Proceedings of the Royal Society of Edinburgh, **129A**, 787–809, *1999*

On the existence of bounded Palais–Smale sequences and application to a Landesman–Lazer-type problem set on $\mathbb{R}^N$

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(MS received 6 April 1998; accepted 22 June 1998)

Using the 'monotonicity trick' introduced by Struwe, we derive a generic theorem. It says that for a wide class of functionals, having a mountain-pass (MP) geometry, almost every functional in this class has a bounded Palais–Smale sequence at the MP level. Then we show how the generic theorem can be used to obtain, for a given functional, a special Palais–Smale sequence possessing extra properties that help to ensure its convergence. Subsequently, these abstract results are applied to prove the existence of a positive solution for a problem of the form

$$\left.\begin{aligned} -\Delta u + Ku &= f(x,u), \\ u \in H^1(\mathbb{R}^N), &\quad K > 0. \end{aligned}\right\} \tag{P}$$

We assume that the functional associated to (P) has an MP geometry. Our results cover the case where the nonlinearity f satisfies (i) $f(x,s)s^{-1} \to a \in]0,\infty]$ as $s \to +\infty$; and (ii) $f(x,s)s^{-1}$ is non decreasing as a function of $s \geqslant 0$, a.e. $x \in \mathbb{R}^N$.

1. Introduction

A first aim of this paper is to study, for a large class of functionals having an MP geometry, the existence of a bounded Palais–Smale sequence at the MP level. Proving the existence of such sequences is a preliminary step when one wants to show that the functionals have a critical point. More precisely, let X be a Banach space, and denote by X^{-1} its dual. By saying that a functional $I \in C^1(X,\mathbb{R})$ possesses an MP geometry, we mean that there are two points (v_1, v_2) in X, such that setting

$$\Gamma = \{\gamma \in C([0,1],X),\ \gamma(0) = v_1,\ \gamma(1) = v_2\},$$

there holds

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) > \max\{I(v_1), I(v_2)\}.$$

Also, a Palais–Smale sequence of I at the level $c \in \mathbb{R}$ is, by definition, a sequence $\{u_n\} \subset X$ satisfying $I(u_n) \to c$ and $I'(u_n) \to 0$ in X^{-1}.

It is well known that if I possesses an MP geometry, the value $c \in \mathbb{R}$, called the MP level, is a good candidate for being a critical value of I. Indeed, assume in addition that the $(\mathrm{PS})_c$ condition holds, namely that all Palais–Smale sequences for I at the level $c \in \mathbb{R}$ possess a convergent subsequence. Then there exists $u \in X$ satisfying $I(u) = c$ and $I'(u) = 0$. This is a celebrated result, known as the MP theorem, due to Ambrosetti and Rabinowitz [3]. Observing the proof given in [3], or alternatively using Ekeland's variational principle [13], one sees that the MP geometry directly implies the existence of a Palais–Smale sequence $\{u_n\} \subset X$ for I at the level $c \in \mathbb{R}$. Thus, to find a critical point, it is sufficient to establish that this particular sequence has a convergent subsequence. Traditionally, this is done in two steps. First one proves that $\{u_n\}$ is bounded and this implies (assuming that X is reflexive) the existence of a $u \in X$, such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in X. Second, one shows that $u_n \to u$ strongly in X and, by continuity of I and I', u then satisfies $I(u) = c$ and $I'(u) = 0$. Note that in many cases, one is interested in finding a (non-trivial) critical point of I, but not necessarily at the MP level. Then, instead of proving that $u_n \to u$ strongly in X, it is sufficient to show that $I'(u) = 0$ (with $I(u) \neq I(0)$). See [10,17,21,29] for some examples.

Concerning the first step, namely the problem of finding conditions on I insuring the existence of a bounded Palais–Smale sequence (a BPS sequence for short), at the MP level, most of the work we know about deals with specific situations. We mean by this that the functional I is introduced in order that its critical points correspond to (weak) solutions of a given PDE or Hamiltonian-type problem. Then particular properties of the underlying problem can directly and crucially be used to prove the existence of a BPS sequence (see, for example, [18,31]). A more systematic approach is due to Ghoussoub [14], where his ideas of using dual sets to localize the critical points of the functionals are often a strong help in concluding the existence of a BPS sequence. Let us also mention the work of Cerami [11], which leads to proof that a sequence $\{u_n\} \subset X$ always exists that satisfies $I(u_n) \to c$ and $\|I'(u_n)\|(1 + \|u_n\|) \to 0$. For this Palais–Smale sequence, called a Cerami sequence, the additional information that $\|I'(u_n)\|(1 + \|u_n\|) \to 0$ has, in several situations, been successfully used to establish that $\{u_n\}$ is bounded. However, probably the most significant contribution is due to Struwe (see also [22]). He introduces a general technique often referred to as the 'monotonicity trick' (see [23,24]), which has been used by Struwe and others to solve difficult variational problems in an MP setting [2,4] and also in minimization problems [26] or in a linking-type situation [25]. Most of these problems have in common the difficulty of establishing the existence of a BPS sequence.

Unfortunately, Struwe's approach has only been used so far on specific examples. Thus, it is not always clear what is the core of the approach and what belongs to the specific problem under study. An initial achievement of our paper is the derivation of a general abstract result based on Struwe's 'monotonicity trick'. Clearly, with respect to the existing works, one advantage is the simplicity of the presentation and the 'ready to use' aspect of the result. We also point out, however, that the possibility of obtaining a result as general as ours, starting from Struwe's work, was not so obvious (M. Struwe, personal communication; see also [23, 9.5, Chapter II]). Roughly speaking, we establish, for a wide class of functionals, a generic result that for states that for almost every functional in this class there exists a BPS sequence at the MP level.

THEOREM 1.1. *Let X be a Banach space equipped with the norm $\|\cdot\|$, and let $J \subset \mathbb{R}^+$ be an interval. We consider a family $(I_\lambda)_{\lambda\in J}$ of C^1-functionals on X of the form*

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \forall \lambda \in J,$$

where $B(u) \geqslant 0$, $\forall u \in X$, and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$. We assume that there are two points (v_1, v_2) in X, such that setting

$$\Gamma = \{\gamma \in C([0,1], X),\ \gamma(0) = v_1,\ \gamma(1) = v_2\},$$

there hold, $\forall \lambda \in J$,

$$c_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\}.$$

Then, for almost every $\lambda \in J$, there is a sequence $\{v_n\} \subset X$, such that

(i) *$\{v_n\}$ is bounded;*

(ii) *$I_\lambda(v_n) \to c_\lambda$; and*

(iii) *$I'_\lambda(v_n) \to 0$ in the dual X^{-1} of X.*

To derive theorem 1.1 we have been inspired by [2] and [27]. In particular, in [27] the authors obtain a conclusion similar to ours for a special family $(I_\lambda)_{\lambda\in J}$. Their result, however, is derived using the precise form of the functional, and it may not be apparent that it is, in fact, very general. In view of theorem 1.1 a natural question to ask is: is the limitation that a BPS sequence exists only for almost every $\lambda \in J$ essential? The answer to this question is yes, and was pointed out to the author (H. Brezis, personal communication; see also [9]). Indeed, at the end of § 2 we give an example of a family $(I_\lambda)_{\lambda\in J}$, satisfying all the assumptions of theorem 1.1, such that for a $\lambda_0 \in J$, all Palais–Smale sequences at the MP level, c_{λ_0}, are unbounded. Note that for a linking-type problem arising in the study of periodic solutions of Hamiltonians systems (see theorem 9.1 in [23]), the fact that a BPS sequence may not exist for every value of $\lambda \in J$ was proved by Ginzburg [15] and Herman [16].

In many situations, one is interested in finding a critical point for a given functional, namely for a given value of $\lambda \in J$. Then, a first step is to prove the existence of a BPS sequence at the MP level or, alternatively, at a level different from $I_\lambda(0)$ to avoid finding $u = 0$ as a critical point. We claim that the generic result, theorem 1.1, is a powerful tool for establishing the existence of such a sequence. This is particularly true if the problem enjoys some compactness properties.

COROLLARY 1.2. *Let X be a Banach space equipped with the norm $\|\cdot\|$, and let $I \in C^1(X, \mathbb{R})$ be of the form*

$$I(u) = A(u) - B(u),$$

where B and B' take bounded sets to bounded sets. Suppose there exists $\varepsilon > 0$, such that, for $J = [1-\varepsilon, 1]$, the family $(I_\lambda)_{\lambda\in J}$ defined by

$$I_\lambda(u) = A(u) - \lambda B(u),$$

satisfies the assumptions of theorem 1.1. Finally, assume that for all $\lambda \in J$, any BPS sequences for I_λ at the level $c_\lambda \in \mathbb{R}$ admits a convergent subsequence. Then there exists

$$\{(\lambda_n, u_n)\} \subset [1-\varepsilon, 1] \times X,$$

with

$$\lambda_n \to 1 \quad \textit{and} \quad \{\lambda_n\} \textit{ is increasing},$$
$$I_{\lambda_n}(u_n) = c_{\lambda_n} \quad \textit{and} \quad I'_{\lambda_n}(u_n) = 0, \quad \textit{in } X^{-1},$$

such that, if $\{u_n\} \subset X$ *is bounded, there hold,*

$$I(u_n) = I_{\lambda_n}(u_n) + (\lambda_n - 1)B(u_n) \to \lim_{n\to\infty} c_{\lambda_n} = c_1,$$
$$I'(u_n) = I'_{\lambda_n}(u_n) + (\lambda_n - 1)B'(u_n) \to 0, \quad \textit{in } X^{-1}.$$

The point of corollary 1.2 is that if $\{u_n\}$ is bounded, it is a BPS sequence for I at the level c_1. Clearly, corollary 1.2 is a direct consequence of theorem 1.1 if we prove that the map $\lambda \to c_\lambda$ is continuous from the left. This is done in lemma 2.3.

