{1-\gamma(x)}|_{p^*(x)} - C'\lambda\int_\Omega h(x)|v_\lambda|^{r(x)}\,\mathrm{d}x \\
&\geq \frac{1}{p^+}\|v_\lambda\|^{p^-} - \frac{1}{1-\gamma^+}|g|_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}}\,||v_\lambda|^{1-\gamma(x)}|_{p(x)} - C'\lambda|h|_{s(x)}||v_\lambda|^{r(x)}|_{s'(x)}\,\mathrm{d}x \\
&\geq \frac{1}{p^+}\|v_\lambda\|^{p^-} - \frac{1}{1-\gamma^+}|g|_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}}\,\min(|v_\lambda|^{1-\gamma^+}_{(1-\gamma(x))p(x)}, |v_\lambda|^{1-\gamma^-}_{(1-\gamma(x))p(x)}) \\
&\qquad - C'\lambda|h|_{s(x)}\min(|v_\lambda|^{r^+}_{s'(x)r(x)}, |v_\lambda|^{r^-}_{s'(x)r(x)}) \\
&\geq \frac{1}{p^+}\|v_\lambda\|^{p^-} - \frac{1}{1-\gamma^+}|g|_{\frac{p^*(x)}{p^*(x)+\gamma-1}}\,\min(C_1^{1-\gamma^+}\|v_\lambda\|^{1-\gamma^+}, C_1^{1-\gamma^-}\|v_\lambda\|^{1-\gamma^-}) \\
&\qquad - C'\lambda|h|_{s(x)}\min(C_2^{r^+}\|v_\lambda\|^{r^+}, C_2^{r^-}\|v_\lambda\|^{r^-}).
\end{aligned}$$

Since $1-\gamma^- < r^+ < p^-$, we infer that $I_\lambda(v_\lambda) \to \infty$ as $\|v_\lambda\| \to \infty$ and $I_{+\lambda}$ is coercive on X. This ends the proof of Lemma 5.1. □

Lemma 5.2. *Under assumptions* (f5) *and* (f6), *there exists* $\varphi \in X$ *such that* $\varphi \geq 0$, $\varphi \neq 0$ *and* $I_{+\lambda}(t\varphi) < 0$ *for* $t > 0$ *small enough.*

Proof. Let $\varphi \in C_0^\infty(\Omega)$ such that $\operatorname{supp}(\varphi) \subset \Omega_1 \Subset \Omega$, $\varphi = 1$ in a subset $\Omega' \subset \operatorname{supp}(\varphi)$ and $0 \leq \varphi \leq 1$ in Ω_1. Using assertions on the functions g and F, we have

$$\begin{aligned} I_{+\lambda}(t\varphi) &= \int_\Omega \frac{1}{p(x)}|\Delta t\varphi|^{p(x)}\,dx - \int_\Omega \frac{1}{1-\gamma(x)} g(x)|t\varphi|^{1-\gamma(x)}\,dx - \lambda \int_\Omega F(x,t\varphi)\,dx \\ &\leq \int_\Omega \frac{t^{p(x)}}{p^-}|\Delta\varphi|^{p(x)}\,dx - \int_\Omega \frac{g(x)}{1-\gamma(x)}|t\varphi|^{1-\gamma(x)}\,dx - \lambda \int_\Omega F(x,t\varphi)\,dx \\ &\leq \frac{t^{p^-}}{p^-}\int_\Omega |\Delta\varphi|^{p(x)}\,dx - t^{1-\gamma^-}\int_\Omega \frac{g(x)|\varphi|^{1-\gamma(x)}}{1-\gamma(x)}\,dx - \lambda \int_{\Omega 1} F(x,t\varphi)\,dx. \end{aligned}$$

Then

$$I_{+\lambda}(t\varphi) \leq \frac{t^{p^-}}{p^-}\int_\Omega |\Delta\varphi|^{p(x)}\,dx - \frac{t^{1-\gamma^-}}{1-\gamma^-}\int_\Omega g(x)|\varphi|^{1-\gamma(x)}\,dx.$$

Since $p^- > 1-\gamma^-$, we have $I_{+\lambda}(t\varphi) < 0$ for $t < \psi^{1/(p^- -(1-\gamma^-))}$ with

$$0 < \psi < \min\left\{1, \frac{\frac{p^-}{1-\gamma^-}\int_\Omega g(x)|\varphi|^{1-\gamma(x)}\,dx}{\rho_{p(x)}(\Delta\varphi)}\right\}.$$

Finally, we point out that $\rho_{p(x)}(\Delta\varphi) > 0$. In fact, if $\rho_{p(x)}(\Delta\varphi) = 0$, then $\|\varphi\| = 0$, and consequently $\varphi = 0$ in Ω, which is a contradiction. □

In the sequel, put $m_\lambda^1 = \inf_{v_\lambda \in X} I_{+\lambda}(v_\lambda)$. As a last proposition, we have the following.

