ON THE EXISTENCE OF CONFORMAL MEASURES

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ABSTRACT. A general notion of conformal measure is introduced and some basic properties are studied. Sufficient conditions for the existence of these measures are obtained, using a general construction principle. The geometric properties of conformal measures relate equilibrium states and Hausdorff measures. This is shown for invariant subsets of S^1 under expanding maps.

1. INTRODUCTION

Let $(X, \mathscr{F})$ be a measurable space and T be a measurable endomorphism of X. In this note, measures m (later called conformal) are studied which have the property that $m \circ T$ is (locally) absolutely continuous with respect to m, and the Radon-Nikodym derivative is a continuous function f. This type of measures plays an important role in studying the equilibrium theory in dynamics and the fractal nature of attractors and nonwandering sets. There are quite a few contributions using these measures, but there has been no attempt to study these measures in more generality.

The main problem is this: Given a continuous function f on the compact metric space X, does there exist a measure m, so that $m \circ T$ is (locally) absolutely continuous with Radon-Nikodym derivative f?

In case of a topologically mixing subshift of finite type (and hence whenever Markov partitions exist) this question has a satisfactory solution by the Schauder-Tychonov theorem (see [1]). If (X, T) is a mixing repeller [16], the existence of conformal measures is easily verified, and this observation avoids constructing Markov partitions to prove, for example, the Bowen-Ruelle formula (first observed by C. Seck). For rational maps on their Julia set X, and if the function f denotes a suitable power of the modulus of the derivative of T, the problem of constructing conformal measures has been investigated in [18]. In the expanding case it leads to some interesting results about the relation between Gibbs states and Hausdorff measures. In all other cases only little is known. First results, after this paper was written, were obtained in [21] and [22].

The term "conformal measures" is chosen from [12] (see also [13]), where this

type of measure is obtained on limit sets of finitely generated discrete subgroups of conformal mappings acting on hyperbolic space. Again, using these measures, it is possible to study the connection between Hausdorff dimension and the exponent of convergence for Poincaré series.

The first result in §2 shows that conformal measures for the function f are fixed points of the adjoint of the transfer operator (Perron-Frobenius operator) defined by f. It follows from this fact that conformal measures were constructed for some one-dimensional transformations in [6], [7] and [23]. The forthcoming papers [4], [21] and [22] show that there are far reaching applications of conformal measures to the theory of equilibrium states and to the study of Hausdorff dimension and measures. §2 also gives some basic properties of conformal measures needed in the sequel.

In §3, a general principle for constructing conformal measures is presented, when T is a continuous, finite-to-one transformation such that there is only a finite number of points, where T is not, locally, a homeomorphism. These points will be called singular. The approach to construct these measures is different from [12], though it has some similarities. It is also different from [6], [7] or [23], where the transfer operator is extended to some larger space, so that it acts on continuous functions and the Schauder-Tychonov theorem applies. It will be shown that $\exp(c - f)$-conformal measures in a weak sense always exist for any continuous function f and some $c \in \mathbf{R}$. These measures turn out to be conformal, if they do not assign positive mass to orbits of singular points. The constant c can be identified with the pressure $P(T, f)$ in case when $P(T, f) > \sup_{x \in X} f(x)$. Theorem 3.13 gives sufficient conditions for this.

One of the applications of the general construction principle and the existence theorem for conformal measures can be found in §4. Let $T\colon S^1 \to S^1$ be a $C^{1+\varepsilon}$-expanding map of the circle. On certain closed, T-invariant subsets X (see [20]) and for a Lipschitz continuous function f there exist, by Theorem 3.13 in §3, an $\exp(P(T, f) - f)$-conformal measure m_f and an absolutely continuous equilibrium state μ_f for f. Then, using the properties of the conformal measure m_f, the relation between μ_f and Hausdorff measures $H_{\tilde{h}}$ (see [15] for a definition) on the topological support of μ_f is investigated, where

$$\tilde{h}(t) = t^\kappa \exp(\sigma \chi^{-1/2} h(-\log t)\sqrt{-\log t}) \qquad (t > 0).$$

(Here κ denotes the Hausdorff dimension of μ_f, χ the Lyapunov exponent, σ^2 some asymptotic variance, and h a positive, nondecreasing function (see §4 for precise definitions).) If h is a lower class function, μ_f is shown to be absolutely continuous with respect to $H_{\tilde{h}}$, and if h is constant, then both measures are orthogonal. In particular, μ_f is singular to the Hausdorff measure corresponding to the function t^κ. This holds whenever $\sigma^2 > 0$. If it is zero, both measures are shown to be equivalent.

This type of problem has been studied in different situations. Fractal prop-

erties of paths of Brownian motion (and even of more general self-similar processes) have been investigated by several authors (see [5] for further references). Much more related to the present study are investigations of harmonic measures on the boundary of an open topological disc on the Riemann sphere (see [9] or [14] for more details). The latter paper [14] deals with harmonic measures and, more generally, with the images of Gibbs states on conformal mixing repellers and uses, in a dynamical context, the theory of upper and lower class functions from probability and Pesin's theory from dynamics. Although there are some similarities of the present work with that paper, the investigations here lead to some new problems caused by the fact, that, in general, the equilibrium states are no longer Gibbs measures. These difficulties are overcome using conformal measures.