At this point, however, one may question the usefulness of corollary 1.2. Indeed, the existence of a Palais–Smale sequence for I at the MP level was already known, and the only remaining problem was, as it is now, to show that it is a bounded sequence. So what progress have we made? In reality, we are now in a more advantageous position since, with respect to a standard Palais–Smale sequence, the sequence $\{u_n\}$ given in corollary 1.2 possesses properties that are very useful when one tries to establish that it is bounded. The difference is that, instead of starting from a sequence of approximate critical points of I (as in the case of a standard Palais–Smale sequence), we now start from a sequence of exact critical points of nearby functionals. The fact that u_n is an exact critical point often provides additional information on the sequence $\{u_n\}$, which helps to show that it is bounded. For example, imagine that I is defined on a Sobolev space and that its critical points (as those of I_{λ_n}) correspond to solutions of a PDE problem. They then possess stronger regularity properties than elements of the ambient space normally do. Also, the use of a maximum principle can often guarantee a given sign for u_n, $\forall n \in \mathbb{N}$ (see § 3 for an application of this idea). Moreover, constraints sometimes exist that u_n must satisfy. Just think of all situations where the solutions of a PDE problem satisfy a Pohozaev-type identity. More globally, for $\lambda \in \mathbb{R}$, let

$$K_\lambda = \{u \in X : I_\lambda(u) = c_\lambda \text{ and } I'_\lambda(u) = 0\},$$

and suppose that $\cup_{\lambda\in[1-\varepsilon,1]} K_\lambda$ is bounded for $\varepsilon > 0$. Then, if for all $\lambda \in [1-\varepsilon, 1]$ any BPS sequence for I_λ at the level $c_\lambda \in \mathbb{R}$ admits a convergent subsequence, the functional I has a critical point.

The idea of constructing Palais–Smale sequences that possess some extra properties that might help to ensure their boundedness, or more generally their convergence, is an old topic. Among some significant contributions in that direction, let us mention [5, 20], where Morse-type information on the sequence proves crucial for ensuring its compactness. This is also the central issue in [14].

REMARK 1.3. It should be clear that the possibility of using theorem 1.1 to construct a special, up to boundedness, Palais–Smale sequence for a given functional exists in a large variety of situations. A particularly important case is the following. Let X be a Banach space with norm $\|\cdot\|$ and $I \in C^1(X, \mathbb{R})$ be such that $I(0) = 0$. Assume that there are two positive constants, r, ρ, and $v \in X$ with $\|v\| > \rho$ satisfying

$$I(u) \geqslant r, \quad \text{if } \|u\| = \rho \quad \text{and} \quad I(v) \leqslant 0.$$

https://doi.org/10.1017/S0308210500013147 Published online by Cambridge University Press

Under these hypotheses, I has an MP geometry and we denote by $c \geqslant r$ the MP level. Now, writing I as

$$I(u) = I(u) - 0\|u\|^2,$$

we see that there exists $\varepsilon > 0$, such that the family

$$I_\lambda(u) = I(u) - \lambda\|u\|^2, \quad \text{for } \lambda \in [0, \varepsilon],$$

satisfies the assumptions of theorem 1.1. Indeed, $I_\lambda(v) \leqslant 0$ for all $\lambda \geqslant 0$ and, since for $\|u\| = \rho$

$$I_\lambda(u) = I(u) - \lambda\|u\|^2 \geqslant r - \lambda\rho^2,$$

the claim holds as soon as $\varepsilon < r\rho^{-2}$. Thus, when, in addition, the family $(I_\lambda)_{\lambda\in[0,\varepsilon]}$ satisfies the compactness conditions of corollary 1.2, we obtain, up to boundedness, a special Palais–Smale sequence for I at the level $\tilde{c} := \lim_{\lambda\to 0^+} c_\lambda$. Note that if the map $\lambda \to c_\lambda$ is discontinuous at $\lambda = 0$ we may have $\tilde{c} < c$. But clearly, also $\tilde{c} > 0 = I(0)$. Thus, as far as the search for a non-trivial critical point is concerned, we can forget that $\tilde{c}$ and c may be different.

In the second part of the paper we apply the abstract results of § 2 to study the existence of solutions of the problem

$$\left.\begin{aligned} -\Delta u(x) + Ku(x) = f(x, u(x)),\\ u \in H^1(\mathbb{R}^N), \quad K > 0. \end{aligned}\right\} \tag{P}$$

Because we shall look for positive solutions, we may assume without restriction that $f(x, s) = 0$, $\forall s < 0$, a.e. $x \in \mathbb{R}^N$. We require that f satisfies the following conditions.

(H1) (i) $f : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ is a Caratheodory function;

(ii) $f(\cdot, s) \in L^\infty(\mathbb{R}^N)$ and $f(\cdot, s)$ is 1-periodic in x_i, $1 \leqslant i \leqslant N$.

(H2) There is $p \in]2, (2N/N-2)[$ if $N \geqslant 3$ and $p > 2$ if $N = 1, 2$, such that $\lim_{s\to\infty} f(x, s)s^{1-p} = 0$, uniformly for $x \in \mathbb{R}^N$.

(H3) $f(x, s)s^{-1} \to 0$ if $s \to 0$, uniformly in $x \in \mathbb{R}^N$.

(H4) There is $a \in]0, \infty]$, such that $f(x, s)s^{-1} \to a$ if $s \to \infty$, uniformly in $x \in \mathbb{R}^N$.

Let $G : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$G(x, s) = \tfrac{1}{2} f(x, s)s - F(x, s), \quad \text{with } F(x, s) = \int_0^s f(x, t)\,\mathrm{d}t.$$

We shall also use

(A1) $G(x, s) \geqslant 0$, $\forall s \geqslant 0$, a.e. $x \in \mathbb{R}^N$, and there is $\delta > 0$, such that

$$f(x, s)s^{-1} \geqslant K - \delta \Longrightarrow G(x, s) \geqslant \delta;$$

(A2) There is $D \in [1, \infty[$, such that, a.e. $x \in \mathbb{R}^N$,

$$G(x, s) \leqslant DG(x, t), \quad \forall (t, s) \in \mathbb{R}^+ \times \mathbb{R}^+, \quad \text{with } s \leqslant t.$$

THEOREM 1.4. (i) *Assume that (H1)–(H4) and (A1) hold with* $a < \infty$ *in (H4). Then, if* $K \in]0, a[$, *there exists a non-trivial positive solution of (P).*

(ii) *Assume that (H1)–(H4) and (A2) hold with* $a = \infty$ *in (H4). Then there exists a non-trivial positive solution of (P).*

REMARK 1.5. If $f(x,s)s^{-1}$ is a non-decreasing function of $s \geqslant 0$, a.e. $x \in \mathbb{R}^N$, both (A1) and (A2) are satisfied. In particular then, (A2) holds with $D = 1$. Note also that (A2) implies (A1) and, thus, the assumption on G is weaker when the nonlinearity is asymptotically linear. Finally, observe that (H2) always hold when $a < \infty$ in (H4).

Theorem 1.4 will be proved using a variational procedure in the spirit of corollary 1.2. For the moment, note that, formally, each critical point of the functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + Ku^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x,$$

is a solution of problem (P). Also, by the weak maximum principle it is a positive solution of (P). As we shall see, when hypotheses (H1)–(H4) hold and $K \in]0, a[$, I possesses an MP geometry.

The existence of solutions of (P), or of closely related problems, have been extensively studied over the last few years (see [7, 30, 32]). In the special case where f is autonomous, namely when the nonlinearity does not depend explicitly on $x \in \mathbb{R}^N$, the existence of one solution of (P) (and even infinitely many) was proved by Berestycki and Lions [7] under hypotheses (H1)–(H4). To obtain the existence of one solution, they develop a subtle Lagrange multiplier procedure that ultimately relies on Pohozaev's identity for (P). The lack of compactness due to the translational invariance of (P) is regained working in the subspace of $H^1(\mathbb{R}^N)$ of radially symmetric functions. In the general case, where f is not autonomous, Pohozaev's identity provides no information and, in the previous work, in addition to (H1)–(H4), it was usually assumed that

$$\exists \mu > 2, \quad \text{such that } 0 \leqslant \mu F(x,s) \leqslant f(x,s)s, \quad \forall s \geqslant 0, \text{ a.e. } x \in \mathbb{R}^N. \tag{SQC}$$

The condition (SQC), from now on referred to as the superquadraticity condition, was originally introduced in [3], and is still present in most work involving the search for critical points of MP type. Roughly speaking, the role of (SQC) is to insure that all Palais–Smale sequences for I at the MP level are bounded.

