ANNALES
POLONICI MATHEMATICI
XLI (1983)

# On the existence of an invariant measure for the dynamical system generated by partial differential equation

by ANTONI LEON DAWIDOWICZ (Kraków)

**Abstract.** In this paper I have shown the existence of a measure invariant with respect to the dynamical systems induced by a differential equation $\partial u/\partial t = \lambda u - x\partial u/\partial x$. This measure is probabilistic, ergodic and positive on open non-empty sets.

**0. Introduction.** In the classical papers of Hopf [4], Prodi [7] and Foiaş [3] it was shown that the phenomenon of turbulence can be explained by the existence of an ergodic invariant measure in the state space of the motion. There are also known certain results on the existence of measures invariant with respect to dynamical systems generated by some partial differential equations [2], [5]. These measures, however, are in general concentrated on small sets and therefore they cannot adequately explain the developed turbulence. The purpose of the present paper is to show that by use of the Avez method [1], [6] it is possible to construct an ergodic invariant measure for a first order partial differential equation, which is positive on open non-empty sets.

**I. Invariant measure for a dynamical system generated by a partial differential equation.** In this section we denote by $X$ the space of functions $x\colon [0, 1] \to R$ satisfying the Lipschitz condition and such that $x(0) = 0$. $X$ will be considered with the topology generated by the basic open sets $U(v_0; x_0; \varepsilon) = \{v \in X \mid \exists c \in R\colon |v(x) - v_0(x) - c| < \varepsilon \text{ for } x \geqslant x_0\}$.

DEFINITION. The family of mappings $T_t\colon X \to X$, $t \in R_+$, is called a *semi-dynamical system* iff $T_{t+s} = T_t \cdot T_s$ for all $t, s \in R_+$ and $((t, x) \to T_t x)\colon R_+ \times X \to X$ is continuous.

DEFINITION. A semi-dynamical system is *surjective* iff $T_t$ is surjective for all $t \in R_+$.

Consider the differential equation

$$(1)\qquad \frac{\partial u}{\partial t} = \lambda u - x\frac{\partial u}{\partial x}$$

in the domain

(2) $$t \geq 0, \quad 0 \leq x \leq 1,$$

with boundary conditions

(3) $$\begin{aligned} u(t, 0) &= 0, & t &\geq 0, \\ u(0, x) &= v(x), & 0 &\leq x \leq 1, \end{aligned}$$

where $v \in X$ is a given function.

Let a semi-dynamical system

(4) $$T_t \colon X \to X$$

be given by the formula

(5) $$(T_t v)(x) = u(t, x),$$

where $u$ is the solution of (1), (3).

THEOREM. *If $\lambda > 1$, then there exists a measure $\mu$ on $X$ satisfying the following conditions:*

(i) *$\mu$ is probabilistic,*

(ii) *$\mu$ is $T_t$-invariant,*

(iii) *$\mu(E) > 0$ for each open non-empty set $E$,*

(iv) *$\mu$ is ergodic,*

(v) *$\mu(E_0) = 0$, where $E_0$ is the set of periodic points.*

**II. Auxiliary lemmas.**

LEMMA 1. *Let $\{p_i\}$ be a sequence of non-negative numbers, such that $p_0 = 0$ and $\sum_{i=1}^{\infty} p_i = 1$. Then the function $\varrho\colon [0, 1] \to [0, 1]$ defined by the formula*

(6) $$\begin{aligned} \varrho(x) &= p_n^{-1}\Big(x - \sum_{i=1}^{n-1} p_i\Big) \quad \textit{for } x \in \Big(\sum_{i=1}^{n-1} p_i, \sum_{i=1}^{n} p_i\Big], \\ \varrho(0) &= 0, \end{aligned}$$

*is ergodic.*

Proof. It is obvious that the Lebesgue measure $m$ is invariant with respect to $\varrho$. Let

$$I_i = \Big(\sum_{j=1}^{i-1} p_j, \sum_{j=1}^{i} p_j\Big], \quad I_0 = \{0\}.$$

Certainly $I = [0, 1] = \bigcup_{i=1}^{\infty} I_i$ and $I_i \cap I_j = \emptyset$ for $i \neq j$. From (6) we obtain for all $i = 0, 1, 2, \ldots$

(7) $$m[\varrho^{-1}(A) \cap I_i] = p_i m(A).$$

Let $A$ be an invariant Borel set, that is, $A = \varrho^{-1}(A)$. We have

(8) $$m(A \cap I_i) = m(A) m(I_i).$$

Define by induction $I_{i_1 \dots i_{n+1}} = I_{i_{n+1}} \cap \varrho^{-1}(I_{i_1 \dots i_n})$. By induction it is easy to prove that

(9) $$m(A \cap I_{i_1 \dots i_n}) = m(A) m(I_{i_1 \dots i_n}).$$

Therefore we obtain

(10) $m(A \cap J) = m(A) m(J)$ for each set $J$ of the form $J = I_{i_1 \dots i_n}$.

