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ON THE EXISTENCE OF AN OPTIMAL PLAN

IN A CONTINUOUS-TIME ALLOCATION PROCESS

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ON THE EXISTENCE OF AN OPTIMAL PLAN
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The object of this discussion is to investigate the attainment of a maximum in the set of admissible plans for a specific class of allocation problems.

Let C be the set of all real-valued functions c which are defined on the unit interval and which satisfy the following properties:

(i) c <u>is bounded and measurable</u>

(ii) $c(t) \geqq 0$ <u>for all</u> t <u>in</u> [0,1]

(iii) $\int_0^1 c(t)dt \leqq \lambda$

where λ is some given non-negative real number. Boundedness and measurability of the function c assure that the Lebesgue integral in (iii) exists.

The problem is to find a function $c^* \epsilon C$ such that

$$(1) \qquad \int_0^1 \alpha(t)g[c^*(t)]dt \geqq \int_0^1 \alpha(t)g[c(t)]dt \quad \text{for all } c \epsilon C \ ,$$

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where α is a bounded, non-negative and continuous real function on $[0,1]$, and g is a concave real function on $[0,\infty)$.

To fix the ideas, we may interpret our problem as follows: c is a consumption plan; C is the set of all admissible consumption plans; λ is the consumer's wealth; g is a utility function associated with the rate of consumption at every moment of time; and α is a subjective discount function. Under this interpretation, the problem is to find an optimal consumption plan for a period of unit length, in a world of zero rate of interest.[1/]

The existence of such an optimal plan c^* in C is our concern here.

It is, of course, possible to give the problem which was stated above, possibly after some modification, numerous other interpretations. Indeed, variants of this problem occur in several discussions of optimization over time. As examples, one might mention Arrow-Karlin [1], Koopmans [4] and Strotz [6].

Chakravarty [2] has a discussion of the existence of solutions for problems of this kind. He concentrates mainly on difficulties which arise in the case of an infinite horizon (i.e. when the functions c are defined on $[0,\infty)$ rather than on $[0,1]$). However, even with a finite horizon (as in the present case) the set C of admissible plan is not, in general,

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1/ The case where the rate of interest is not zero is considered below.

compact (in any of the usual topologies) and therefore the attainment of a maximum in it is not a matter which goes without saying. Indeed, many examples involving particular shapes for the functions g and α can be given for which a maximum is not attained.

Several aspects of the problem are discussed in a well known contribution of S. Karlin [3, Vol. II, pp. 210-214]. Karlin proves the existence of a (unique) solution in the case where g is strictly concave and bounded. The present discussion will take an entirely different approach and will generalize Karlin's results by dropping the requirements that g be strictly concave and bounded. On the other hand, our approach will tell us little if anything about the nature of the optimal plan (whenever one exists) whereas Karlin's approach entails, in effect, finding a characterization of the solution and then exhibiting its maximal property.

To start the investigation, we point out that without loss of generality we may take g as non-decreasing. We glorify this well known fact by giving it the name --

Lemma 1: Suppose that for some $K \geqq 0$, $g(x)$ is non-decreasing for $0 \leqq x < K$ and strictly decreasing for $x > K$. Let $\hat{g}$ be defined as follows:

$$\hat{g}(x) = g(x) \qquad 0 \leqq x \leqq K \tag{2}$$

$$\qquad = g(K) \qquad K \leqq x < \infty \quad .$$

<u>If the set of solutions to the problem</u>

$$(3) \qquad \max_{c\epsilon C} \int_0^1 \alpha(t)\hat{g}[c]dt$$

<u>is non-empty, then a function</u> $\hat{c}$ <u>can be picked from this set in a way that</u> $\hat{c}$ <u>will also be a solution to the problem</u>

$$(4) \qquad \max_{c\epsilon C} \int_0^1 \alpha(t)g[c]dt \quad .$$

<u>Proof:</u> Let $\overline{c}$ be a solution of (3). Define $\hat{c}$ as follows:

$$(5) \qquad \hat{c}(t) = \min(\overline{c}(t),K) \qquad 0 \leqq t \leqq 1 \quad .$$

Clearly, $\hat{c}$ is also a solution of (3):

$$(6) \qquad \int_0^1 \alpha(t)\hat{g}[\hat{c}(t)]dt \geqq \int_0^1 \alpha(t)\hat{g}[c(t)]dt \qquad \text{for all } c\epsilon C \quad .$$