Lemma 5.3. *Let* $\lambda \geq 0$, $\gamma \in C(\bar{\Omega}, (0,1))$,

$$g \in L^{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}}(\Omega)$$

with $g(x) > 0$ *for almost every* $x \in \Omega$, *and assume that assertions* (f5) *and* (f6) *hold. Then* $I_{+\lambda}$ *reaches its global minimizer in* X, *that is, there exists* $v_\lambda \in X$ *such that* $I_{+\lambda}(v_\lambda) = m_\lambda^1 < 0$.

Proof. The proof of Lemma 5.3 is word for word as the one of Lemma 4.3. □

Proof of Theorem 3.4. From Lemma 5.3, v_λ is a local minimizer for $I_{+\lambda}$, with $I_{+\lambda}(v_\lambda) = m_\lambda < 0$, which implies that v_λ is nontrivial. Now, we prove that v_λ is a positive solution of problem $(\mathrm{P}_{+\lambda})$. Our proof is inspired by Saoudi and Ghanmi in [11].

Let $\phi \in X$ and $0 < \epsilon < 1$. We define $\Psi \in X$ by $\Psi := (v_\lambda + \epsilon\phi)^+$, where $(v_\lambda + \epsilon\phi)^+ = \max\{v_\lambda + \epsilon\phi, 0\}$. Since v_λ is a local minimizer for $I_{+\lambda}$, one has

$$\begin{aligned} 0 &\leq \int_\Omega |\Delta v_\lambda|^{p(x)-2}\Delta v_\lambda \Delta\Psi\,dx - \int_\Omega g(x)v_\lambda^{-\gamma(x)}\Psi\,dx - \lambda\int_\Omega f(x,v_\lambda)\Psi\,dx \\ &= \int_{\{x:v_\lambda+\epsilon\phi>0\}} |\Delta v_\lambda|^{p(x)-2}\Delta v_\lambda \Delta(v_\lambda+\epsilon\phi)\,dx - \int_{\{x:v_\lambda+\epsilon\phi>0\}} g(x)u^{-\gamma(x)}(v_\lambda+\epsilon\phi)\,dx \\ &\quad - \lambda \int_{\{x:v_\lambda+\epsilon\phi>0\}} f(x,v_\lambda)(v_\lambda+\epsilon\phi)\,dx \\ &= \int_\Omega |\Delta v_\lambda|^{p(x)-2}\Delta v_\lambda \Delta(v_\lambda+\epsilon\phi)\,dx - \int_\Omega g(x)u^{-\gamma(x)}(v_\lambda+\epsilon\phi)\,dx - \lambda\int_\Omega f(x,v_\lambda)(v_\lambda+\epsilon\phi)\,dx \\ &\quad - \int_{\{x:v_\lambda+\epsilon\phi\leq 0\}} |\Delta v_\lambda|^{p(x)-2}\Delta v_\lambda \Delta(v_\lambda+\epsilon\phi)\,dx - \int_{\{x:v_\lambda+\epsilon\phi\leq 0\}} g(x)u^{-\gamma(x)}(v_\lambda+\epsilon\phi)\,dx \\ &\quad - \lambda\int_{\{x:v_\lambda+\epsilon\phi\leq 0\}} f(x,v_\lambda)(v_\lambda+\epsilon\phi)\,dx \end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega |\Delta v_\lambda|^{p(x)}\,dx - \int_\Omega g(x)v_\lambda^{1-\gamma(x)}\,dx - \lambda\int_\Omega f(x,v_\lambda)v_\lambda\,dx \\
&\quad + \epsilon\Big(\int_\Omega |\Delta v_\lambda|^{p(x)-2}\Delta v_\lambda \nabla v_\lambda \nabla\phi\,dx - \int_\Omega g(x)v_\lambda^{-\gamma(x)}\phi\,dx - \lambda\int_\Omega f(x,v_\lambda)\phi\,dx\Big) \\
&\quad + \epsilon^2\int_\Omega |\Delta v_\lambda|^{p(x)-2}\Delta v_\lambda\Delta\phi\,dx - \int_{\{x:v_\lambda+\epsilon\phi\le 0\}} |\Delta v_\lambda|^{p(x)-2}\Delta v_\lambda\Delta(v_\lambda+\epsilon\phi)\,dx \\
&\quad - \int_{\{x:v_\lambda+\epsilon\phi\le 0\}} g(x)v_\lambda^{-\gamma(x)}(v_\lambda+\epsilon\phi)\,dx - \lambda\int_{\{x:v_\lambda+\epsilon\phi\le 0\}} f(x,v_\lambda)(v_\lambda+\epsilon\phi)\,dx \\
&= \epsilon\Big(\int_\Omega |\Delta v_\lambda|^{p(x)-2}\Delta v_\lambda \nabla u\nabla\phi\,dx - \int_\Omega g(x)v_\lambda^{-\gamma(x)}\phi\,dx - \lambda\int_\Omega f(x,v_\lambda)\phi\,dx\Big) \\
&\quad + \epsilon^2\int_\Omega |\Delta v_\lambda|^{p(x)-2}\Delta v_\lambda\Delta\phi\,dx - \int_{\{x:v_\lambda+\epsilon\phi\le 0\}} |\Delta v_\lambda|^{p(x)-2}\Delta v_\lambda\Delta(v_\lambda+\epsilon\phi)\,dx \\
&\quad - \int_{\{x:v_\lambda+\epsilon\phi\le 0\}} g(x)v_\lambda^{-\gamma(x)}(v_\lambda+\epsilon\phi)\,dx - \lambda\int_{\{x:v_\lambda+\epsilon\phi\le 0\}} f(x,v_\lambda)(v_\lambda+\epsilon\phi)\,dx \\
&\le \epsilon\Big(\epsilon\int_\Omega |\Delta v_\lambda|^{p(x)-2}\Delta v_\lambda\Delta\phi\,dx - \int_\Omega g(x)v_\lambda^{-\gamma(x)}\phi\,dx - \lambda\int_\Omega f(x,v_\lambda)\phi\,dx\Big) \\
&\quad + \epsilon\int_\Omega |\Delta v_\lambda|^{p(x)-2}\Delta v_\lambda\nabla v_\lambda\nabla\phi\,dx - \epsilon^2\int_{\{x:v_\lambda+\epsilon\phi\le 0\}} |\Delta v_\lambda|^{p(x)-2}\Delta v_\lambda\Delta\phi\,dx \\
&\le \epsilon\Big(\int_\Omega |\Delta v_\lambda|^{p(x)-2}\Delta v_\lambda\Delta\phi\,dx - \int_\Omega g(x)v_\lambda^{-\gamma(x)}\phi\,dx - \lambda\int_\Omega f(x,v_\lambda)\phi\,dx\Big) \\
&\quad - \epsilon\int_{\{x:v_\lambda+\epsilon\phi\le 0\}} |\Delta v_\lambda|^{p(x)-2}\Delta v_\lambda\Delta\phi\,dx.
\end{aligned}
$$