Since these sets generate the $\sigma$-algebra of Borel sets, we have all Borel sets $B$

(11) $$m(A \cap B) = m(A) m(B).$$

In particular, setting $B = A' = X \setminus A$, we get

(12) $$m(A) m(A') = m(A \cap A') = m(\emptyset) = 0,$$

which finishes the proof.

LEMMA 2. *Let a continuous surjective semi-dynamical system $T_t$ be given on a $\sigma$-compact metrizable space $X$ and let $\mu$ be a probabilistic Borel measure on $X$, such that*

(13) $$\mu(T_1^{-1}(E)) = \mu(E) \quad \textit{for } E \subset X,\ E \textit{ Borel};$$

*then the function $f(t) = \mu(T_t^{-1}(E))$ is measurable on $[0, 1]$ for every Borel set $E$.*

Proof. Since the family of all $E$ for which $f$ is measurable is a $\sigma$-algebra, it is sufficient to prove that $f(t)$ is measurable in the case of a closed set $E$.

Write $F = T_1^{-1}(E)$. Then

$$T_s^{-1}(E) = \{u\colon T_s u \in E\} = \{T_{1-s} v\colon T_1 v \in E\} = T_{1-s}(F),$$

because $T_t$ is surjective.

Since $X$ is $\sigma$-compact and $F$ is closed, there exists an increasing sequence of compact sets $\{F_n\}$, such that $F = \bigcup_{n=1}^{\infty} F_n$. Thus we can assume that $F$ is compact.

Further, denote:

(14) $$H = T_{1-t_0}(F), \qquad H_t = T_{1-t_0+t}(F) = T_t(H),$$
$$\mu(H) = f(t_0), \qquad \mu(H_t) = f(t_0 - t).$$

Evidently, $H$ and $H_t$ are compact as the images of compact sets by a continuous function. Now, let $v \in \bigcap_{t>0} \bigcup_{0<s<t} H_s$. Then there exists $v_t \in H$ such that $T_{s(t)} v_t = v$, $s(t) < t$. Since $H$ is compact, there exists a sequence $\{t_n\}$

such that $t_n \to 0$, $s(t_n) \to 0$ and $v_{t_n}$ is convergent to $w \in H$. It results from the continuity of the system that $T_{s(t_n)} v_{t_n} \to T_0 w = w$; on the other hand, $T_{s(t_n)} v_{t_n} = v$, and so $v = w \in H$. Thus

$$\bigcap_{t>0} \bigcup_{0<s<t} H_s \subset H \tag{15}$$

and

$$\overline{\lim_{t \to 0}} f(t_0 - t) \leqslant f(t_0). \tag{16}$$

Hence $f$ is left-sidedly lower semi-continuous, and this guarantees measurability.

LEMMA 3. *For a fixed natural number $n_0$, the set of polynomials with rational coefficients, of degree different from $n_0$, and with the coefficient at the greatest power equal to* 1, *is dense in* $C[0, 1]$.

Proof. Let $f \in C[0, 1]$ and let $\varepsilon > 0$. There exists a polynomial $P$ with rational coefficients with $\deg P = k$ and $\sup_{x \in [0,1]} |f(x) - P(x)| < \frac{1}{2}\varepsilon$. Since the sequence $a_n(x) = x^n - x^{n-1}$ is uniformly convergent to 0, there exists $n > k$ such that $n \neq n_0$ and $\sup_{x \in [0,1]} |x^n - x^{n-1}| < \frac{1}{2}\varepsilon$.

We define $R(x) = P(x) + x^n - x^{n-1}$. Evidently, $R$ fulfils the conditions stated in the assertion and $\sup_{x \in [0,1]} |R(x) - f(x)| < \varepsilon$.

**III. The proof of the theorem.** It is easy to prove that the unique solution of (1), (3) is given by the formula

$$T_t v(x) = u(t, x) = e^{\lambda t} v(xe^{-t}) \tag{17}$$

and it is also easy to show that $T_t\colon X \to X$ is a continuous semi-dynamical system.