We also know, since $\hat{g} \geqq g$ and α is non-negative, that

$$(7) \qquad \int_0^1 \alpha(t)\hat{g}[c(t)]dt \geqq \int_0^1 \alpha(t)g[c(t)]dt \qquad \text{for all } c\epsilon C \quad .$$

But since, by (5) $\hat{c}(t) \leqq K$ for all t, we conclude that $\hat{g}[\hat{c}] = g[\hat{c}]$ so that

$$(8) \qquad \int_{o}^{1} \alpha(t)g[\hat{c}(t)]dt \geqq \int_{o}^{1} \alpha(t)g[c(t)]dt \quad \text{for all} \quad c \epsilon C ,$$

as was to be shown.

Next, having taken g to be non-decreasing, we observe that the set C of admissible plans may now be restricted to a subset C' defined by

$$(9) \qquad C' = \left\{ c: \ c \epsilon C; \ \int_{o}^{1} c(t)dt = \lambda \right\} .$$

This fact may be summarized as --

<u>Lemma 2</u>: <u>If</u> $\sup_{C'} \int \alpha(t)g[c]dt$ <u>is attained in</u> C' , <u>then</u> $\sup_{C} \int \alpha(t)g[c]dt$ <u>is attained in</u> C , <u>and the function</u> c^* <u>which attains the supremum in</u> C' <u>also attains it in</u> C .

<u>Proof</u>: By the fact that $C' \subset C$, we have

$$(10) \qquad \sup_{C'} \int_{o}^{1} \alpha(t)g[c]dt \leqq \sup_{C} \int_{o}^{1} \alpha(t)g[c]dt .$$

Now, for each $c \epsilon C$, define λ_c as follows

$$(11) \qquad \int_{o}^{1} c(t)dt = \lambda_c \leqq \lambda .$$

By the monotonicity of g :

(12) $$\int_0^1 \alpha(t)g[c(t)]dt \leqq \int_0^1 \alpha(t)g[c(t)+\lambda-\lambda_c]dt \quad \text{for all } c\epsilon C ,$$

(13) $$\sup_C \int_0^1 \alpha(t)g[c(t)]dt \leqq \sup_C \int_0^1 \alpha(t)g[c(t)+\lambda-\lambda_c]dt \quad .$$

But by construction --

(14) $$\sup_C \int_0^1 \alpha(t)g[c(t)+\lambda-\lambda_c]dt = \sup_{C'} \int_0^1 \alpha(t)g[c(t)]dt \quad .$$

This implies that

(15) $$\sup_C \int_0^1 \alpha(t)g[c(t)]dt \leqq \sup_{C'} \int_0^1 \alpha(t)g[c(t)]dt \quad .$$

Inequalities (10) and (15) say that the two suprema are equal. Since $C' \subset C$, if $c^* \epsilon C'$ attains the supremum in C' , then c^* also attains it in C .

It suffices, then, to investigate the existence of a solution when g is non-decreasing and the set of admissible plans is C' .

As is well known, in maximization problems of this kind, the function g is arbitrary up to a linear translation. Therefore, the discount function α may be normalized in such a way that

(16) $$\int_0^1 \alpha(t)dt = 1$$

We may now proceed to the main part of the investigation. One result is readily obtainable:

<u>Theorem 1</u>: <u>If</u> α <u>is constant then an optimal plan always exists</u>.

<u>Proof</u>: Set $\alpha(t) = 1$, $0 \leqq t \leqq 1$. By concavity of g , one obtains

$$(17) \qquad \int_0^1 g[c(t)]dt \leqq g[\int_0^1 c(t)dt] = g(\lambda) ,$$

for all $c \epsilon C'$. However, setting $c(t) = \lambda$ for all t , we see immediately that the level $g(\lambda)$ is actually attained:

$$(18) \qquad \int_0^1 g(\lambda)dt = g(\lambda) .$$

Hence, a maximum is attained in C' and the function c such that $c(t) = \lambda$ for all t in [0,1] is a solution.

The important case, of course, is one in which α is not a constant. We proceed now to the case where α is assumed to be non-increasing on [0,1]. The results which one obtains in this case apply with certain modifications to other shapes of α as well.