Since the measure of the domain of integration $\{x : v_\lambda + \epsilon\phi \le 0\}$ tends to zero as $\epsilon \to 0^+$, it follows as $\epsilon \to 0^+$ that

$$\int_{\{x:v_\lambda+\epsilon\phi\le 0\}} |\Delta v_\lambda|^{p(x)-2}\Delta v_\lambda\Delta\phi\,dx \to 0.$$

Dividing by ϵ and letting $\epsilon \to 0^+$, we get

$$\int_\Omega |\Delta v_\lambda|^{p(x)-2}\Delta v_\lambda\Delta\phi\,dx - \int_\Omega g(x)v_\lambda^{-\gamma(x)}\phi\,dx - \lambda\int_\Omega f(x,v_\lambda)\phi\,dx \ge 0.$$

Since the equality holds if we replace ϕ by $-\phi$, which implies that v_λ is a positive solution of problem $(\mathrm{P}_{+\lambda})$, this completes the proof of Theorem 3.3. □

6 An example

In this section, we give an example to illustrate our results.

Example 6.1. Let Ω be a smooth bounded domain in $\mathbb{R}^N$ ($N \ge 3$), let p be a Lipschitz continuous function on $\overline{\Omega}$ with $1 < p^- \le p^+ < \frac{N}{2}$ and $p^*(x) = \frac{Np(x)}{N-2p(x)}$, let s, s_1, r and r_1 be continuous functions on $\overline{\Omega}$ such that $1 < \max(r(x), r_1(x)) < p(x) < \frac{N}{2} < \min(s(x), s_1(x))$ for all $x \in \Omega$, let $\gamma : \overline{\Omega} \to (0, 1)$ be a continuous function, let

$$g \in L^{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}}(\Omega),$$

and let h and h_1 be two positive functions such that $h \in L^{s(x)}(\Omega)$ and $h_1 \in L^{s_1(x)}(\Omega)$. Put

$$f(x,t) = \begin{cases} h(x)|t|^{\beta(x)-1}, & |t| \le 1, \\ h_1(x)|t|^{\alpha(x)-1}, & |t| > 1, \end{cases}$$

with $r(x) < \beta(x)$ and $\alpha(x) < r_1(x)$ for all $x \in \Omega$. Then conditions (I1), (I2) and (I3) are satisfied, so for any $\lambda \ge 0$ problem $(\mathrm{P}_{-\lambda})$ has a weak solution.

Moreover, if we suppose that $f(x,t) = Ch(x)|t|^{r(x)-2}t$ for all $x \in \Omega$, then assumptions (f5) and (f6) hold, and consequently, for any $\lambda \ge 0$, problem $(\mathrm{P}_{+\lambda})$ has at least one nontrivial weak solution in $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$.

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