Now let $\{\sigma_n\}$ be a sequence of polynomials with rational coefficients, of degree different from $\lambda$ and such that the coefficients at the greatest power are equal to 1. Moreover, the sequence may be chosen in such a way that

$$\forall_n \|\sigma_n\| \leqslant n; \quad \|\sigma_n'\| \leqslant n; \quad \sigma(0) = 0. \tag{18}$$

Since the topology of $X$ is weaker than the topology of uniform convergence, $\{\sigma_n\}$ may be chosen so as to be dense in $X$. Now we define the functions $T\colon X \to X$ and $S_n\colon X \to X$ by the formulae

$$T = T_{\ln 2}, \tag{19}$$

$$S_n v(x) = \begin{cases} 2^{-\lambda} v(2x), & 0 \leqslant x \leqslant \frac{1}{2}, \\ 2^{-\lambda} v(1) + 2^{-\lambda} \sigma_n(2x-1), & \frac{1}{2} \leqslant x \leqslant 1. \end{cases} \tag{20}$$

We have

(21) $$TS_n = \mathrm{id}_X.$$

Let a sequence of natural numbers $\{k_n\}$ be given such that, for almost every $n$, $k_n \leqslant n$. We claim that in this case

(22) $$\operatorname{card} \bigcap_{n=1}^{\infty} S_{k_1} \dots S_{k_n}(X) = 1.$$

Write

(23) $$\varphi(x) = 2^{-\lambda n} \sigma_{k_n}(2^n x - 1) + s_n \quad \text{for } x \in [2^{-n}, 2^{-n+1}],$$

where

(24) $$s_l = \sum_{n=l+1}^{\infty} 2^{-\lambda n} \sigma_{k_n}(1).$$

The convergence of the series (24) results from

$$\sum_{n=1}^{\infty} |2^{-\lambda n} \sigma_{k_n}(1)| \leqslant \Big| \sum_{n=1}^{p} 2^{-\lambda n} \sigma_{k_n}(1) \Big| + \sum_{n=p+1} |n/2^{\lambda n}| < \infty,$$

where $p$ is sufficiently large. To prove that $\varphi$ fulfils the Lipschitz condition it is sufficient to check that for every $n$ it satisfies the Lipschitz condition on the interval $[2^{-n}, 2^{-n+1}]$ with a constant $M$ independent on $n$. Thus let $x, y \in [2^{-n}, 2^{-n+1}]$. We have

$$|\varphi(x) - \varphi(y)| = 2^{-\lambda n} |\sigma_{k_n}(2^n x - 1) - \sigma_{k_n}(2^n y - 1)|$$
$$\leqslant \sup_{0 \leqslant x \leqslant 1} |\sigma'_{k_n}(x)| 2^{n - \lambda n} |x - y|.$$

Since $\sup_{0 \leqslant x \leqslant 1} |\sigma'_{k_n}(x)| \leqslant n$ and $n/2^{(\lambda - 1)n} \to 0$, then indeed there exists an $M$ independent of $n$ such that $|\varphi(x) - \varphi(y)| \leqslant M|x - y|$.

The continuity of $\varphi$ and the equality $\varphi(0) = 0$ follow from the definition of $\varphi$. Moreover, from the definition of $\varphi$ it follows that if $f \in S_{k_1} \dots S_{k_n}(X)$, then

(25) $$f - \varphi | [2^{-n}, 1] = \text{const}.$$

Further, since $(0, 1] = \bigcup_{n=1}^{\infty} [2^{-n}, 1]$, the equality

(26) $$f - \varphi | (0, 1] = \text{const}$$

holds for $f \in \bigcap_{n=1}^{\infty} S_{k_1} \dots S_{k_n}(X)$.

Now from the continuity of $f$, $\varphi$ and from the condition $f(0) = 0$ it follows that $f = \varphi$. This completes the proof of the claim.

Suppose that we are given a sequence $\{p_l\}$ of rational numbers and a function $\varrho^*: I \to I$ satisfying the conditions of Lemma 1, and such that

(27) $$\sum_{n=1}^{\infty} R_n < \infty,$$

where

$$R_n = 1 - \sum_{i=1}^{n-1} p_i.$$

Denote by $J$ the set of all irrational numbers in [0, 1] and let $\varrho$ be the restriction of $\varrho^*$ to $J$.