Having assumed that α is non-increasing, we now proceed to show that the class C' of admissible plans can be restricted further without loss of generality to the set of non-increasing members of C' . This requires a somewhat lengthy argument, which is summarized as Lemma 3 below.

Before stating the lemma, it will be convenient to define a functional U as follows:

$$(19) \qquad U(c) = \int_0^1 \alpha(t)g[c(t)]dt \quad .$$

<u>Lemma 3</u>: <u>Let</u> C^* <u>be the set of all non-increasing members of</u> C' . <u>If</u> α <u>is non-increasing then for every</u> $c\epsilon C'$ <u>there exists</u> $c^*\epsilon C^*$ <u>such that</u> $U(c^*) \geqq U(c)$.

To prove this lemma we need the concept of a <u>monotone re-arrangement</u> of a function $c\epsilon C'$. This concept is used quite often in the economic literature (e.g. in the explanation for the negative slope of the marginal-efficiency-of-investment schedule) but it is rarely given a rigorous definition. As might be expected, the Lebesgue integral provides a natural framework for such a definition.

Consider a function $c\epsilon C'$. By definition, c is measurable and bounded. $0 \leqq c(t) < M$ for all t in $[0,1]$ and for some real number M . Let us divide the interval $[0,M)$ into 2^n sub-intervals of equal lengths, where n is a non-negative integer. We enumerate these sub-intervals starting at the right of $[0,M)$. Let the k-th sub-interval be denoted $I_{n,k}$

$$(20) \qquad I_{n,k} = \left[\frac{2^n-k}{2^n} M \, , \; \frac{2^n-k+1}{2^n} M \right) \qquad k = 1, 2, \ldots, 2^n \quad .$$

A family of 2^n sets from the unit interval may now be defined as follows:

(21)
$$S_{n,k} = \left\{t: \ c(t) \epsilon I_{n,k}\right\} \qquad k = 1,2, \ldots, 2^n \ .$$

By the fact that c is a measurable function, the sets $S_{n,k}$ are measurable for any n and for all $k = 1, 2, \ldots, 2^n$. For any n, the sets $S_{n,k}$ and $S_{n,j}$ are disjoint if $k \neq j$ and we have

(22)
$$\bigcup_{k=1}^{2^n} S_{n,k} = [0,1]$$

$$\sum_{k=1}^{2^n} \mu S_{n,k} = 1 \ ,$$

where μ denotes Lebesgue measure.

Define 2^n+1 points in the unit interval, to be denoted $t_{n,k}$ for $k = 0, 1, \ldots, 2^n$ as follows:

(23)
$$t_{n,0} = 0$$

$$t_{n,k} - t_{n,k-1} = \mu S_{n,k} \qquad k = 1,2,\ldots,2^n \ .$$

For each non-negative integer n, define two functions, $\overline{c}_n$ and $\underline{c}_n$, both on $[0,1)$, in the following manner:

(24) $$\bar{c}_n(t) = \frac{2^n-k+1}{2^n} M \quad \text{for } t_{n,k-1} \leq t < t_{n,k}; \quad k = 1,2,\ldots,2^n$$

$$\underline{c}_n(t) = \frac{2^n-k}{2^n} M \quad \text{for } t_{n,k-1} \leq t < t_{n,k}; \quad k = 1,2,\ldots,2^n \quad .$$

We define $\bar{c}_n(1)$ and $\underline{c}_n(1)$ as follows:

(25) $$\bar{c}_n(1) = \inf_{t\in[0,1)} \bar{c}_n(t)$$

$$\underline{c}_n(1) = \inf_{t\in[0,1)} \underline{c}_n(t) ,$$

so that $\bar{c}_n$ and $\underline{c}_n$ are now defined on the closed unit interval.

The following facts about $\bar{c}_n$ and $\underline{c}_n$ are immediate:

(i) $\bar{c}_n(t) - \underline{c}_n(t) = M/2^n$ for all t in $[0,1]$;

(ii) $\underline{c}_n(t) \leq \underline{c}_{n+1}(t) \leq \bar{c}_{n+1}(t) \leq \bar{c}_n(t) \leq M$ for all t in $[0,1]$ and for all n ;

(iii) $\bar{c}_n$ and $\underline{c}_n$ are integrable and non-increasing for all n .