The following notations will be used throughout the paper. Let X be a measurable or metric space. For a subset $A \subset X$, the closure of A is denoted by $\overline{A}$ and the boundary of A by ∂A. For $x \in X$, $r > 0$ and $n \geq 1$ let

$$B_n(x, r) = \{y \in X: \operatorname{dist}(T^j(x), T^j(y)) < r \ \forall 0 \leq j < n\}.$$

If $n = 1$, $B_1(x, r) = B(x, r)$ is the ball with radius r around x. Now let $(X, \mathscr{F})$ and $(Y, \mathscr{F}')$ be measurable spaces and let $T\colon X \to Y$ be a measurable map. T will be called a measurable isomorphism, if T is invertible and both T and T^{-1} are measurable. For a measurable function g on Y set $S_n g = \sum_{0 \leq k < n} g \circ T^j$. Occasionally, $\mu(g)$ will be written for $\int_Y g \, d\mu$, where μ is a measure on Y and $g \in L_1(\mu)$. Finally, $\mathbf{R}_+ = [0, \infty)$ denotes the set of nonnegative reals and $\mathbf{N}$ the set of nonnegative integers. The cardinality of a set E will be denoted by $\sharp E$. Throughout most of the paper the conformal measures will be probabilities.

2. CONFORMAL MEASURES

Consider a measurable endomorphism $T\colon X \to X$ on a measurable space $(X, \mathscr{F})$ and a measurable nonnegative function f on X.

Definition 2.1. A measure m on $(X, \mathscr{F})$ is called f-conformal, if

$$(2.1) \qquad m(TA) = \int_A f(z) m(dz),$$

whenever $A \in \mathscr{F}$ is a measurable set, for which TA is measurable and $T\colon A \to TA$ is invertible.

In the sequel, a set A as in the definition will be called special. Note that (2.1) implies that $m \circ T$ is absolutely continuous with respect to m on the σ-algebra $\mathscr{F} \cap A$ for every special set A such that $T\colon A \to TA$ is a measurable isomorphism. In this case, (2.1) is equivalent to the fact that the corresponding Radon-Nikodym derivative $dm \circ T/dm = f_{|A}$.

Examples. There are several examples of conformal measures appearing in the literature.

(1) Let (X, T) denote a topologically mixing subshift of finite type and f a Hölder continuous function on X. Denote by $L\varphi(x) = \sum_{T(y)=x} \varphi(y) \exp(f(y))$ the Perron-Frobenius operator associated to f, which acts on the space of continuous functions φ on X. By the Schauder-Tychonov theorem, there exists a fixed point of the map $\mathscr{L}(\mu) = (L^*\mu(1))^{-1} L^*\mu$ (see [1, p. 14]). This fixed point is a conformal measure with respect to the function $\exp(P(T, f) - f)$. $(P\cdot, \cdot)$ denotes the pressure function (see [3]).) It is not too difficult to see this, and, if the pressure vanishes, it follows immediately from Proposition 2.2 below.

(2) For any piecewise differentiable map T on an interval, on S^1 or on $\mathbf{R}^n$, Lebesgue measures is conformal with respect to the modulus of the derivative of T. In [6] and [7] conformal measures were constructed for piecewise differentiable maps on the unit interval and certain continuous functions, in particular all Lipschitz continuous functions. These measures are shown to be fixed points of the adjoint of the transfer operator and thus, by Proposition 2.2 below, conformal. For β-transformations the corresponding result has been obtained earlier [23].

(3) Lebesgue measure is conformal with respect to $|T'|$ for any rational map T on the Riemann sphere, but not necessarily for this map restricted to its Julia set. In the latter situation conformal measures were obtained in [18] with respect to the function $|T'|^\delta$ for some $\delta \in (0, 2]$. In the expanding case, δ is known to be the Hausdorff dimension of the Julia set [18].

(4) Let Γ be a finitely generated discrete subgroup of conformal maps acting on hyperbolic space. In this case, one can construct a measure, which is conformal with respect to the functions $j(g)^\delta$ for any map $g \in \Gamma$ acting on the limit set of Γ (see [12] and also [13]), where j denotes the conformal distortion. δ is known to be the exponent of convergence of the Poincaré series associated to Γ.

Let m be an f-conformal measure and define $f^{(1)} = f$ and

$$f^{(n)} = (f^{(n-1)} \circ T)f \quad \text{for } n \geq 2.$$

It follows immediately from (2.1), that for every $n \geq 1$, m is also $f^{(n)}$-conformal with respect to the transformation T^n and

$$(2.2) \qquad \int_{TA} g \, dm = \int_A (g \circ T) f \, dm$$

for any measurable function g, which is nonnegative or integrable on TA.

Let T be a measurable endomorphism of $(X, \mathscr{F})$, which is finite-to-one (i.e. the numbers of preimages of points is uniformly bounded). For a measurable function $f\colon X \to \mathbf{R}_+$ the Perron-Frobenius operator

$$(2.3) \qquad L_f\varphi(x) = L\varphi(x) = \sum_{T(y)=x} \varphi(y)/f(y)$$

is well defined for (real-valued) measurable functions φ and all $x \in X$ such that $f(y) > 0$ for all preimages y of x. In this situation one has

Proposition 2.2. *Assume that there exists a finite partition of X into special sets X_i $(1 \leq i \leq s)$, such that $T: X_i \to TX_i$ is a measurable isomorphism. Let $f: X \to \mathbf{R}_+$ be a measurable function and m be a probability measure on $(X, \mathscr{F})$. Then m is f-conformal if and only if L_f acts on $L^1(m)$ and $L^*m = m$.*

Proof. Let M be f-conformal. By (2.1), for $1 \leq i \leq s$, $m(TA_i) = 0$ where $A_i = \{x \in X_i : f(x) = 0\}$. Whence $L\varphi$ is a.e. well defined. Moreover, if φ is an integrable function, (2.2) implies

$$\int_{TX_i} \frac{\varphi}{f} \circ (T_{|X_i})^{-1} \, dm = \int_{X_i} \varphi \, dm \qquad (1 \leq i \leq s).$$

Summing over i yields

$$\int_X L_f \varphi \, dm = \int_X \varphi \, dm.$$

Conversely, assume that L_f acts on $L^1(m)$ and that m is a fixed point of L^*. Let A be a special set. Then, by (2.3),

$$\int_A f \, dm = \int_A f \, dL^*m = \int_X L(1_A f) \, dm$$

$$= \int_X \sum_{T(y)=x} 1_A(y) m(dx) = m(TA).$$

Thus m is f-conformal.