In theorem 1.4, we replace (SQC) by (A1) if $a < \infty$ or by (A2) if $a = \infty$ in (H4). A simple calculation shows that (SQC) implies that $f(x,\cdot)$ must increase at least as $s^{\mu-1}$ for $s \to \infty$. So, when $a < \infty$, it is not possible that (SQC) holds. When $a = \infty$, it may happen that (SQC) is satisfied, but our requirements on f do not imply it. For example, (SQC) is not true for the nonlinearity $f(x,s) = f(s) = s\ln(s+1)$ for $s \geqslant 0$, which satisfies (H1)–(H4) and (A2).

To the best of our knowledge, when $a = \infty$ in (H4) there is no general result on (P) without assuming the (SQC) condition. We believe, however, that the method applied in [1] to deal with an equation of the type (P), set on a bounded domain of $\mathbb{R}^N$, could be extended to cover the case of $\mathbb{R}^N$. However, in addition to (H1)–(H4), it is required in [1] that $f(x,s)s^{-1}$ is convex, and this is substantially stronger than (A2). When $a < \infty$ in (H4) we just know two results [30, 32] which can be compared to theorem 1.4. In [32], (P) is studied assuming that f is radial as a function of $x \in \mathbb{R}^N$. A similar hypothesis is present in [30] on a problem related to (P) arising from a model of self-trapping of an electro-magnetic wave. There, as in many papers dealing with a nonlinearity that is not superquadratic, an abstract critical point theorem due to Bartolo, Benci and Fortunato [6] is used, which is based on the work of Cerami [11]. Thanks to the radial assumption, the problems are somehow set on $\mathbb{R}$, and possess a much stronger compactness. It is not clear to us how the arguments developed in [30, 32] could be extended to treat a general problem on $\mathbb{R}^N$. Also, in addition to (H1)–(H4), the assumptions that $f(x,s)s^{-1}$ is non-decreasing and that $G(x,s) \to +\infty$ as $s \to \infty$, a.e. $x \in \mathbb{R}^N$ are needed both in [30] and [32]. Finally, f has to satisfy a superquadraticity condition for $s \geqslant 0$ small. Namely, for some $\delta > 0$, there is a $\mu > 2$ such that

$$0 \leqslant \mu F(x,s) \leqslant f(x,s)s, \quad \forall s \in [0,\delta], \text{ a.e. } x \in \mathbb{R}^N.$$

For all these reasons we believe that theorem 1.4, both in the cases $a = \infty$ and $a < \infty$ that we treat in a unified way, strongly generalize the previous existence results.

Let us now sketch the proof of theorem 1.4. We start by noticing that I is of the form

$$I(u) = A(u) - B(u),$$

with $A(u) \to +\infty$ as $\|u\| \to \infty$ and $B(u) \geqslant 0$, $\forall u \in H^1(\mathbb{R}^N)$. Then, thanks to lemmas 3.1 and 3.2, we show that the family of functionals defined by

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in [1,2],$$

satisfies the assumptions of theorem 1.1. Thus we get that for almost every $\lambda \in [1,2]$ there exists a bounded sequence $\{v_m\} \subset H^1(\mathbb{R}^N)$, such that

$$I_\lambda(v_m) \to c_\lambda \quad \text{and} \quad I'_\lambda(v_m) \to 0, \quad \text{in } H^{-1}(\mathbb{R}^N).$$

Using the translational invariance of (P), we establish in lemma 3.5 that there is a sequence $\{y_m\} \subset \mathbb{Z}^N$, such that $u_m(x) := v_m(x - y_m)$ satisfies $u_m \rightharpoonup u_\lambda \neq 0$ weakly in $H^1(\mathbb{R}^N)$ with $I_\lambda(u_\lambda) \leqslant c_\lambda$ and $I'_\lambda(u_\lambda) = 0$. From the weak maximum principle we get that $u_\lambda \geqslant 0$ a.e. $x \in \mathbb{R}^N$. At this point, we have proved the existence of a sequence $\{(\lambda_n, u_n)\} \subset [1,2] \times H^1(\mathbb{R}^N)$ with $u_n \geqslant 0$ a.e. $x \in \mathbb{R}^N$, such that

(i) $\lambda_n \to 1$ and $\{\lambda_n\}$ is decreasing;

(ii) $u_n \neq 0$, $I_{\lambda_n}(u_n) \leqslant c_{\lambda_n}$ and $I'_{\lambda_n}(u_n) = 0$.

In lemma 3.6, assuming that $\{u_n\} \subset H^1(\mathbb{R}^N)$ is bounded we show how to obtain a non-trivial critical point of I corresponding to a positive solution of (P). To prove the boundedness of $\{u_n\}$, we develop an original approach, relying somehow on the work of Lions [19] on the concentration compactness principle, which, we believe, could be applied to a large variety of problems where (SQC) does not hold. The proof, by contradiction, assumes that $\|u_n\| \to \infty$. Then, setting $w_n = u_n\|u_n\|^{-1}$ (and using, if necessary, the translational invariance of (P)), there is a subsequence of $\{w_n\}$ with $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^N)$ satisfying one of the two following alternatives.

https://doi.org/10.1017/S0308210500013147 Published online by Cambridge University Press

(1) (Non-vanishing) $\exists \alpha > 0$, $R < \infty$, such that

$$\lim_{n\to\infty} \int_{B_R} w_n^2 \, dx \geqslant \alpha > 0;$$

(2) (Vanishing)

$$\lim_{n\to\infty} \sup_{y\in\mathbb{Z}^N} \int_{y+B_R} w_n^2 \, dx = 0, \quad \forall R < \infty.$$

We shall prove that neither of the two cases can occur and this will give us the desired contradiction. If we assume that $\{w_n\}$ does not vanish, then $w \neq 0$. To eliminate this alternative, we distinguish the cases $a < \infty$ and $a = \infty$ in (H4). When $a < \infty$, we show in lemma 3.7 that $w \neq 0$ satisfies the equation

$$-\Delta w + K w = a w, \quad x \in \mathbb{R}^N.$$

Since the operator $-\Delta$ has no eigenvector in $H^1(\mathbb{R}^N)$, this is a contradiction. When $a = \infty$, we show in lemma 3.8 that the condition $f(x,s)s^{-1} \to \infty$ as $s \to \infty$ a.e. $x \in \mathbb{R}^N$ prevents the set $\Omega = \{x \in \mathbb{R}^N : w(x) > 0\}$ having a non-zero Lebesgue measure. But this is the case since $w \neq 0$. To eliminate alternative (2), we again distinguish between the cases $a < \infty$ and $a = \infty$. Noticing that, $\forall n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N} G(x, u_n) \, dx \leqslant \frac{c_{\lambda_n}}{\lambda_n} \leqslant c,$$

we show in lemma 3.9 that when $a < \infty$ and (A1) holds, the integral goes to $+\infty$. Finally, when $a = \infty$ we show in lemma 3.10 that the vanishing of $\{w_n\}$ is incompatible with the 'nice' radial behaviour of I, which is ensured by (A2). Having proved the boundedness of $\{u_n\} \subset H^1(\mathbb{R}^N)$, the proof of theorem 1.4 is completed.

Notation

Throughout the article, the letter C will denote various positive constants whose exact value may change from line to line but are not essential to the analysis of the problem. Also if we take a subsequence of a sequence $\{u_n\}$ we shall denote it again $\{u_n\}$.

2. Abstract results

In this section we give the proof of theorem 1.1 and we show that it is sharp in the sense that a BPS sequence cannot be found for every $\lambda \in J$. Since $J \subset \mathbb{R}^+$ and $B(u) \geqslant 0$, $\forall u \in X$, the map $\lambda \to c_\lambda$ is non-increasing. Thus, c'_λ, the derivative of c_λ with respect to λ, exists almost everywhere. Theorem 1.1 will be proved if we establish that the existence of c'_λ implies that I_λ has a BPS sequence at the level c_λ.

Let $\lambda \in J$ be an arbitrary but fixed value where c'_λ exists. Let $\{\lambda_n\} \subset J$ be a strictly increasing sequence such that $\lambda_n \to \lambda$.

PROPOSITION 2.1. *There exists a sequence of paths* $\{\gamma_n\} \subset \Gamma$ *and* $K = K(c'_\lambda) > 0$, *such that*

(i) $\|\gamma_n(t)\| \leqslant K$ *if* $\gamma_n(t)$ *satisfies*

$$I_\lambda(\gamma_n(t)) \geqslant c_\lambda - (\lambda - \lambda_n); \tag{2.1}$$

(ii) $\max_{t\in[0,1]} I_\lambda(\gamma_n(t)) \leqslant c_\lambda + (-c'_\lambda + 2)(\lambda - \lambda_n)$.