It follows from these observations that the sequences $\bar{c}_n$ and $\underline{c}_n$ both converge uniformly. For any $\epsilon > 0$, pick N such that $M/2^N < \epsilon$. If $n \geq N$ and $m \geq N$ then $|\bar{c}_n(t) - \bar{c}_m(t)| < \epsilon$ for all t in $[0,1]$, and similarly for $\underline{c}_n$. Furthermore, $\bar{c}_n$ and $\underline{c}_n$ clearly converge to the same function. Call it c^*. c^* is necessarily non-increasing and measurable. Furthermore, since the sequences $\overline{c}_n$ and $\underline{c}_n$ are uniformly bounded, we have, by a theorem of Lebesgue on the integrability of the limit of a sequence of functions, that

$$(25) \qquad \int_0^1 c^*(t)dt = \lim_{n\to\infty} \int_0^1 \overline{c}_n(t)dt = \lim_{n\to\infty} \int_0^1 \underline{c}_n(t)dt \quad .$$

But $\overline{c}_n$ and $\underline{c}_n$ are constructed precisely in such a way that

$$(26) \qquad \lim_{n\to\infty} \int_0^1 \overline{c}_n(t)dt = \lim_{n\to\infty} \int_0^1 \underline{c}_n(t)dt = \int_0^1 c(t)dt = \lambda \; .$$

Hence, c^* is in the set C^*. It remains to be shown that $U(c^*) \geqq U(c)$. To show this, let x be any bounded measurable function on [0,1] and let x^* be the non-increasing re-arrangement of x according to the procedure of the foregoing paragraphs. We wish to show that

$$(27) \qquad \int_0^1 \alpha(t)x^*(t)dt \geqq \int_0^1 \alpha(t)x(t)dt \quad .$$

It follows from the construction of x^* that for any t_o in [0,1],

$$(28) \qquad \int_o^{t_o} (x^*-x)dt = -\int_{t_o}^1 (x^*-x)dt \geqq 0 \; .$$

To prove (27) we write

$$(29) \qquad \int_0^1 \alpha(t)(x^*-x)dt = \int_0^{t_0} \alpha(t)(x^*-x)dt + \int_{t_0}^1 \alpha(t)(x^*-x)dt ,$$

and since α is assumed to be continuous, there exist two points t_{oo} in $[0,t_o]$ and t_{o1} in $[t_o,1]$ such that

$$(30) \qquad \int_0^1 \alpha(t)(x^*-x)dt = \alpha(t_{oo}) \int_0^{t_o} (x^*-x)dt + \alpha(t_{o1}) \int_{t_o}^1 (x^*-x)dt$$

$$= \Big(\alpha(t_{oo})-\alpha(t_{o1})\Big) \int_0^{t_o} (x^*-x)dt \quad .$$

But α is non-increasing and $t_{oo} \leq t_{o1}$, so (27) is proved. Now $g[c^*]$ is the non-increasing re-arrangement of $g[c]$, because g is non-decreasing. Hence, the proof of Lemma 3 is complete.

The upshot of the foregoing argument is that we may now restrict the search for a function c^* which maximizes the functional U to the set C^* of all functions c on the unit interval which have the following properties: they are bounded, non-increasing and satisfy $\int c dt = \lambda$. Actually, it is easy to see that we can restrict the choice even further, to those members of C^* which are, say, right-continuous. Any function in C^* can be transformed into a right-continuous function by changing at most a denumerable number of values of the function, and this does not affect the value of the functional U , nor does it affect the integral $\int c dt$.

At this point it might be well to note that the assertion that a maximum is _always_ attained in C^* is clearly false. For consider simply the case where $g(x) = x$ for all $x \geqq 0$. If α is anything but a constant, a maximum of U is not attained in C^*. Hence, we must now seek specific conditions under which a maximum is attained. Theorem 2 provides the general framework in which such conditions might be sought.

Theorem 2: For the maximum of the functional U _to be attained in the set_ C^* _it is necessary and sufficient that there exist a real number_ K _such that for any_ $c \epsilon C^*$ _there is a_ $c^* \epsilon C^*$ _such that_ $c^*(t) \leqq K$ _for all_ t, _and_ $U(c^*) \geqq U(c)$.

In other words, the maximum is attained in C^* if and only if the search for the optimal plan can be restricted to a uniformly bounded subset of C^*.