Definition 2.3. Let X be a compact metric space and $T: X \to X$ be a continuous map, which is finite-to-one. Denote by X_0 the set of all points $x \in X$, such that for all sufficiently small $\varepsilon > 0$ the set $TB(x, \varepsilon)$ is not open. A point $x \in X$ is called critical (for T), if there does not exist an open neighborhood of x, which is a special set, and it is called singular, if it is critical or if it belongs to X_0.

Denote by $\mathrm{Crit}(T)$ (resp. $\mathrm{Sing}(T) = X_0 \cup \mathrm{Crit}(X)$) the set of critical (resp. singular) points for G. If T has only a finite number of singular points, it follows that for every nonsingular point $x \in X$ there exists an open neighbourhood U of x, such that $T_{|U}$ is a homeomorphism onto the open set TU. Moreover, if m is a Borel measure on the Borel σ-field $\mathscr{F}$, then there exists a finite measurable partition of $X \pmod m$ into special sets on which T acts as a measurable isomorphism. In fact, this can be proved, for example, applying Zorn's lemma to the system

$$\{(A_1, \ldots, A_s): m(A_i \cap A_j) = 0 \ (i \neq j); \ T_{|A_i} \text{ is a measurable isomorphism}\},$$

where s is sufficiently large, and where the ordering is given by inclusion $\mathrm{mod}\, m$.

For the remaining part of this section, let X be a compact metric space and $T: X \to X$ be a continuous map, which is finite-to-one, and which has at most a finite number of singular points.

Lemma 2.4. *Let m be a measure on $(X, \mathscr{F})$ and let $f \in L^1(m)$. Assume that* (2.1) *holds for every special set $C \in \mathscr{F}$ with $C \cap \mathrm{Sing}(T) = \varnothing$ and $m(\partial C) = m(\partial TC) = 0$. Then for any special set $A \in \mathscr{F}$ satisfying $A \cap \mathrm{Sing}(T) = \varnothing$,* (2.1) *continues to hold.*

Proof. Let $\varepsilon > 0$. For each $x \in A$ there exists an open set $C(x)$ such that $T_{|C(x)}$ is a homeomorphism, $m(\partial C(x)) = m(\partial TC(x)) = 0$, $C(x) \cap \mathrm{Sing}(T) = \varnothing$, and such that

$$\int_{\bigcup C(x) \setminus A} f \, dm < \varepsilon.$$

The existence of sets with these properties follows from well-known measure theoretic arguments, since T is open at $x \in A$. Choose a countable family $\{C_k\}$ from $\{C(x)\}$ which covers A and define recursively $A_1 = C_1$ and $A_n = C_n \setminus \bigcup_{k<n} C_k$.

By definition and by the assumption in the lemma, each set A_k satisfies (2.1) and hence

$$m(TA) = m\left(\bigcup_{k=1}^{\infty} T(A \cap A_k) \right) \leq \sum_{k=1}^{\infty} m(TA_k)$$

$$= \sum_{k=1}^{\infty} \int_{A_k} f \, dm = \int_A f \, dm + \sum_{k=1}^{\infty} \int_{A_k \setminus A} f \, dm$$

$$\leq \int_A f \, dm + \varepsilon.$$

If $\varepsilon \to 0$, it follows that $m(TB) \leq \int_B f \, dm$ for any special set B not containing any singular point.

Using this fact, the lower bound for $m(TA)$ is obtained from the following estimate, if $\varepsilon \to 0$:

$$m(TA) = m\left(\bigcup_{k=1}^{\infty} T(A \cap A_k) \right) = \sum_{k=1}^{\infty} m(T(A \cap A_k))$$

$$= \sum_{k=1}^{\infty} [m(TA_k) - m(T(A_k \setminus A))]$$

$$\geq \sum_{k=1}^{\infty} \left[\int_{A_k} f \, dm - \int_{A_k \setminus A} f \, dm \right]$$

$$= \int_{\bigcup_{k \geq 1} A_k} f \, dm - \int_{\bigcup_{k \geq 1} A_k \setminus A} f \, dm \geq \int_A f \, dm - \varepsilon.$$

This proves the lemma.

Theorem 2.5. *Let $T: X \to X$ be a continuous, finite-to-one map of the compact metric space X having at most a finite number of singular points, and let f_n be measurable functions. For each $n \in \mathbf{N}$, let m_n be an f_n-conformal normalized*

Borel measure, such that

$$K := \sup_{n \geq 1} \left\| \frac{f_n}{f_n \circ T} \right\|_{L^\infty(m_n)} < \infty, \quad or \quad \sup_{n \geq 1} \|f_n\|_{L^1(m_n)} < \infty,$$

and such that $(m_n: n \geq 1)$ *converges weakly to some measure* m. *Moreover, suppose that* m *satisfies the following condition:*

(2.4) $m(\{x\}) = 0 \Rightarrow m(\{T(x)\}) = 0 \qquad (\forall x \in \text{Sing}(T)).$

Then m *is* f-*conformal for some* $f \in L^1(m)$.