Proof. Let $\{\gamma_n\} \subset \Gamma$ be an arbitrary sequence such that

$$\max_{t\in[0,1]} I_{\lambda_n}(\gamma_n(t)) \leqslant c_{\lambda_n} + (\lambda - \lambda_n). \tag{2.2}$$

Note that such a sequence exists since the class of paths Γ is independent of λ. We shall prove that, for $n \in \mathbb{N}$ sufficiently large, $\{\gamma_n\}$ is a sequence such as we are looking for. When $\gamma_n(t)$ satisfies (2.1), we have

$$\begin{aligned}\frac{I_{\lambda_n}(\gamma_n(t)) - I_\lambda(\gamma_n(t))}{\lambda - \lambda_n} &\leqslant \frac{c_{\lambda_n} + (\lambda - \lambda_n) - c_\lambda + (\lambda - \lambda_n)}{\lambda - \lambda_n} \\ &= \frac{c_{\lambda_n} - c_\lambda}{\lambda - \lambda_n} + 2.\end{aligned}$$

Since c'_λ exists, there is $n(\lambda) \in \mathbb{N}$ such that $\forall n \geqslant n(\lambda)$

$$-c'_\lambda - 1 \leqslant \frac{c_{\lambda_n} - c_\lambda}{\lambda - \lambda_n} \leqslant -c'_\lambda + 1, \tag{2.3}$$

and, thus, $\forall n \geqslant n(\lambda)$,

$$\frac{I_{\lambda_n}(\gamma_n(t)) - I_\lambda(\gamma_n(t))}{\lambda - \lambda_n} \leqslant -c'_\lambda + 3.$$

Consequently,

$$B(\gamma_n(t)) = \frac{I_{\lambda_n}(\gamma_n(t)) - I_\lambda(\gamma_n(t))}{\lambda - \lambda_n} \leqslant -c'_\lambda + 3.$$

Also

$$\begin{aligned}A(\gamma_n(t)) &= I_{\lambda_n}(\gamma_n(t)) + \lambda_n B(\gamma_n(t)) \\ &\leqslant c_{\lambda_n} + (\lambda - \lambda_n) + \lambda_n(-c'_\lambda + 3) \\ &\leqslant C.\end{aligned}$$

Using our assumption that either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$, the uniform boundedness of $A(\gamma_n(t))$ and $B(\gamma_n(t))$ proves (i). To prove (ii), observe that from (2.3), we have, $\forall n \geqslant n(\lambda)$,

$$c_{\lambda_n} \leqslant c_\lambda + (-c'_\lambda + 1)(\lambda - \lambda_n). \tag{2.4}$$

Using (2.2), (2.4) and since

$$I_{\lambda_n}(v) \geqslant I_\lambda(v), \quad \forall v \in X,$$

we get

$$\begin{aligned}I_\lambda(\gamma_n(t)) &\leqslant I_{\lambda_n}(\gamma_n(t)) \\ &\leqslant c_{\lambda_n} + (\lambda - \lambda_n) \\ &\leqslant c_\lambda + (-c'_\lambda + 2)(\lambda - \lambda_n).\end{aligned}$$

Thus point (ii) also holds. □

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Roughly speaking, proposition 2.1 says that there exists a sequence of paths $\{\gamma_n\} \subset \Gamma$, such that

$$\max_{t\in[0,1]} I_\lambda(\gamma_n(t)) \to c_\lambda,$$

for which, for all $n \in \mathbb{N}$ sufficiently large, starting from a level strictly below c_λ, all the 'top' of the path is contained in the ball centred at the origin of fixed radius $K = K(c'_\lambda) > 0$. Now, for $\alpha > 0$, we define

$$F_\alpha = \{u \in X : \|u\| \leqslant K+1 \text{ and } |I_\lambda(u) - c_\lambda| \leqslant \alpha\},$$

where the constant $K > 0$ is given in proposition 2.1.

PROPOSITION 2.2. *For all $\alpha > 0$,*

$$\inf\{\|I'_\lambda(u)\| : u \in F_\alpha\} = 0. \tag{2.5}$$

Proof. Seeking a contradiction, we assume that (2.5) does not hold. Then there exists $\alpha > 0$, such that for any $u \in F_\alpha$ one has

$$\|I'_\lambda(u)\| \geqslant \alpha, \tag{2.6}$$

and, without loss of generality, we can assume that

$$0 < \alpha < \tfrac{1}{2}[c_\lambda - \max\{I_\lambda(v_1), I_\lambda(v_2)\}].$$

A classical deformation argument then says that there exist $\varepsilon \in]0,\alpha[$ and a homeomorphism $\eta : X \to X$, such that

$$\eta(u) = u, \quad \text{if } |I_\lambda(u) - c_\lambda| \geqslant \alpha, \tag{2.7}$$

$$I_\lambda(\eta(u)) \leqslant I_\lambda(u), \quad \forall u \in X, \tag{2.8}$$

$$I_\lambda(\eta(u)) \leqslant c_\lambda - \varepsilon, \quad \forall u \in X \text{ satisfying } \|u\| \leqslant K \quad \text{and} \quad I_\lambda(u) \leqslant c_\lambda + \varepsilon. \tag{2.9}$$

Let $\{\gamma_n\} \subset \Gamma$ be the sequence obtained in proposition 2.1. We choose and fix $m \in \mathbb{N}$ sufficiently large in order that

$$(-c'_\lambda + 2)(\lambda - \lambda_m) \leqslant \varepsilon. \tag{2.10}$$

Clearly, by (2.7), $\eta(\gamma_m) \in \Gamma$. Now if $u = \gamma_m(t)$ satisfies

$$I_\lambda(u) \leqslant c_\lambda - (\lambda - \lambda_m),$$

then (2.8) implies that

$$I_\lambda(\eta(u)) \leqslant c_\lambda - (\lambda - \lambda_m). \tag{2.11}$$

On the other hand, if $u = \gamma_m(t)$ satisfies

$$I_\lambda(u) > c_\lambda - (\lambda - \lambda_m),$$

then proposition 2.1 and (2.10) implies that u is such that $\|u\| \leqslant K$ with $I_\lambda(u) \leqslant c_\lambda + \varepsilon$. Applying (2.9), one has

$$I_\lambda(\eta(u)) \leqslant c_\lambda - \varepsilon \leqslant c_\lambda - (\lambda - \lambda_m). \tag{2.12}$$

Thus, combining (2.11) and (2.12), we get

$$\max_{t\in[0,1]} I_\lambda(\eta(\gamma_m(t))) \leqslant c_\lambda - (\lambda - \lambda_m),$$

which contradicts the variational characterization of c_λ. □

Proof of theorem 1.1. Since proposition 2.2 is true, there exists a Palais–Smale sequence for I_λ at the level $c_\lambda \in \mathbb{R}$, which is contained in the ball of radius $K+1$ centred at the origin. This proves the theorem. □

LEMMA 2.3. *The map $\lambda \to c_\lambda$ is continuous from the left.*

Proof. Seeking a contradiction, we assume that there are $\lambda_0 \in J$ and $\{\lambda_n\} \subset J$ with $\lambda_n < \lambda_0$, $\forall n \in \mathbb{N}$ and $\lambda_n \to \lambda_0$ for which

$$c_{\lambda_0} < \lim_{n\to\infty} c_{\lambda_n}.$$

Let $\delta = \lim_{n\to\infty} c_{\lambda_n} - c_{\lambda_0} > 0$. By definition of c_{λ_0}, there is $\gamma_0 \in \Gamma$ such that

$$\max_{t\in[0,1]} I_{\lambda_0}(\gamma_0(t)) < c_{\lambda_0} + \tfrac{1}{3}\delta.$$

Using the fact that $I_\lambda(u) = I_{\lambda_0}(u) + (\lambda_0 - \lambda)B(u)$, $\forall \lambda \in J$, $\forall u \in X$, we get, $\forall \lambda < \lambda_0$,

$$\max_{t\in[0,1]} I_\lambda(\gamma_0(t)) < c_{\lambda_0} + \tfrac{1}{3}\delta + (\lambda_0 - \lambda) \max_{t\in[0,1]} B(\gamma_0(t)).$$

But B being continuous, we have $\max_{t\in[0,1]} B(\gamma_0(t)) \leqslant C$ for $C > 0$, and, thus, for any $n \in \mathbb{N}$ sufficiently large,

$$\max_{t\in[0,1]} I_{\lambda_n}(\gamma_0(t)) < c_{\lambda_0} + \tfrac{2}{3}\delta.$$

We reach a contradiction noticing that, by definition of c_{λ_n},

$$\max_{t\in[0,1]} I_{\lambda_n}(\gamma_0(t)) \geqslant c_{\lambda_n}.$$

□

We end this section by presenting a family $(I_\lambda)_{\lambda\in J}$ for which there does not exist a BPS sequence for every $\lambda \in J$. As we already mentioned, this example was provided for us by Brezis, and it shows that theorem 1.1 is sharp. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x,y) = x^2 - (x-1)^3 y^2.$$