Proof: (i) Necessity: If no such K exists, then it is clearly possible to pick a sequence c_n from C^* which will not converge in C^* and such that the sequence $U(c_n)$ will be strictly increasing.

(ii) Sufficiency: By concavity of g and monotonicity of α, we have

$$(31) \qquad U(c) = \int_0^1 \alpha(t)g[c(t)]dt \leqq g[\int_0^1 \alpha(t)c(t)dt] \leqq g[\lambda\alpha(0)] \text{ for all } c \epsilon C^* .$$

Hence, $\sup U(c)$ is finite. Let c_n be a sequence of functions from C^* such that

(32) $$\lim_{n\to\infty} U(c_n) = \sup_{c\in C^*} U(c) \quad .$$

Since the functions of the sequence c_n are all monotone and uniformly bounded, we can apply Helly's Theorem[2] which states that a subsequence of c_n can be found which converges at every point of the interval $[0,1]$. In other words, Helly's theorem says that there exists a sequence $c_{n'}$ which is a subsequence of c_n, such that

(33) $$\lim_{n'\to\infty} c_{n'}(t) = c^*(t) \quad \text{for all } t \text{ in } [0,1],$$

where the limit function c^* is non-increasing and bounded. To show that $c^* \epsilon C^*$, we need only prove that $\int c^*(t)dt = \lambda$. But by the theorem of Lebesgue mentioned above[3] we have

(34) $$\int_0^1 c^*(t)dt = \lim_{n'\to\infty} \int_0^1 c_{n'}(t)dt = \lambda \quad ,$$

so c^* is in C^*. Finally, by applying the same theorem of Lebesgue to the functional U, one obtains

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[2] See Natanson [5, Vol I, pp. 220-223]

[3] Lebesgue's theorem states that the integral of the limit equals the limit of the integrals given convergence in measure. However, in the present case pointwise convergence implies convergence in measure.

$$(35) \qquad \sup_{c \in C^*} U(c) = \lim_{n' \to \infty} U(c_{n'}) = \lim_{n' \to \infty} \int_0^1 \alpha(t) g[c_{n'}(t)]dt = \int_0^1 \alpha(t) g[c^*(t)]dt = U(c^*).$$

Hence the maximum of U is attained at c^* . This completes the proof.

With the aid of Theorem 2 it is possible to determine conditions on g , α and λ which would ensure the attainment of a maximum. Before proceeding along these lines, let us recall the differentiability properties of the function g . As is well known, the concavity of g implies that g has a derivative at all but a denumerable number of points. Furthermore, whenever a derivative fails to exist, left and right derivatives exist and the left derivative exceeds the right derivative. We may therefore define a function g' to be equal to the derivative of g wherever it exists and to the right derivative of g whenever no derivative exists. With this definition in mind, we shall refer to g' simply as the derivative of g .

The derivative g' is a non-negative and non-increasing function on $[0,\infty)$. Hence, $\lim_{x \to \infty} g'(x)$ must exist. Call it simply $g'(\infty)$.

<u>Theorem 3</u>: <u>Under the hypothesis that</u> α <u>is non-increasing, a sufficient condition for the maximum of</u> U <u>to be attained in</u> C^* <u>is</u> --

$$(36) \qquad \alpha(0)g'(\infty) < \alpha(t^*)g'\left(\frac{\lambda}{t^*}\right) \quad \underline{\text{for some}}\ t^*\ \underline{\text{in}}\ [0,1].$$

In other words, a sufficient condition for the existence of a solution is that the quantity $\alpha(t)g'(\lambda/t)$ not have a maximum at $t = 0$.

Before going on to the proof of Theorem 3, we note that the condition of this theorem, condition (36), generalizes Karlin's result, mentioned above. Specifically, if g is strictly concave and bounded, as in Karlin's case, we see immediately that condition (36) is satisfied, because then $\alpha(0)g'(\infty) = 0$. We note, however, that condition (36) is satisfied in a great many other cases. For instance, whenever g is unbounded but $g'(\infty) = 0$, a maximum is always attained. Finally, there is nothing in Theorem 3 to require strict concavity of g. When g is weakly concave but condition (36) is satisfied, a solution will exist, although it may not be unique.