Proof. It suffices to prove the theorem under the assumption

$$K := \sup_{n \geq 1} \left\| \frac{f_n}{f_n \circ T} \right\|_{L^\infty(m_n)} < \infty.$$

The other case is proved similarly, but in a simpler way.

Let $\{X_1, \ldots, X_s\}$ be a finite measurable partition $(\bmod\, m_n)$ into special sets. (This can be obtained simultaneously for all $n \in \mathbf{N}$ by the remark preceding Lemma 2.4.) Define measures μ_n by $\mu_n(A) = \int_A f_n \, dm_n$ $(m \geq 1)$. Then

$$\mu_n(X) = \int_X f_n \, dm_n = \sum_{i=1}^{s} \int_{X_i} f_n \, dm_n$$

$$= \sum_{i=1}^{s} m_n(TX_i) \leq s.$$

Therefore there is no loss of generality assuming that the measures μ_n converge weakly to some measure μ with $\mu(X) \leq s$.

Next, using (2.2),

$$\|f_n\|_2^2 = \sum_{i=1}^{s} \int_{X_i} f_n^2 \, dm_n = \sum_{i=1}^{s} \int_{TX_i} f_n \circ (T_{|X_i})^{-1} \, dm_n$$

$$\leq K \sum_{i=1}^{s} \int_{TX_i} f_n \, dm_n \leq sK \int_X f_n \, dm_n$$

$$= sK \sum_{i=1}^{s} \int_{X_i} f_n \, dm_n = sK \sum_{i=1}^{s} m_n(TX_i)$$

$$\leq s^2 K < \infty.$$

For any set $U \in \mathscr{F}$ satisfying $\mu(\partial U) = m(\partial U) = 0$ it follows that

(2.5) $\mu(U) = \lim_{n \to \infty} \mu_n(U) = \lim_{n \to \infty} \int_U f_n \, dm_n$

$$\leq \lim_{n \to \infty} \|f_n\|_2 (m_n(U))^{1/2} \leq (s^2 K m(U))^{1/2}.$$

Clearly, this suffices to show that μ is absolutely continuous with respect to m, and so, by the Radon-Nikodym theorem, there exist a function $\tilde{f} \in L^1(m)$ satisfying $\mu(A) = \int_A \tilde{f} \, dm$ for $A \in \mathscr{F}$.

Define f by setting $f(x) = \check{f}(x)$ if $x \in X$ is nonsingular. This defines f a.e., if m assigns measure zero to $\text{Sing}(T)$. If $x \in X$ is singular and $m(\{x\}) > 0$, then set

$$(2.6) \qquad f(x) = m(\{T(x)\})/m(\{x\}).$$

It is left to show that m is f-conformal. First note that for a set $C \in \mathscr{F}$ with $C \cap \text{Sing}(T) = \varnothing$ and $\mu(\partial C) = m(\partial T C) = 0$, one has

$$(2.7) \qquad m(TC) = \lim_{n\to\infty} m_n(TC) = \lim_{n\to\infty} \int_C f_n \, dm_n$$

$$= \lim_{n\to\infty} \mu_n(C) = \mu(C) = \int_C \check{f} \, dm = \int_C f \, dm.$$

By (2.4) and (2.6), it suffices to check the condition (2.1) for special sets A not containing any singular point. By (2.7) the assumptions in Lemma 2.4 are satisfied, whence the theorem.

Corollary 2.6. *Assume that the measure m in Theorem 2.5 assigns measure zero to $\text{Sing}(T)$. If f_n converges uniformly to some function f, then m is f-conformal.*

Proof. This follows from (2.5).

Corollary 2.7. *If in Corollary 2.6, $f_n = s_n h$ for some continuous function h and some $s_n \in \mathbf{R}_+$, and if $s = \lim_{n\to\infty} s_n$ exists, then m is sh-conformal.*

It is possible to obtain a similar theorem about weak convergence to some conformal measure using a version of the proof of Proposition 3.11 below. We state it without proof. (In fact, its proof is contained in the proof of Theorem 3.13.)

Theorem 2.8. *Let $T: X \to X$ be a continuous, finite-to-one map of the compact metric space X having at most a finite number of singular points. Assume that $X_0 \cap \text{Crit}(T) = \varnothing$ and that T has no critical periodic point. Let f_n, f be continuous functions so that $(f_n: n \geq 1)$ converges uniformly to f. If $\inf_{x \in X} f(x) > 1$ and if the f_n-conformal measures m_n converge weakly to m, then m is f-conformal.*

3. CONSTRUCTIONS OF CONFORMAL MEASURES

The main purpose of this section is to provide a general scheme for constructing conformal measures. Although the basic idea goes back to [12], the construction turns out to be different, and is, in fact, more closely related to the proof of the variational principle for the pressure in [11]. Besides this, several sufficient conditions for the existence of conformal measures are presented.