The space $\mathbb{R}^2$ is equipped with the Euclidean norm $\|(x,y)\| = \sqrt{x^2+y^2}$. Around the origin, F behaves as $\|(x,y)\|^2$. Moreover, taking $x > 0$ sufficiently large, we see that $F(x,1) < 0$. In particular F has an MP geometry and, as we notice in remark 1.3, there exists $\varepsilon > 0$ such that the family of functions $(F_\lambda)_{\lambda\in[0,\varepsilon]}$ defined by

$$F_\lambda(x,y) = F(x,y) - \lambda(x^2+y^2),$$

satisfies the assumptions of theorem 1.1. In fact, it is even possible to assume that $\lambda \in [-\varepsilon, \varepsilon]$. Let us show that there is no BPS sequence for $F = F_0$ at the MP level. We have

$$\left.\begin{aligned} F_x &= 2x - 3(x-1)^2 y^2 \\ F_y &= -2(x-1)^3 y. \end{aligned}\right\} \tag{2.13}$$

Thus, any sequence $\{(x_n, y_n)\} \subset \mathbb{R}^2$, such that $\|F'(x_n,y_n)\| \to 0$ must satisfy

$$2x_n - 3(x_n-1)^2 y_n^2 \to 0, \tag{2.14}$$

$$(x_n-1)^3 y_n \to 0. \tag{2.15}$$

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Without restriction, we can assume that $x_n \to x \in [-\infty, \infty]$ and $y_n \to y \in [-\infty, \infty]$. We distinguish two cases:

(I) $x_n \not\to 1$. Then, from (2.15), we get that $y_n \to 0$ and since

$$(x_n - 1)^2 y_n^2 = [(x_n - 1)^3 y_n]^{2/3} y_n^{4/3} \to 0,$$

it follows from (2.14) that $x_n \to 0$;

(II) $x_n \to 1$. Then, from (2.14), $(x_n - 1)^2 y_n^2 \to \frac{2}{3}$ and, in particular, $|y_n| \to \infty$.

In the first case $F(x_n, y_n) \to 0$ and in the second $F(x_n, y_n) \to 1$. We deduce that the MP level for F is $c = 1$ and that there is no BPS sequence for F at this level. Analysing the Palais–Smale sequences of F_λ for $\lambda \in [-\varepsilon, \varepsilon] \setminus \{0\}$, we find that a critical point always exists at the MP level $c_\lambda = (1-\lambda)(1-\lambda^{1/3})^2$. We have $c_\lambda \to 1$ as $\lambda \to 0$ and, thus, c_λ is continuous on $[-\varepsilon, \varepsilon]$. Moreover,

$$c'_\lambda = (1 - \lambda^{1/3})(\tfrac{5}{3}\lambda^{1/3} - 1 - \tfrac{2}{3}\lambda^{-2/3}), \quad \text{for } \lambda \in]-\varepsilon, \varepsilon[\setminus\{0\}$$

and, thus, c'_λ exists for all $\lambda \in]-\varepsilon, \varepsilon[\setminus\{0\}$. On the contrary, we can check that c'_λ for $\lambda = 0$ does not exist as we already know from theorem 1.1.

3. Applications

The main aim of this section is to prove theorem 1.4 applying the abstract variational approach of § 2. In the proofs that follow, we shall routinely take $N \geqslant 3$. The proofs for $N = 1$ or $N = 2$ are not more complicated. Our working space is the Sobolev space $H^1(\mathbb{R}^N)$, equipped with the norm

$$\|u\| = \left\{ \int_{\mathbb{R}^N} (|\nabla u|^2 + K u^2) \, dx \right\}^{1/2},$$

which, since $K > 0$, is equivalent to the usual one. We denote by $\| \cdot \|_p$, for each $p \in [1, \infty]$, the standard norm of the Lebesgue space $L^p(\mathbb{R}^N)$. As we mentioned in the introduction, proving theorem 1.4 amounts to finding a non-trivial critical point of the functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$, defined by

$$I(u) = \tfrac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x, u) \, dx.$$

A proof that, under (H1)–(H3), I is a C^1-functional is given in [12, proposition 2.1]. Let us show that I has an MP geometry. Since $I(0) = 0$, this is a consequence of following two results.

LEMMA 3.1. *Assume that (H1)–(H3) hold. Then* $I(u) = \frac{1}{2}\|u\|^2 + o(\|u\|^2)$ *as* $u \to 0$.

Proof. By (H3) we know that $f(x, s)s^{-1} \to 0$ as $s \to 0$ uniformly in $x \in \mathbb{R}^N$. Thus, for any $\varepsilon > 0$, it follows by (H2) that there exists a $C_\varepsilon > 0$ such that

$$f(x, s) \leqslant \varepsilon s + C_\varepsilon s^{p-1}, \quad \forall s \geqslant 0, \text{ a.e. } x \in \mathbb{R}^N, \tag{3.1}$$

or, equivalently, that

$$F(x, s) \leqslant \tfrac{1}{2}\varepsilon s^2 + \frac{C_\varepsilon}{p} s^p, \quad \forall s \geqslant 0, \text{ a.e. } x \in \mathbb{R}^N. \tag{3.2}$$

We deduce that

$$\int_{\mathbb{R}^N} F(x,u)\,dx \leqslant \tfrac{1}{2}\varepsilon\|u\|^2 + C\|u\|_p^p,$$

and this implies that

$$\int_{\mathbb{R}^N} F(x,u)\,dx = o(\|u\|^2),$$

as $u \to 0$. □

LEMMA 3.2. *Assume that (H1), (H2) and (H4) hold, and that $K \in]0,a[$. Then we can find a $v \in H^1(\mathbb{R}^N)$, $v \neq 0$ satisfying $I(v) \leqslant 0$.*

Proof. Without loss of generality, we can assume that $a < \infty$ in (H4). The proof is based in the construction of a family of testing functions that we borrow from [32] (see also [28]). Let

$$d^2(N) = \int_{\mathbb{R}^N} e^{-2|x|^2}\,dx \quad \text{and} \quad D(N) = 4[d(N)]^{-2}\int_{\mathbb{R}^N} |x|^2 e^{-2|x|^2}\,dx.$$

For $\alpha > 0$, we set

$$w_\alpha(x) = [d(N)]^{-1}\alpha^{N/4}e^{-\alpha|x|^2}.$$

Straightforward calculations show that

$$\|w_\alpha\|_2 = 1 \quad \text{and} \quad \|\nabla w_\alpha\|_2^2 = \alpha D(N).$$

Thus, in particular if we fix $\alpha \in (0, [(a-K)/D(N)])$ we get that

$$\|\nabla w_\alpha\|_2^2 < (a-K). \tag{3.3}$$

On the other hand, by (H4),

$$\lim_{s\to\infty}(F(x,s)/s^2) = \tfrac{1}{2}a, \quad \text{uniformly in } x \in \mathbb{R}^N.$$

and, since for every $x \in \mathbb{R}^N$, $tw_\alpha(x) \to +\infty$ as $t \to +\infty$, it follows that

$$\lim_{t\to+\infty}(F(x,tw_\alpha)/t^2w_\alpha^2) = \tfrac{1}{2}a, \text{ a.e. } x \in \mathbb{R}^N.$$

Now observe that (H1), (H3) and (H4) imply the existence of a constant $C < \infty$, such that $\forall s \geqslant 0$, a.e. $x \in \mathbb{R}^N$,

$$0 \leqslant (F(x,s)/s^2) \leqslant C. \tag{3.4}$$

Thus, using (3.4), it follows by Lebesgue's theorem that

$$\lim_{t\to+\infty}\int_{\mathbb{R}^N}\frac{F(x,tw_\alpha)}{t^2}\,dx = \tfrac{1}{2}a\int_{\mathbb{R}^N} w_\alpha^2\,dx = \tfrac{1}{2}a.$$

Now, using (3.3), we get

$$\begin{aligned}\lim_{t\to+\infty}\frac{I(tw_\alpha)}{t^2} &= \tfrac{1}{2}\|\nabla w_\alpha\|_2^2 + \tfrac{1}{2}K\|w_\alpha\|_2^2 - \lim_{t\to+\infty}\int_{\mathbb{R}^N}\frac{F(x,tw_\alpha)}{t^2}\,dx \\ &= \tfrac{1}{2}\|\nabla w_\alpha\|_2^2 + \tfrac{1}{2}K - \tfrac{1}{2}a < 0,\end{aligned}$$

and the lemma is proved. □

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We define on $H^1(\mathbb{R}^N)$ the family of functionals

$$I_\lambda(u) = \tfrac{1}{2}\|u\|^2 - \lambda \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x, \quad \lambda \in [1,2].$$

LEMMA 3.3. *Assume that (H1)–(H4) hold. The family (I_λ) with $\lambda \in [1,2]$ satisfies the hypotheses of theorem 1.1. In particular, for almost every $\lambda \in [1,2]$ there exists a bounded sequence $\{v_m\} \subset H^1(\mathbb{R}^N)$ satisfying*

$$I_\lambda(v_m) \to c_\lambda \quad \textit{and} \quad I'_\lambda(v_m) \to 0, \quad \textit{in } H^{-1}(\mathbb{R}^N).$$

Proof. For the $v \in H^1(\mathbb{R}^N)$ obtained in lemma 3.2, $I_\lambda(v) \leqslant 0$ for all $\lambda \geqslant 1$ since

$$\int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x \geqslant 0, \quad \forall u \in H^1(\mathbb{R}^N).$$

Also, from lemma 3.1, we know that

$$\int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x = o(\|u\|^2), \quad \text{as } u \to 0.$$

Thus, setting

$$\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)),\ \gamma(0) = 0 \text{ and } \gamma(1) = v\},$$

we have, $\forall \lambda \in [1,2]$,

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > 0.$$

An application of theorem 1.1 now completes the proof. □

In the rest of the paper we shall often use the following terminology. Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be an arbitrary bounded sequence. If it is possible to translate each u_n in $\mathbb{R}^N$ such that the translated sequence (still denoted $\{u_n\}$) satisfies, up to a subsequence, $\exists \alpha > 0$, $R < \infty$ such that

$$\lim_{n\to\infty} \int_{B_R} u_n^2\,\mathrm{d}x \geqslant \alpha > 0,$$

we say that $\{u_n\}$ does not vanish. If it is not the case, then one necessarily has

$$\lim_{n\to\infty} \sup_{y \in \mathbb{Z}^N} \int_{y+B_R} u_n^2\,\mathrm{d}x = 0, \quad \forall R < \infty,$$

and, in this case, we say that $\{u_n\}$ vanishes.