Evidently,

(28) $$\varrho: J \to J.$$

Define functions $\pi: J \to N$ and $k_n: J \to N$ by the formulae

(29) $$\pi(x) = \sup\{n: \sum_{k=1}^{n-1} p_n < x\}$$

and

(30) $$k_n(x) = \pi\varrho^n(x), \quad n = 1, 2, \ldots$$

We are going to estimate the measure of the set $P$ of all points $x$ for which $k_n(x) \leqslant n$ for infinitely many $n$, i.e., of the set

$$P = \bigcap_{p=1}^{\infty} \{x: \exists n \geqslant p: k_n(x) > n\} = \bigcap_{p=1}^{\infty} \bigcup_{n=p}^{\infty} \{x: k_n(x) \geqslant n\}.$$

Then $m(P) \leqslant \lim_{p\to\infty} \sum_{n=p}^{\infty} R_n = 0$. On the set $F = J \setminus P$ of the full measure we have $k_n(x) \leqslant n$ for $n$ sufficiently large. For any $x \in F$ denote by $\Phi x$ the only function which belongs to $\bigcap_{n=1}^{\infty} S_{k_1(x)} \ldots S_{k_n(x)}(X)$. This construction defines a certain mapping

(31) $$\Phi: F \to X.$$

Let now $x_\nu \to x_0$. For $\nu$ sufficiently large (dependent only on $n$) $k_n(x_\nu) = k_n(x_0)$ and

(32) $$(\Phi x_\nu - \Phi x_0)|[2^{-n}, 1] = \text{const}$$

which guarantees the convergence of $\Phi x_\nu$ to $\Phi x_0$ in the topology of $X$, and, by the same, the continuity of $\Phi$.

Once we have settled this, we may define a Borel measure on $X$ by the formula

(33) $$\bar{\mu}(E) = m(\Phi^{-1}(E)).$$

In turn, we have

$$\{T\Phi x\} = \bigcap_{n=2}^{\infty} S_{k_2(x)} \dots S_{k_n(x)}(X) = \bigcap_{n=1}^{\infty} S_{k_1(\varrho x)} \dots S_{k_n(\varrho x)}(X) = \{\Phi(\varrho x)\},$$

and

$$T\Phi = \Phi\varrho. \tag{34}$$

Hence

$$\bar{\mu}(T^{-1}(E)) = m(\Phi^{-1}(T^{-1}(E))) = m((T\Phi)^{-1}(E)) = m((\Phi\varrho)^{-1}(E))$$
$$= m(\varrho^{-1}(\Phi^{-1}(E))) = m(\Phi^{-1}(E)) = \bar{\mu}(E).$$

Thus $\bar{\mu}$ is a measure invariant with respect to $T$ on $X$. As a simple corollary from the Weierstrass approximation theorem it follows that $\Phi(F)$ is dense in $X$. Therefore if $E$ is an open non-empty set, then $\Phi^{-1}(E)$ has the same properties and consequently $\bar{\mu}(E) > 0$.

Let now $T^{-1}(E) = E$. We have

$$\varrho^{-1}(\Phi^{-1}(E)) = (\Phi\varrho)^{-1}(E) = (T\Phi)^{-1}(E) = \Phi^{-1}(T^{-1}(E)) = \Phi^{-1}(E),$$

whence, by Lemma 1, $m(\Phi^{-1}(E)) = 0$ or 1, i.e. $\bar{\mu}(E)\bar{\mu}(E') = 0$.

Finally, let $\Phi x$ be an eventually periodic point for the system $T_t$, i.e., one for which there exist numbers $t$ and $s > 0$ such that for all $k \in N$

$$T_{ks+t}\Phi x = T_t \Phi x.$$

Then

$$e^{\lambda t + \lambda s} v(ye^{-t-s}) = e^{\lambda t} v(ye^{-t}). \tag{35}$$

In other words, there exist $r$, $q$ such that

$$e^{\lambda t + \lambda s} 2^{-\lambda q} \sigma_{k_q(x)}(2^q ye^{-t-s} - 1) = e^{\lambda t} 2^{-\lambda r} \sigma_{k_r(x)}(2^r ye^{-t} - 1) + \text{const}$$

on an interval of positive length. Therefore

$$e^{\lambda s} 2^{-\lambda q} \sigma_{k_q(x)}(2^q ze^{-s} - 1) = 2^{-\lambda r} \sigma_{k_r(x)}(2^r z - 1) + \text{const}. \tag{36}$$