Throughout this section let X be a compact metric space, equipped with its Borel σ-field $\mathscr{F}$, and let $T: X \to X$ be a continuous map, such that the set $\text{Sing}(T)$ of singular points is finite. Moreover, let $f: X \to \mathbf{R}$ be a continuous

function and fix (except otherwise stated) a sequence $(E_n : n \geq 1)$ of finite subsets of X satisfying

$$(3.1) \qquad\qquad T^{-1}E_n \subset E_{n+1} \quad \forall n \in \mathbf{N}.$$

The construction principle for conformal measures will make use of the following simple analytical fact. For a sequence $(a_n : n \geq 1)$ of reals, the number $c = \limsup_{n \to \infty} a_n / n$ will be called the transition parameter of $(a_n : n \geq 1)$. It is uniquely determined by the fact that $\sum_{n \in \mathbf{N}} \exp(a_n - ns)$ converges for $s > c$ and diverges for $s < c$. For $s = c$ the sum may converge or diverge. By a simple argument (see e.g. [12] for a similar consideration) one obtains

Lemma 3.1. *There exists a sequence* $(b_n : n \geq 1)$ *of positive reals such that*

$$\sum_{n=1}^{\infty} b_n \exp(a_n - ns) \begin{cases} < \infty, & s > c, \\ = \infty, & s \leq c, \end{cases}$$

and $\lim_{n \to \infty} b_n / b_{n+1} = 1$.

Proof. If $\sum \exp(a_n - nc) < \infty$, choose a sequence $n_k \in \mathbf{N}$ $(k \geq 1)$ satisfying

$$\lim_{k \to \infty} n_k n_{k+1}^{-1} = 0 \quad \text{and} \quad \varepsilon_k := a_{n_k} n_k^{-1} - c \to 0.$$

Setting

$$b_n = \exp\left(n\left(\frac{n_k - n}{n_k - n_{k-1}} \varepsilon_{k-1} + \frac{n - n_{k-1}}{n_k - n_{k-1}} \varepsilon_k \right) \right) \quad \text{for } n_{k-1} \leq n < n_k,$$

it is easy to check the claim.

Construction Principle. Given the function f and the finite sets E_n $(n \geq 1)$, the general construction is done as follows: Let $a_n = \log \sum_{x \in E_n} \exp S_n f(x)$, where $S_n f = \sum_{0 \leq k < n} f \circ T^k$. Denote by c the transition parameter of this sequence. Choose a sequence $(b_n : n \geq 1)$ of positive reals as in Lemma 3.1 for the sequence $(a_n : n \geq 1)$. For $s > c$ define

$$(3.2) \qquad\qquad M_s = \sum_{n=1}^{\infty} b_n \exp(a_n - ns)$$

and the normalized measure

$$(3.3) \qquad m_s = \frac{1}{M_s} \sum_{n=1}^{\infty} \sum_{x \in E_n} b_n \exp(S_n f(x) - ns)\delta_x,$$

where δ_x denotes the unit mass at the point $x \in X$.

Let $A \in \mathcal{F}$ be a special set. Using (3.1) and (3.3) it follows that

$$m_s(TA) = \frac{1}{M_s} \sum_{n=1}^{\infty} \sum_{x \in E_n \cap TA} b_n \exp(S_n f(x) - ns)$$

$$= \frac{1}{M_s} \sum_{n=1}^{\infty} \sum_{x \in A \cap T^{-1} E_n} b_n \exp(S_n f(T(x)) - ns)$$

(3.4)
$$= \frac{1}{M_s} \sum_{n=1}^{\infty} \sum_{x \in A \cap E_{n+1}} b_n \exp[S_{n+1} f(x) - (n+1)s] \exp(s - f(x))$$

$$- \frac{1}{M_s} \sum_{n=1}^{\infty} \sum_{x \in A \cap (E_{n+1} \setminus T^{-1} E_n)} b_n \exp(S_n f(T(x)) - ns).$$

Define

$$\Delta_A(s) = \left| \frac{1}{M_s} \sum_{n=1}^{\infty} \sum_{x \in A \cap E_{n+1}} b_n \exp[S_{n+1} f(x) - (n+1)s] \exp(s - f(x)) \right.$$

$$\left. - \int_A \exp(c - f) dm_s \right|$$

and observe that

$$\Delta_A(s) = \frac{1}{M_s} \left| \sum_{n=1}^{\infty} \sum_{x \in A \cap E_{n+1}} \exp[S_{n+1} f(x) - (n+1)s] \right.$$

$$\left. \cdot \exp(-f(x))[b_n e^s - b_{n+1} e^c] - b_1 \sum_{x \in A \cap E_1} e^{c-s} \right|$$

$$\leq \frac{1}{M_s} \sum_{n=1}^{\infty} \sum_{x \in A \cap E_{n+1}} \left| \frac{b_n}{b_{n+1}} - e^{c-s} \right| b_{n+1} \exp(s - f(x))$$

$$\cdot \exp[S_{n+1} f(x) - (n+1)s] + \frac{1}{M_s} b_1 \exp(c - s) \sharp(A \cap E_1)$$

$$\leq \frac{1}{M_s} \sum_{n=1}^{\infty} \sum_{x \in E_{n+1}} \left| \frac{b_n}{b_{n+1}} - e^{c-s} \right| b_{n+1} \exp(s - f(x))$$

$$\cdot \exp[S_{n+1} f(x) - (n+1)s] + \frac{1}{M_s} b_1 \exp(c - s) \sharp E_1.$$

By Lemma 3.1, $\lim_{n \to \infty} b_{n+1}/b_n = 1$ and $\lim_{s \downarrow c} M_s = \infty$. Therefore

(3.5)
$$\lim_{s \downarrow c} \Delta_A(s) = 0$$

uniformly for all special sets $A \in \mathcal{F}$.

Definition 3.2. Any weak accumulation point, when $s \downarrow c$, of the measures $(m_s: s > c)$ defined in the construction principle will be called a limit measure (associated to the function f and the sequence $(E_n: n \geq 1)$).