<u>Proof of Theorem 3</u>: Suppose condition (36) holds. It is then possible to find a real number K and a point t^* in [0,1] such that

$$\alpha(0)g'(K) < \alpha(t^*)g'\left(\frac{\lambda}{t^*}\right) . \tag{37}$$

We shall show that for every $c \in C^*$ there exists $c^* \in C^*$ such that $c^*(t) \leqq K$ for all t in [0,1], and such that $U(c^*) \geqq U(c)$. Theorem 2 will do the rest.

In the first place, select K in such a way that $K > \lambda/t^*$. This is clearly possible. Let c be a function in the set C^* such that $c(t) > K$ for some $t > 0$. By monotonicity of c, there exists a point t_o in [0,1] such that

$$(38) \qquad c(t) > K \quad \text{for} \quad 0 \leqq t < t_o$$
$$\leqq K \quad \text{for} \quad t_o < t \leqq 1 \ .$$

Furthermore, there also exists in [0,1] a point t_1 such that

$$(39) \qquad c(t) > \frac{\lambda}{t^*} \quad \text{for} \quad 0 \leqq t < t_1$$
$$\leqq \frac{\lambda}{t^*} \quad \text{for} \quad t_1 < t \leqq 1 \ .$$

Since $K > \lambda/t^*$, we have $t_o \leqq t_1$. Now define the function c^* as follows:

$$(40) \qquad c^*(t) = K \quad \text{for} \quad 0 \leqq t < t_o$$
$$= c(t) \quad \text{for} \quad t_o \leqq t < t_1$$
$$= \frac{\lambda}{t^*} \quad \text{for} \quad t_1 \leqq t < t_2$$
$$= c(t) \quad \text{for} \quad t_2 \leqq t \leqq 1$$

where t_2 is determined so as to have $\int c^*(t)dt = \lambda$. In other words, t_2 is determined from the equation

$$(41) \qquad \int_o^{t_o} \Big(c(t)-K\Big)dt = \int_{t_1}^{t_2} \left(\frac{\lambda}{t^*} - c(t)\right) dt \ .$$

Note that $t_2 \leqq t^*$. This is true because $c^*(t) \geqq \lambda/t^*$ for $0 \leqq t < t_2$ and if we had $t_2 > t^*$, then we would have to have $\int c^*(t)dt > \lambda$ which is a contradiction (see Figure 1).