Both sides of equality (36) are polynomials which are equal on an interval. Thus from the identity principle for polynomials they must be equal everywhere and in particular they must have the same degree $p$.. Using the same argument $\sigma_{k_r(x)}$ and $\sigma_{k_q(x)}$, we conclude that these also have degree $p$. Since $p \neq \lambda$ and the coefficients of $\sigma_{k_r(x)}$ and $\sigma_{k_q(x)}$ at the $p$-th power are equal to 1, we have

$$e^s 2^{-\lambda q} 2^{pq} e^{-ps} = 2^{-\lambda r} 2^{pr} \tag{37}$$

which means that

$$e^{(\lambda - p)s} 2^{(p-\lambda)q} = 2^{(p-\lambda)r}.$$

Since $p \neq \lambda$, we finally obtain

(38) $$s = \ln 2(r-q).$$

Therefore the period $s$ of the point $T_t \Phi x$ is a multiple of $\ln 2$. Consequently there exists an integer $d$ such that

(39) $$k_{n+d}(x) = k_n(x).$$

But there exist at most countably many sequences satisfying (39) and consequently

(40) $$\bar{\mu}(E'_0) = 0,$$

where $E'_0$ is the set of all eventually periodic points.

Let now $E$ be any Borel set in $X$. Define:

(41) $$\mu(E) = \frac{1}{\ln 2} \int_0^{\ln 2} \bar{\mu}(T_t^{-1}(E))\,dt;$$

$\mu$ is a probabilistic measure, because

(42) $$\mu(X) = \frac{1}{\ln 2} \int_0^{\ln 2} 1\,dt = 1.$$

Let $E$ be an open non-empty set. For any $t > 0$ the set $T_t^{-1}(E)$ is also open and non-empty. Thus $\bar{\mu}(T_t^{-1}(E)) > 0$ and consequently $\mu(E) > 0$.

This proves that $\mu$ satisfies (iii).

Let $E$ be an invariant set, i.e. such that $T_t^{-1}(E) = E$ for all $t > 0$. Then, in particular, $T_{\ln 2}^{-1}(E) = E$, and in consequence $\bar{\mu}(E) = 0$ or 1. But since $T_t^{-1}(E) = E$ we see that also $\bar{\mu}(T_t^{-1}(E)) = \bar{\mu}(E)$ which implies (iv).

Finally, let $\varphi \in E_0$. Then for any $t > 0$ and $\psi$ satisfying $T_t \psi = \varphi$ the point $\psi$ is eventually periodic, that is $T_t^{-1}(E_0) \subset E'_0$, which in turn implies (v).

It remains to prove (ii). Let $0 < s < \ln 2$. We have

$$\mu(T_s^{-1}(E)) = \frac{1}{\ln 2} \int_0^{\ln 2} \bar{\mu}(T_{t+s}^{-1}(E))\,dt = \frac{1}{\ln 2} \int_s^{\ln 2} \bar{\mu}(T_u^{-1}(E))\,du + $$
$$+ \frac{1}{\ln 2} \int_{\ln 2}^{\ln 2 + s} \bar{\mu}(T_{u-\ln 2}(E))\,du$$
$$= \frac{1}{\ln 2} \int_0^{\ln 2} \bar{\mu}(T_u^{-1}(E))\,du = \mu(E).$$

Let now $s = s' + k \ln 2$; we have

$$\mu(T_s^{-1}(E)) = \mu(T^{-k} T_{s'}^{-1}(E)) = \mu(E).$$

Thus $\mu$ is the desired measure which satisfies (i)–(v).

In conclusion, I wish to thank my professor A. Lasota for the help in solving the problem.

**References**

[1] A. Avez, *Propriétés ergodiques des endomorphismes dilatants des variétés compactes*, C.R. Acad. Sci. Paris Sér. A, 266 (1968), p. 610–612.

[2] T. Butler, *A mathematical example by Hopf with features of turbulence*, J. of Math. Anal. and Appl. 9 (1964), p. 215–233.

[3] C. Foiaş, *Statistical study of Navier–Stokes equations*, Rendiconti del Sem. Mat. Univ. di Padova 48 and 49 (1972, 1973).

[4] E. Hopf, *A mathematical displying features of turbulence*, Commun. Pure Appl. Math. 1 (1948), p. 303–322.

[5] A. Lasota, *Invariant measure and a linear model of turbulence*, Rendiconti del Seminario Matematico Universitate di Padova (to be published).

[6] — and G. Pianigiani, *Invariant measure on topological spaces*, Bolletino U.M.I. (5) 15-B (1977), p. 592–603.

[7] G. Prodi, *Theorem ergodici per le equazioni della idrodinamica*, C.I.M.E., Roma 1960.

*Reçu par la Rédaction le 21.9.1978*