In order to find conformal measures among the limit measures, it is necessary to examine (3.4) in greater detail. For a subset $D \in \mathscr{F}$ consider the condition

$$(3.6) \qquad \lim_{s \downarrow c} \frac{1}{M_s} \sum_{n=1}^{\infty} \sum_{x \in D \cap (E_{n+1} \setminus T^{-1} E_n)} b_n \exp[S_n f(T(x)) - ns] = 0.$$

Lemmas 3.3–3.5 collect a few useful facts about the behaviour of a limit measure at singular points and on special sets, which do not contain a singular point.

Lemma 3.3. *Let m be a limit measure. Assume that every special set $D \in \mathscr{F}$ with $m(\partial D) = m(\partial T D) = 0$, and $\overline{D} \cap \mathrm{Sing}(T) = \varnothing$ satisfies condition (3.6). Then for any special set $A \in \mathscr{F}$ with $A \cap \mathrm{Sing}(T) = \varnothing$,*

$$m(TA) = \int_A \exp(c - f)\, dm.$$

Proof. Let $D \in \mathscr{F}$ be a special set such that $\overline{D} \cap \mathrm{Sing}(T) = \varnothing$ and $m(\partial D) = m(\partial T D) = 0$. It follows immediately from (3.4)–(3.6) that

$$(3.7) \qquad m(TD) = \int_D \exp(c - f)\, dm.$$

Now, let A be as in the lemma, and let $C \subset A$ be compact. A slight modification of the proof of Lemma 2.4 shows that (3.7) continues to hold for C. Since every Borel measure is regular, and since T is continuous, (3.7) also holds for A by approximation with compact sets contained in A.

Lemma 3.4. *Assume that condition (3.6) is satisfied for $D = X$. Then for $x \in \mathrm{Sing}(T) \setminus \mathrm{Crit}(T)$*

$$m(\{T(x)\}) \geq \exp(c - f(x)) m(\{x\}).$$

Proof. Let $(C_n : n \geq 1)$ be a sequence of compact neighbourhoods of $x \in \mathrm{Sing}(T) \setminus \mathrm{Crit}(T)$, decreasing to x, so that each C_n is special and satisfies $m(\partial C_n) = 0$. From (3.4), (3.5), and the assumption, one obtains

$$\lim_{s \in J} \left| m_s(TC_n) - \int_{C_n} \exp(c - f)\, dm_s \right| = 0 \quad \forall n \geq 1,$$

where J denotes the subsequence along which m_s converges to m. Since TC_n is compact, this clearly implies

$$m(TC_n) \geq \limsup_{s \in J} m_s(TC_n)$$

$$= \lim_{s \in J} \int_{C_n} \exp(c - f)\, dm_s$$

$$= \int_{C_n} \exp(c - f)\, dm.$$

Letting $n \to \infty$, the lemma follows.

Lemma 3.5. *Assume that condition (3.6) holds for all sets $D \in \mathscr{F}$ satisfying $\overline{D} \cap X_0 = \varnothing$. Then for every limit measure m and every point $x \in \mathrm{Crit}(T) \setminus X_0$,*

one has $m(\{x\}) = 0 \Leftrightarrow m(\{T(x)\}) = 0$. *More precisely,*

$$m(\{T(x)\}) \leq \exp(c - f(x))m(\{x\}) \leq q(x)m(\{T(x)\})$$

where $q = q(x)$ *denotes the number of preimages of* T *restricted to a sufficiently small neighbourhood of* x.

Proof. Since $x \notin X_0$, there exist open neighbourhoods U of x of arbitrarily small diameter, containing no other singular point and satisfying $m(\partial U) = m(\partial TU) = 0$. Observe that there exists a decomposition of U into special sets $U_1, \ldots, U_q$. Then, using (3.4)–(3.6),

$$\lim_{s \in J} \sum_{i=1}^{q} m_s(TU_i) = \lim_{s \in J} \sum_{i=1}^{q} \int_{U_i} \exp(s - f) \, dm_s$$

$$= \lim_{s \in J} \int_U \exp(s - f) \, dm_s = \int_U \exp(c - f) \, dm,$$

where J denotes the subsequence determined by the convergence to m. Since $m_s(TU) \leq \sum m_s(TU_i) \leq q m_s(TU)$ for every $s > c$, it follows that

$$m(TU) \leq \int_U \exp(c - f) \, dm \leq q m(TU),$$

and the lemma follows letting $U \downarrow x$.

Remark 3.6. It seems important to note the following consequence of the foregoing three lemmas. If m is a limit measure and if (3.6) is satisfied for $D = X$, then m satisfies (2.1) for all special sets not intersecting $\mathrm{Sing}(T)$, and

(3.8) $m(\{T(x)\}) \geq \exp(c - f(x))m(\{x\})$ $(x \in \mathrm{Sing}(T) \backslash \mathrm{Crit}(T))$,

$m(\{T(x)\}) \leq \exp(c - f(x))m(\{x\})$

$\leq q(x)m(\{T(x)\})$ $(x \in \mathrm{Crit}(T) \backslash X_0)$.

This implies that m is $\exp(c - f)$-conformal, if $m(T\,\mathrm{Sing}(T)) = 0$ and the measure on orbits of points in $\mathrm{Crit}(T) \cap X_0$ is zero.

Definition 3.7. A Borel probability measure m is called weakly $\exp(c - f)$-conformal, if

(3.9) $$m(TA) = \int_A \exp(c - f) \, dm$$

for all special sets A not intersecting $\mathrm{Sing}(T)$.