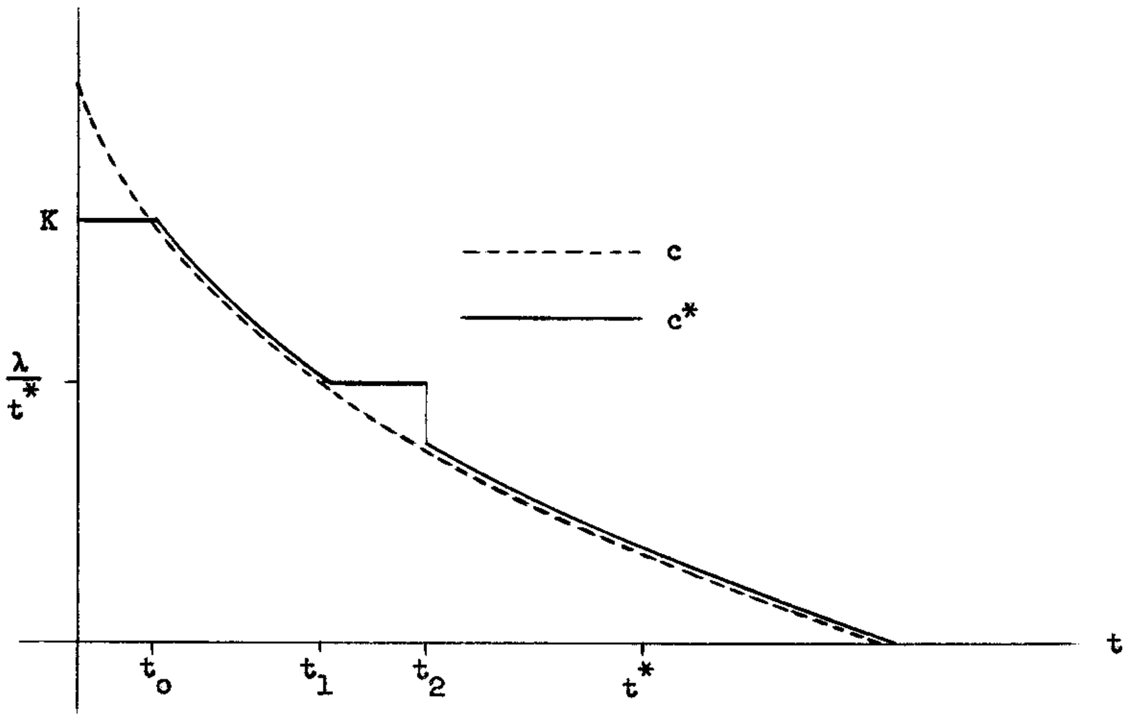

Figure 1

To show that $U(c^*) \geqq U(c)$ one proceeds as follows:

$$(42) \qquad U(c^*)-U(c) = \int_{t_1}^{t_2} \alpha(t)\left\{g\left(\frac{\lambda}{t^*}\right) - g[c(t)]\right\} dt - \int_{o}^{t_o} \alpha(t)\left\{g[c(t)]-g(K)\right\} dt \; .$$

Consider each of these integrals separately: For t in $[t_1, t_2]$ monotonicity of α implies $\alpha(t) \gtreqless \alpha(t^*)$ and concavity of g implies $g(\lambda/t^*)-g[c(t)] \geqq g'(\lambda/t^*)[(\lambda/t^*) - c(t)]$. Hence

$$(43) \qquad \int_{t_1}^{t_2} \alpha(t)\left\{g\left(\frac{\lambda}{t^*}\right) - g[c(t)]\right\} dt \geqq \alpha(t^*) g'\left(\frac{\lambda}{t^*}\right) \int_{t_1}^{t_2} \left(\frac{\lambda}{t^*} - c(t)\right) dt \; .$$

Next, for t in $[0,t_o]$, monotonicity of α implies $\alpha(t) \leqq \alpha(0)$ and concavity of g implies $g[c(t)] - g(K) \leqq g'(K)(c(t)-K)$. Hence,

$$\text{(44)} \qquad \int_0^{t_o} \alpha(t)\Big\{g[c(t)] - g(K)\Big\}\,dt \leqq \alpha(0)g'(K)\int_0^{t_o}\Big(c(t) - K\Big)dt\ .$$

Looking back at equation (41), we see that

$$\text{(45)} \qquad U(c^*) - U(c) \geqq \left[\alpha(t^*)g'\left(\frac{\lambda}{t^*}\right) - \alpha(0)g'(K)\right]\int_0^{t_o}\Big(c(t)-K\Big)dt\ ,$$

which completes the proof of the theorem.

<u>Corollary 1</u>: <u>For small λ, condition (36) of Theorem 3 is both necessary and sufficient to ensure attainment of a maximum.</u>

<u>Proof</u>: Only necessity has to be shown. Let $\lambda \to o$ in condition (36) and suppose that

$$\text{(46)} \qquad \alpha(0)g'(\infty) \geqq \alpha(t)g'(0) \quad \text{for all } t \text{ in } (0,1].$$

If we now let $t \to 0$, we obtain

$$\text{(47)} \qquad g'(\infty) = g'(0)\ ,$$

that is, g must be a linear function, in which case we know that a maximum is not attained.

Corollary 2: The only case in which the maximum is not attained for any λ is the case in which g is a linear function.

This follows immediately from the proof of Corollary 1.

Corollary 2 may also be stated as follows: If the derivative g' of the function g is not constant, then there will exist a $\lambda > 0$ for which a maximum is attained.

It will of course be convenient to interpret condition (36) as a condition on λ. Since in general condition (36) is sufficient but not necessary, one should expect that the greatest value of λ which is consistent with (36) will understate the greatest value of λ for which the maximum is attained. To illustrate this point, we consider an example. This example will also serve to show how the non-negativity condition enters in the solution of the variational problem.

Example: Let $\alpha(t) = 2(1-t)$ for $0 \leqq t \leqq 1$, and let $g(x) = x - e^{-x}$ for $x \geqq 0$. Condition (36) now says that a solution exists if

$$(1-t)\,(1+e^{-\lambda/t}) > 1 \quad \text{for some } t \text{ in } [0,1] . \tag{48}$$

This condition is equivalent to

$$\lambda < \max_{t\in[0,1]} t \log \frac{1-t}{t} , \tag{49}$$

and (49), in turn, yields $\lambda < .278$ approximately. For values of λ below .278 we know that a maximum must exist. Let us proceed now to carry out the maximization.