LEMMA 3.4. *Assume that (H1)–(H3) hold. Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be an arbitrary bounded sequence which vanishes. Then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} G(x,u_n)\,\mathrm{d}x = 0.$$

Proof. It is known that if $\{u_n\} \subset H^1(\mathbb{R}^N)$ vanishes, then $u_n \to 0$ strongly in $L^q(\mathbb{R}^N)$ for all $q \in]2, 2N/(N-2)[$. A proof of this result is given in lemma 2.18

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of [12]. It is a special case of lemma I.1 of [19]. Now, by (3.1) and (3.2), we know that $\forall\varepsilon > 0$, $\exists C_\varepsilon > 0$, such that

$$\int_{\mathbb{R}^N} f(x, u_n)u_n \,\mathrm{d}x \leqslant \varepsilon\|u_n\|_2^2 + C_\varepsilon\|u_n\|_p^p$$
$$\int_{\mathbb{R}^N} F(x, u_n) \,\mathrm{d}x \leqslant \tfrac{1}{2}\varepsilon\|u_n\|_2^2 + \frac{C_\varepsilon}{p}\|u_n\|_p^p.$$

Thus, if $\{u_n\} \subset H^1(\mathbb{R}^N)$ vanishes, both

$$\int_{\mathbb{R}^N} f(x, u_n)u_n \,\mathrm{d}x \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} F(x, u_n) \,\mathrm{d}x \to 0,$$

and the lemma follows from the definition of G. □

LEMMA 3.5. *Assume that (H1)–(H4) and either (A1) or (A2) hold. Let* $\lambda \in [1, 2]$ *be fixed. Then, for all bounded sequences* $\{v_m\} \subset H^1(\mathbb{R}^N)$ *satisfying*

(I) $0 < \lim_{m\to\infty} I_\lambda(v_m) \leqslant c_\lambda$;

(II) $I'_\lambda(v_m) \to 0$ *in* $H^{-1}(\mathbb{R}^N)$;

there exists $\{y_m\} \subset \mathbb{Z}^N$, *such that, up to a subsequence,* $u_m(x) := v_m(x - y_m)$ *satisfies* $u_m \rightharpoonup u_\lambda \neq 0$ *with* $I_\lambda(u_\lambda) \leqslant c_\lambda$ *and* $I'_\lambda(u_\lambda) = 0$.

Proof. Since $\{v_m\} \subset H^1(\mathbb{R}^N)$ is bounded, we have

$$\int_{\mathbb{R}^N} G(x, v_m) \,\mathrm{d}x = I_\lambda(v_m) - \tfrac{1}{2}I'_\lambda(v_m)v_m \to \lim_{m\to\infty} I_\lambda(v_m) > 0.$$

Thus, we see, by lemma 3.4, that $\{v_m\} \subset H^1(\mathbb{R}^N)$ does not vanish and there is $\{y_m\} \subset \mathbb{Z}^N$ such that, up to a subsequence, $u_m(x) := v_m(x - y_m)$ satisfies: $\exists\alpha > 0$, $R < \infty$ such that

$$\lim_{m\to\infty} \int_{B_R} u_m^2 \,\mathrm{d}x \geqslant \alpha > 0. \tag{3.5}$$

Moreover, since problem (P) is invariant under the translation group associated to the periodicity of $f(\cdot, s)$, we still have

(I) $0 < \lim_{m\to\infty} I_\lambda(u_m) \leqslant c_\lambda$;

(II) $I'_\lambda(u_m) \to 0$ in $H^{-1}(\mathbb{R}^N)$.

We have, up to a subsequence, $u_m \rightharpoonup u_\lambda$ for a $u_\lambda \in H^1(\mathbb{R}^N)$, and to complete the proof of the lemma we just need to show that $u_\lambda \neq 0$,

$$I'_\lambda(u_\lambda) = 0 \quad \text{and} \quad I_\lambda(u_\lambda) \leqslant c_\lambda.$$

Step 1: $u_\lambda \neq 0$

Since (3.5) holds, we get by the compactness of the Sobolev embedding

$$H^1(B_R) \hookrightarrow L^2(B_R)$$

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that

$$\|u_\lambda\|_2^2 \geqslant \int_{B_R} u_\lambda^2 \, dx = \lim_{m\to\infty} \int_{B_R} u_m^2 \, dx \geqslant \alpha > 0.$$

Thus, $u_\lambda \neq 0$ and step 1 is completed.

Step 2: $I'_\lambda(u_\lambda) = 0$

Noting that $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, it suffices to check that $I'_\lambda(v)\varphi = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Let $(\cdot,\cdot)$ denote the inner product on $H^1(\mathbb{R}^N)$ associated with our chosen norm. Then

$$I'_\lambda(u_m)\varphi - I'_\lambda(u_\lambda)\varphi = (u_m - u_\lambda, \varphi) - \int_{\mathbb{R}^N} (f(x,u_m) - f(x,u_\lambda))\varphi \, dx \to 0,$$

since $u_m \rightharpoonup u_\lambda$ weakly in $H^1(\mathbb{R}^N)$ and strongly in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for $q \in [2, 2N/(N-2)[$. Thus, recalling that $I'_\lambda(u_m) \to 0$, we indeed have that $I'_\lambda(u_\lambda) = 0$.

Step 3: $I_\lambda(u_\lambda) \leqslant c_\lambda$

Observe that either (A1) or (A2) implies that

$$G(x,s) \geqslant 0, \quad \forall s \geqslant 0, \text{ a.e. } x \in \mathbb{R}^N. \tag{3.6}$$

Thus, using Fatou's lemma, we get using step 2

$$\begin{aligned} c_\lambda &\geqslant \lim_{m\to\infty} [I_\lambda(u_m) - \tfrac{1}{2} I'_\lambda(u_m)u_m] \\ &= \lim_{m\to\infty} \int_{\mathbb{R}^N} G(x,u_m) \, dx \\ &\geqslant \int_{\mathbb{R}^N} G(x,u_\lambda) \, dx \\ &= I_\lambda(u_\lambda) - \tfrac{1}{2} I'_\lambda(u_\lambda)u_\lambda = I_\lambda(u_\lambda). \end{aligned}$$

This ends the proof of the lemma. □

At this point, combining lemmas 3.3 and 3.5 we deduce the existence of a sequence $\{(\lambda_n, u_n)\} \subset [1,2] \times H^1(\mathbb{R}^N)$ with $u_n \geqslant 0$ a.e. $x \in \mathbb{R}^N$, such that

(I) $\lambda_n \to 1$ and $\{\lambda_n\}$ is decreasing;

(II) $u_n \neq 0$, $I_{\lambda_n}(u_n) \leqslant c_{\lambda_n}$ and $I'_{\lambda_n}(u_n) = 0$.