The preceding lemmas imply the following results.

Proposition 3.8. *There exists a weakly* $\exp(p - f)$-*conformal Borel probability measure* m *for some* $p \in \mathbf{R}$. *Moreover,* m *satisfies*

$m(\{T(x)\}) \geq \exp(p - f(x))m(\{x\})$ $(x \in \mathrm{Sing}(T) \backslash \mathrm{Crit}(T))$,

$m(\{T(x)\}) \leq \exp(p - f(x))m(\{x\})$

$\leq q(x)m(\{T(x)\})$ $(x \in \mathrm{Crit}(T) \backslash X_0)$.

Proof. Since T is continuous and X is compact, there exists some point $z \in \bigcap_{n=0}^{\infty} T^n X$. Setting $E_n = T^{-n}\{z\}$, and defining p by the transition parameter associated to $(E_n: n \geq 1)$ and f, the proposition follows immediately from Lemmas 3.3, 3.4 and 3.5.

Theorem 3.9. *Let T be an open map and let m be a limit measure (associated to the function f and the sequence $(E_n: n \geq 1)$) assigning measure zero to any critical periodic point. Moreover, assume that condition (3.6) is satisfied for $D = X$. Then there exists an $\exp(c - f)$-conformal measure $\tilde{m}$. Moreover, if $m(\mathrm{Crit}(T)) = 0$, then $m = \tilde{m}$.*

Remark. If T is a rational map on the Riemann sphere, there exists an $\exp(p - f)$-conformal measure on the Julia set of T for some $p \in \mathbf{R}$. This follows from the previous theorem and proposition, and the fact that no critical periodic point of a rational map can be contained in the Julia set.

Proof. By Lemmas 3.3, 3.4 and 3.5 the measure m has the properties stated in (3.8). Since T is open, m is $\exp(c - f)$-conformal, if $m(\mathrm{Crit}(T)) = 0$. (See Remark 3.6.)

Denote by Y the union of all forward and backward orbits of critical points. If $m(Y) < 1$, it is easy to see that, restricting m to $X \backslash Y$, gives an $\exp(c - f)$-conformal measure.

In the remaining case, it is not too difficult to define a conformal measure on the orbit of a critical point. For simplicity in notation, let us assume that x_0 is a critical point and that its orbit $O(x_0)$ contains no other critical point. Again, inducing if necessary, w.l.o.g. $m(O(x_0)) = 1$. By Lemma 3.5 there exists $\alpha > 0$ such that $m(\{T(x_0)\}) = \alpha m(\{x_0\})$. Define a new measure $\tilde{m}$ by setting

$$\tilde{m}(\{x\}) = \begin{cases} \beta m(\{x\}) & \text{if } T^n(x) = x_0 \text{ for some } n \geq 0, \\ \beta\alpha^{-1}\exp(c - f(x_0))m(\{x\}) & \text{if not,} \end{cases}$$

where

$$\beta^{-1} = m(\{x: T^n(x) = x_0 \text{ for some } n \geq 0\})$$
$$+ m(\{x \in O(x_0): T^n(x) \neq x_0 \text{ for all } n \geq 0\})\alpha^{-1}\exp(c - f(x_0)).$$

It is easy to see that $\tilde{m}$ is $\exp(c - f)$-conformal.

Example 3.10. The previous theorem is not true, if the map T is not open. There is a very simple example for this. For every integer $n \geq 1$ let $x_n = \exp(2\pi i/n)$ and $X = \{x_n: n \geq 1\}$. Define the transformation $T: X \to X$ setting

$$T(x_n) = \begin{cases} x_{n-1} & \text{for } n \geq 2, \\ x_n & \text{for } n = 1. \end{cases}$$

Clearly, X is compact and T is continuous, surjective and even invertible for all x_n $(n \geq 2)$. For f being the constant function 0 there is no $\exp c$-conformal measure for any $c \in \mathbf{R}$. Assume that m is e^c-conformal. Since $m(\{x_j\}) > 0$

for some $j \geq 1$, $m(\{x_1\}) = m(\{T^{j-1}(x_j)\}) = \exp((j-1)c)m(\{x_j\}) > 0$. Hence $m(\{x_1\}) = m(\{T(x_1)\}) = e^c m(\{x_1\})$ implies that $c = 0$. But then $m(\{x_n\}) = m(\{x_1\})$ for all $n \geq 1$, which is impossible.

Proposition 3.8 implies that conformal measures exist whenever a limit measure is not concentrated on orbits of singular points. In particular this is true, if the limit measure is atom-free. The following proposition ensures this property, even in the case when $X_0 \neq \varnothing$.

Proposition 3.11. *Let condition* (3.6) *be satisfied for* $D = X$. *In addition, suppose that* $c > \sup_{x \in X} f(x)$, $X_0 \cap \mathrm{Crit}(T) = \varnothing$, *and that no periodic point is critical. Then every limit measure* m *is an atom-free* $\exp(c - f)$-*conformal measure.*

Proof. Suppose that $m(\{x\}) > 0$ for some $x \in X$. Since $X_0 \cap \mathrm{Crit}(T) = \varnothing$, Lemmas 3.3–3.5 imply that $m(\{T^n(x)\}) > 0$ for all $n \geq 0$. Since no periodic point is critical and since $c > \sup_{x \in X} f(x)$, these lemmas also show that $\lim_{n \to \infty} m(\{T^n(x)\}) = \infty$, a contradiction. Thus any limit measure is atom-free, and the proposition follows from Lemma 3.3.