Let the optimal plan be denoted x. The Euler differential equation for a problem of this kind is given by[4/]

$$\frac{dx}{dt} = -\frac{1}{\alpha}\frac{d\alpha}{dt}\frac{g'(x)}{g''(x)} \tag{50}$$

which in this case reduces to

$$\frac{dx}{dt} = -\frac{1}{1-t}\;\frac{1+e^{-x}}{e^{-x}}\,, \tag{51}$$

$$\frac{dx}{1+e^{x}} = -\frac{dt}{1-t}\;.$$

The solution of this equation is

$$x(t) = \log\frac{1-t}{k+t}\;, \quad 0 \leqq t \leqq 1\,, \tag{52}$$

where k is a constant which is constrained to be positive. To evaluate this constant, one has to solve the equation $\int x(t)dt = \lambda$. Taking the integral of (52), we obtain

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4/ For a full discussion, see [7].

(53) $$\int_0^1 x(t)dt = -\Big[(k+t)\log(k+t) + (1-t)\log(1-t)\Big]_0^1$$

$$= k \log k - (k+1)\log(k+1) < 0 \quad \text{for all } k > 0 .$$

It seems as if no solution of the Euler equation is consistent with a positive λ . However, we know by Theorem 3 that solutions of the maximization problem must exist at least for $\lambda < .278$. The clue to this apparent inconsistency lies in the fact that the solution of the maximization problem does not satisfy Euler's equation on the entire interval [0,1]. Rather, it satisfies Euler's equation up to some point in the interval and is zero thereafter. Looking back at equation (52), we see that $x(t) \geqq 0$ if and only if $1-t \geqq k+t$, i.e. if and only if $t \leqq (1-k)/2$. Hence, the solution x of the maximization problem will satisfy equation (52) on the interval $\left[0, \frac{1-k}{2}\right]$ and will continue at zero thereafter. The constant k is now restricted to satisfy $(1-k)/2 \geqq 0$, which means $k \leqq 1$. We evaluate k by solving the equation

(54) $$\int_0^{\frac{1-k}{2}} x(t)dt = \lambda \quad ,$$

that is,

(55) $$k \log k - (k+1) \log\left(\frac{k+1}{2}\right) = \lambda , \quad 0 < k \leqq 1 .$$

Equation (55) has a solution in the interval $(0,1]$ so long as $\lambda < \log 2$, i.e. $\lambda < .693$. This shows that condition (36) of theorem 3 may lead to a fairly drastic understatement of the values of λ for which the maximum is attained. It may therefore be useful to look for a condition which would be stronger than (36). My own efforts in this direction have not as yet led me to any appreciably stronger results.

Going back now to the main line of the discussion, we make the remark that only a slight modification of the results is called for if α is not a non-increasing function. In fact, if α is any continuous function of bounded variation, then the optimal plan c^* will mimic the monotonicity of α (i.e. wherever α is monotone, c^* is monotone in the same direction) and Theorem 2 will still hold. As for Theorem 3, it too holds, except that condition (36) has to be modified to read as follows: Consider the quantity

$$\alpha(t)g'(\lambda/|t_{max}-t|) \tag{56}$$

where t_{max} is the point at which α has an absolute maximum in $[0,1]$; if this quantity does not have a maximum at $t = t_{max}$, then the maximum is attained.

Finally, the constraint $\int c(t)dt = \lambda$ can clearly be replaced by any linear constraint, say

$$\int_0^1 r(t)c(t) = \lambda \qquad (r(t) > 0) \tag{57}$$

without changing much of the foregoing analysis. This does away with the "zero rate of interest" assumption. It can be shown that in the case where (57) is the constraint, the optimal plan c^* mimics the monotonicity of α/r rather than of α. Furthermore, so long as α/r is a continuous function of bounded variation, Theorem 2 holds. In Theorem 3, the sufficient condition for attainment of a maximum is that the quantity

$$(\alpha(t)/r(t))g'(\lambda/|\int_t^{t_{max}} r(\tau)d\tau|) \tag{58}$$

not have a maximum at $t = t_{max}$, where t_{max} is a point at which α/r has a maximum.

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