Since

$$\tfrac{1}{2}\|u_n\|^2 - \lambda_n \int_{\mathbb{R}^N} F(x,u_n) \, dx \leqslant c_{\lambda_n} \quad \text{and} \quad \|u_n\|^2 = \lambda_n \int_{\mathbb{R}^N} f(x,u_n)u_n \, dx,$$

we have in particular that

$$\int_{\mathbb{R}^N} G(x,u_n) \, dx \leqslant \frac{c_{\lambda_n}}{\lambda_n}.$$

Clearly, c_{λ_n}/λ_n is increasing and is bounded by $c = c_1$. It follows that

$$\int_{\mathbb{R}^N} G(x,u_n) \, dx \leqslant c, \quad \forall n \in \mathbb{N}. \tag{3.7}$$

LEMMA 3.6. *Assume that (H1)–(H4) and either (A1) or (A2) hold. If the sequence* $\{u_n\} \subset H^1(\mathbb{R}^N)$ *given above is bounded, there exists* $u \neq 0$ *such that* $I'(u) = 0$. *In particular,* u *is a non-trivial positive solution of (P).*

Proof. First, notice that

$$I'(u_n)v = I'_{\lambda_n}(u_n)v + (\lambda_n - \lambda) \int_{\mathbb{R}^N} f(x, u_n)v \, dx \to 0, \quad \forall v \in H^1(\mathbb{R}^N).$$

Now, knowing that

$$I(u_n) = I_{\lambda_n}(u_n) + (\lambda_n - \lambda) \int_{\mathbb{R}^N} F(x, u_n) \, dx,$$

we distinguish two cases. Either $\limsup_{n\to\infty} I_{\lambda_n}(u_n) > 0$ or $\limsup_{n\to\infty} I_{\lambda_n}(u_n) \leqslant 0$. In the first case, we get $\limsup_{n\to\infty} I(u_n) > 0$ and the result follows from lemma 3.5. In the second case, we define the sequence $\{z_n\} \subset H^1(\mathbb{R}^N)$ by $z_n = t_n u_n$ with $t_n \in [0, 1]$ satisfying

$$I_{\lambda_n}(z_n) = \max_{t\in[0,1]} I_{\lambda_n}(tu_n). \tag{3.8}$$

(If, for $n \in \mathbb{N}$, t_n defined by (3.8) is not unique, we choose the smallest value.) By construction, $\{z_n\} \subset H^1(\mathbb{R}^N)$ is bounded. Moreover, on one hand, $I'_{\lambda_n}(z_n)z_n = 0$, $\forall n \in \mathbb{N}$ and, thus,

$$\lambda_n \int_{\mathbb{R}^N} G(x, z_n) \, dx = I_{\lambda_n}(z_n) - \tfrac{1}{2} I'_{\lambda_n}(z_n)z_n = I_{\lambda_n}(z_n). \tag{3.9}$$

On the other hand, it is easily seen, following the proof of lemma 3.1, that $I'_{\lambda_n}(u)u = \|u\|^2 + o(\|u\|^2)$ as $u \to 0$, uniformly in $n \in \mathbb{N}$. Thus, since $I'_{\lambda_n}(u_n) = 0$, there is $\alpha > 0$ such that $\|u_n\| \geqslant \alpha$, $\forall n \in \mathbb{N}$. Writing that $\limsup_{n\to\infty} I_{\lambda_n}(u_n) \leqslant 0$, we then obtain from lemma 3.1 and (3.8) that $\liminf_{n\to\infty} I_{\lambda_n}(z_n) > 0$ and, from (3.9), it follows that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} G(x, z_n) \, dx = \liminf_{n\to\infty} I_{\lambda_n}(z_n) > 0.$$

Lemma 3.4 then shows that $\{z_n\}$ does not vanish and, thus, neither does $\{u_n\}$. At this point, we conclude by repeating steps 1 and 2 of the proof of lemma 3.5. □

In view of lemma 3.6, to complete the proof of theorem 1.4 we just need to check that $\{u_n\} \subset H^1(\mathbb{R}^N)$ is bounded. This is the purpose of our last four lemmas. Seeking a contradiction, we assume that $\|u_n\| \to \infty$ and define the sequence $\{w_n\} \subset H^1(\mathbb{R}^N)$ by

$$w_n = u_n / \|u_n\|.$$

Clearly, $\|w_n\| = 1$ and, thus, $w_n \rightharpoonup w$ up to a subsequence. Either $\{w_n\} \subset H^1(\mathbb{R}^N)$ vanishes or it does not vanish. Using (A1) when $a < \infty$ or (A2) when $a = \infty$ in (H4), we shall prove that none of these alternatives can occur and this contradiction will prove that $\{u_n\} \subset H^1(\mathbb{R}^N)$ is bounded. Assume, first, that $\{w_n\} \subset H^1(\mathbb{R}^N)$ does not vanish. Then, as in the proof of lemma 3.5, using, if necessary, the translation invariance of problem (P), we get that $w_n \rightharpoonup w \neq 0$. Also, we can assume without loss of generality, that $w_n \to w$ a.e. $x \in \mathbb{R}^N$. At this point, the proof bifurcates to cover separately the cases $a < \infty$ and $a = \infty$ in (H4).

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LEMMA 3.7. *Assume that (H1)–(H4) hold with $a < \infty$ in (H4) and that $K \in]0, a[$. Then the non-vanishing of $\{w_n\} \subset H^1(\mathbb{R}^N)$ is impossible.*

Proof. We shall prove that $0 \neq w \in H^1(\mathbb{R}^N)$ satisfies the eigenvalue problem

$$-\Delta w(x) + Kw(x) = aw(x), \quad x \in \mathbb{R}^N. \tag{3.10}$$

This gives us the desired contradiction since it is well known that the operator $-\Delta$ has no eigenvalue in $H^1(\mathbb{R}^N)$. To prove that (3.10) holds, it suffices to check that, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} [\nabla w \nabla \varphi + Kw\varphi]\,\mathrm{d}x = \int_{\mathbb{R}^N} [aw\varphi]\,\mathrm{d}x. \tag{3.11}$$

Recall that $I'_{\lambda_n}(u_n) = 0$. Thus, we have

$$-\Delta u_n + Ku_n = \lambda_n f(x, u_n), \quad \text{in } H^{-1}(\mathbb{R}^N).$$

Consequently, $\{w_n\} \subset H^1(\mathbb{R}^N)$ satisfies

$$-\Delta w_n + Kw_n = \lambda_n (f(x, u_n)/u_n) w_n, \quad \text{in } H^{-1}(\mathbb{R}^N),$$

and this implies that, $\forall \varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} [\nabla w_n \nabla \varphi + Kw_n\varphi]\,\mathrm{d}x = \int_{\mathbb{R}^N} \left[\lambda_n \frac{f(x, u_n)}{u_n} w_n \varphi\right] \mathrm{d}x. \tag{3.12}$$

Since $w_n \rightharpoonup w$ weakly in $H^1(\mathbb{R}^N)$, we have, $\forall \varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} [\nabla w_n \nabla \varphi + Kw_n\varphi]\,\mathrm{d}x \to \int_{\mathbb{R}^N} [\nabla w \nabla \varphi + Kw\varphi]\,\mathrm{d}x. \tag{3.13}$$

We claim that

$$\lambda_n (f(x, u_n)/u_n) w_n \to aw, \text{ a.e. } x \in \mathbb{R}^N. \tag{3.14}$$

To prove (3.14), it is convenient to distinguish the cases $w(x) = 0$ and $w(x) \neq 0$ (without loss of generality we can assume that $w \neq 0$ is defined everywhere on $\mathbb{R}^N$). Let $x \in \mathbb{R}^N$ be such that $w(x) = 0$. Using the assumptions (H1), (H3) and (H4) we see that there exists $C < \infty$ such that

$$0 \leqslant (f(x, s)/s) \leqslant C, \quad \forall s \geqslant 0, \text{ a.e. } x \in \mathbb{R}^N. \tag{3.15}$$

Thus, since $\{\lambda_n\} \subset \mathbb{R}$ is bounded and $w_n(x) \to w(x)$ a.e. $x \in \mathbb{R}^N$, we have for such $x \in \mathbb{R}^N$ that

$$0 \leqslant \lambda_n (f(x, u_n(x))/u_n(x)) w_n(x) \leqslant \lambda_n C w_n(x) \to 0 = aw(x).$$

Now let $x \in \mathbb{R}^N$ be such that $w(x) \neq 0$. Then we necessarily have $u_n(x) \to \infty$ and, thus, using (H4), we get, since $\lambda_n \to 1$,

$$\lambda_n (f(x, u_n(x))/u_n(x)) \to a.$$

Consequently, also in this case,

$$\lambda_n (f(x, u_n(x))/u_n(x)) w_n(x) \to aw(x) \tag{3.16}$$

and (3.14) is established. Now let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be arbitrary but fixed, and let $\Omega \subset \mathbb{R}^N$ be a compact set such that $\operatorname{supp} \varphi \subset \Omega$. By the compactness of the

Sobolev embedding $H^1(\Omega) \hookrightarrow L^1(\Omega)$, we have $w_n \to w$ strongly in $L^1(\Omega)$. Thus, in particular, there is $h \in L^1(\Omega)$ such that $w_n(x) \leqslant h(x)$ a.e. $x \in \Omega$ (see [8, theorem IV.9]), and, using (3.15) again, we have

$$0 \leqslant \lambda_n(f(x,u_n)/u_n)w_n \leqslant Cw_n \leqslant Ch, \quad \text{a.e. } x \in \Omega. \tag{3.17}$$

Now (3.14) and (3.17) allow us to apply the Lebesgue theorem, and we get

$$\int_{\mathbb{R}^N} \left[\lambda_n \frac{f(x,u_n)}{u_n} w_n \varphi\right] \mathrm{d}x \to \int_{\mathbb{R}^N} [aw\varphi] \,\mathrm{d}x. \tag{3.18}$$

Since (3.18) holds for an arbitrary $\varphi \in C_0^\infty(\mathbb{R}^N)$, combining (3.13) and (3.18) we indeed get (3.11). Thus, (3.10) holds and the lemma is proved. □

LEMMA 3.8. *Assume that (H1)–(H4) hold with $a = \infty$ in (H4). Then the non-vanishing of $\{w_n\} \subset H^1(\mathbb{R}^N)$ is impossible.*

Proof. From

$$-\Delta u_n + Ku_n = \lambda_n f(x,u_n),$$

we deduce that

$$-\Delta w_n + Kw_n = \lambda_n (f(x,u_n)/\|u_n\|). \tag{3.19}$$

Multiplying (3.19) by an arbitrary $v \in H^1(\mathbb{R}^N)$ and integrating, it follows that

$$\int_{\mathbb{R}^N} [\nabla w_n \nabla v + Kw_n v] \,\mathrm{d}x = \lambda_n \int_{\mathbb{R}^N} \frac{f(x,u_n)}{\|u_n\|} v \,\mathrm{d}x.$$

Thus, if $w_n \rightharpoonup w$, we have, $\forall v \in H^1(\mathbb{R}^N)$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{f(x,u_n)}{\|u_n\|} v \,\mathrm{d}x = \int_{\mathbb{R}^N} [\nabla w \nabla v + Kwv] \,\mathrm{d}x$$

and, in particular, setting $v = w$ we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{f(x,u_n)}{\|u_n\|} w \,\mathrm{d}x = \|w\|^2 < \infty. \tag{3.20}$$

But on $\Omega = \{x \in \mathbb{R}^N : w(x) \neq 0\}$ we have, since $a = \infty$,

$$\frac{f(x,u_n)}{\|u_n\|} w = \frac{f(x,u_n)}{u_n} w_n w \to +\infty, \quad \text{a.e. } x \in \mathbb{R}^N.$$

Thus, taking into account that $|\Omega| > 0$ and using Fatou's lemma, we deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{f(x,u_n)}{\|u_n\|} w \,\mathrm{d}x = +\infty.$$

This contradicts (3.20). □

Now we shall prove that the vanishing of $\{w_n\} \subset H^1(\mathbb{R}^N)$ is forbidden. Here, we also distinguish the cases $a < \infty$ and $a = \infty$ and (H4).

LEMMA 3.9. *Assume that (H1)–(H4) hold with $a < \infty$ in (H4). Then, if (A1) holds, the vanishing of $\{w_n\} \subset H^1(\mathbb{R}^N)$ is impossible.*

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Proof. We have

$$-\Delta u_n + K u_n = \lambda_n f(x, u_n).$$

Thus,

$$-\Delta w_n + K w_n = \lambda_n (f(x, u_n)/u_n) w_n. \tag{3.21}$$

Multiplying (3.21) by w_n and integrating, we get

$$\int_{\mathbb{R}^N} [|\nabla w_n|^2 + K w_n^2] \, dx = \lambda_n \int_{\mathbb{R}^N} \frac{f(x, u_n)}{u_n} w_n^2 \, dx,$$

and we deduce from the normalization of $\{w_n\} \subset H^1(\mathbb{R}^N)$ that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{f(x, u_n)}{u_n} w_n^2 \, dx = 1. \tag{3.22}$$

We define for $\delta > 0$ given in (A1)

$$\Omega_n = \{x \in \mathbb{R}^N : (f(x, u_n)/u_n) \leqslant K - \tfrac{1}{2}\delta\}.$$

Then, since $1 = \|w_n\|^2 = \|\nabla w_n\|_2^2 + K\|w_n\|_2^2$, we have

$$\begin{aligned} \int_{\Omega_n} \frac{f(x, u_n)}{u_n} w_n^2 \, dx &\leqslant (K - \tfrac{1}{2}\delta) \int_{\Omega_n} w_n^2 \, dx \\ &\leqslant \frac{1}{K}(K - \tfrac{1}{2}\delta). \end{aligned}$$

Consequently, we see, using (3.22), that necessarily

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N \setminus \Omega_n} \frac{f(x, u_n)}{u_n} w_n^2 \, dx > 0. \tag{3.23}$$

We claim that

$$\limsup_{n\to\infty} |\mathbb{R}^N \setminus \Omega_n| = \infty. \tag{3.24}$$

Seeking a contradiction, we assume that

$$\limsup_{n\to\infty} |\mathbb{R}^N \setminus \Omega_n| < \infty. \tag{3.25}$$

Note that by (3.15),

$$\int_{\mathbb{R}^N \setminus \Omega_n} \frac{f(x, u_n)}{u_n} w_n^2 \, dx \leqslant C \int_{\mathbb{R}^N \setminus \Omega_n} w_n^2 \, dx. \tag{3.26}$$

But, since $\{w_n\} \subset H^1(\mathbb{R}^N)$ vanishes, taking (3.25) into account we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N \setminus \Omega_n} w_n^2 \, dx \to 0,$$

and, thus, (3.26) contradicts (3.23). The contradiction proves that (3.24) is true. Now observe that, by (A1), $G(x, s) \geqslant 0$, $\forall s \geqslant 0$, a.e. $x \in \mathbb{R}^N$ and, thus, $\forall n \in \mathbb{N}$,

$$\begin{aligned} \int_{\mathbb{R}^N} G(x, u_n) \, dx &= \int_{\Omega_n} G(x, u_n) \, dx + \int_{\mathbb{R}^N \setminus \Omega_n} G(x, u_n) \, dx \\ &\geqslant \int_{\mathbb{R}^N \setminus \Omega_n} G(x, u_n) \, dx. \end{aligned}$$

Taking (3.7) into account, we deduce that, $\forall n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N \setminus \Omega_n} G(x, u_n)\,\mathrm{d}x \leqslant C. \tag{3.27}$$

But, on $\mathbb{R}^N \setminus \Omega_n$, we have $(f(x, u_n)/u_n) \geqslant K - \frac{1}{2}\delta$ and, thus, by (A1),

$$G(x, u_n) \geqslant \delta, \text{ a.e. } x \in \mathbb{R}^N \setminus \Omega_n. \tag{3.28}$$

Combining (3.24) and (3.28), we get a contradiction with (3.27). □

LEMMA 3.10. *Assume that (H1)–(H4) hold with $a = \infty$ in (H4). Then, if (A2) holds, the vanishing of $\{w_n\} \subset H^1(\mathbb{R}^N)$ is impossible.*

Proof. We again use the sequence $\{z_n\} \subset H^1(\mathbb{R}^N)$ introduced in lemma 3.6. We claim that, under our assumptions and since we assume that $\|u_n\| \to \infty$,

$$\lim_{n\to\infty} I_{\lambda_n}(z_n) = +\infty. \tag{3.29}$$

Seeking a contradiction, we assume that for $M < \infty$

$$\liminf_{n\to\infty} I_{\lambda_n}(z_n) \leqslant M, \tag{3.30}$$

and we define, for the corresponding subsequence, $\{k_n\} \subset H^1(\mathbb{R}^N)$ by

$$k_n = \sqrt{4M}(u_n/\|u_n\|).$$

Now, since $\{k_n\} \subset H^1(\mathbb{R}^N)$ vanishes and is bounded, from the proof of lemma 3.4, we get that

$$\int_{\mathbb{R}^N} F(x, k_n)\,\mathrm{d}x \to 0.$$

It follows that, for $n \in \mathbb{N}$ sufficiently large,

$$I_{\lambda_n}(k_n) = 2M - \lambda_n \int_{\mathbb{R}^N} F(x, k_n)\,\mathrm{d}x \geqslant \tfrac{3}{2}M. \tag{3.31}$$

Since k_n and z_n correspond, for all $n \in \mathbb{N}$, to the same direction, we see using the definition of z_n that (3.31) contradicts (3.30). Thus, (3.29) holds. Now we have $I'_{\lambda_n}(z_n)z_n = 0$, $\forall n \in \mathbb{N}$, and, thus,

$$I_{\lambda_n}(z_n) = I_{\lambda_n}(z_n) - \tfrac{1}{2}I'_{\lambda_n}(z_n)z_n = \lambda_n \int_{\mathbb{R}^N} G(x, z_n)\,\mathrm{d}x. \tag{3.32}$$

Combining (3.29) and (3.32), we see that

$$\int_{\mathbb{R}^N} G(x, z_n)\,\mathrm{d}x \to +\infty.$$

But, from (A2) and (3.7), we also have

$$\int_{\mathbb{R}^N} G(x, z_n)\,\mathrm{d}x \leqslant D \int_{\mathbb{R}^N} G(x, u_n)\,\mathrm{d}x \leqslant C. \tag{3.33}$$

This contradiction proves the lemma. □

https://doi.org/10.1017/S0308210500013147 Published online by Cambridge University Press

Acknowledgments

I thank F. Alessio, P. Caldiroli and P. Montecchiari for fruitful discussions and constant support during the development of this work. In a preliminary version, only the case $a = \infty$ in (H4) was considered. I am grateful to Professor C. A. Stuart who kindly pointed out to me the possibility of also treating linear cases asymptotically. Finally, I thank Professor E. Séré and Professor M. Struwe whose pertinent remarks have made it possible to simplify and clarify initial proofs of theorem 1.1.

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(*Issued 13 August 1999*)