Remark. Conditions of the form $c > \sup_{x \in X} f(x)$ appear frequently in the literature. It is, for example, satisfied if

$$\limsup_{n \to \infty} \frac{1}{n} \log \sharp E_n > \sup_{x \in X} f(x) - \inf_{x \in X} f(x).$$

In the remaining part of this section the transition parameter c will be determined, when the sets E_n are separated (see [3]). Sufficient conditions will be given to identify the transition parameter with the pressure $P(T, f)$ of f. The following notation will be used.

Let $\varepsilon > 0$ and let $F_n(\varepsilon)$ $(n \geq 1)$ be a sequence of maximal (n, ε)-separated sets. Define

$$(3.10) \qquad P(\varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{x \in F_n(\varepsilon)} \exp(S_n f(x)).$$

Note that $P(\varepsilon)$ is the transition parameter associated to f and the sequence $E_n = F_n(\varepsilon)$ $(n \geq 1)$ if $T^{-1}F_n(\varepsilon) \subset F_{n+1}(\varepsilon)$ (this cannot always be achieved). Then the pressure of f is defined by $P = P(T, f) = \lim_{\varepsilon \to 0} P(\varepsilon)$. Recall that $P(T, f)$ does not depend on the particular choice of the maximal separated sets, that $P(T, f) < \infty$, if the topological entropy is finite, and that $P(T, f) = P(\delta)$ for any expansive constant $\delta > 0$, if T is expansive.

The existence of $\exp(P - f)$-conformal measures will be proved in the following theorems.

Theorem 3.12. *There exists a weakly* $\exp(P(T, f) - f)$-*conformal measure* m *for any expansive map* T. *If, in addition,* T *is an open map then* m *is* $\exp(P(T, f) - f)$-*conformal.*

Proof. Let $\delta > 0$ be an expansive constant. Then, by expansiveness, every two preimages of a point in X are at a distance at least δ. Thus there exists a

sequence $(E_n: n \geq 1)$ of maximal (n, δ)-separated sets E_n satisfying $T^{-1}E_n \subset E_{n+1}$ $(n \geq 1)$, and hence, by the foregoing remarks, it suffices to show the condition (3.6) for sets D as in Lemma 3.3.

Let $D \in \mathscr{F}$ satisfy $\overline{D} \cap \text{Sing}(T) = \varnothing$. Note that there exists a constant $\gamma > 0$ such that

$$(3.11) \qquad\qquad TB(x, \delta) \supset B(T(x), \gamma) \quad \forall x \in D.$$

By expansiveness,

$$\lim_{n \to \infty} \left(\sup_{x \in X} B_n(T(x), \delta) \right) = 0.$$

Hence there exists some $n_0 \geq 1$ such that for all $n \geq n_0$ and $x \in X$

$$(3.12) \qquad\qquad \text{diam} B_n(x, \delta) < \gamma.$$

In order to prove (3.6) it suffices to show

$$(3.13) \qquad\qquad D \cap (E_{n+1} \backslash T^{-1}E_n) = \varnothing \quad \forall n \geq n_0.$$

Assume that $x \in D \cap (E_{n+1} \backslash E_n)$ for some $n \geq n_0$. Since E_n is maximal separated, there exists a point $y \in E_n$ with $T(x) \in B_n(y, \delta)$, equivalently $y \in B_n(T(x), \delta)$. By (3.11) and (3.12) there is $z \in B(x, \delta)$ with $T(z) = y$. Consequently $x, z \in B_{n+1}(x, \delta) \cap E_{n+1}$, contradicting the separation property of E_{n+1}. This proves (3.13) and the first part of the theorem. The additional statement follows, e.g., from Remark 3.6.

Theorem 3.13. *Let G be of finite topological entropy and such that $X_0 \cap \text{Crit}(T) = \varnothing$. Moreover, assume that G has no critical periodic point and that for all sufficiently small $\varepsilon > 0$ there exists a sequence $(F_n(\varepsilon): n \geq 1)$ of maximal (n, ε)-separated sets satisfying (3.1) and*

$$(3.14) \qquad \max \left(0 ; \sup_{\varepsilon > 0} \limsup_{n \to \infty} \frac{1}{n} \log \sharp(F_{n+1}(\varepsilon) \backslash T^{-1}F_n(\varepsilon)) \right)$$
$$< P(T, f) - \sup_{x \in X} f(x).$$

Then there exists an atom-free $\exp(P(T, f) - f)$-conformal measure on $(X, \mathscr{F})$.

Proof. Let $\varepsilon > 0$ sufficiently small so that $P(\varepsilon) > d := \sup_{x \in X} f(x)$, and that the $F_n(\varepsilon)$ satisfy the equation (3.14), where the right-hand side is replaced by $P(\varepsilon) - d - \eta$ for some $\eta > 0$. The main step in the proof is to show the existence of an $\exp(P(\varepsilon) - f)$-conformal measure m_ε.

Since $P(\varepsilon)$ is the transition parameter associated to f and the sequence $(F_n(\varepsilon): n \geq 1)$, in view of Proposition 3.11 the existence of m_ε is guaranteed, once condition (3.6) is verified for $D = X$. By (3.14) and the definition of b_n, there is an $n_0 \geq 1$ such that

$$b_n \leq \exp \left(\frac{n}{2} \eta \right) \quad (\forall n \geq n_0)$$

and

$$\sharp(F_{n+1}(\varepsilon) \backslash T^{-1}F_n(\varepsilon)) < \exp[n(P(\varepsilon) - d - \eta)] \quad (\forall n \geq n_0).$$

It follows for $n \geq n_0$ and $s > P(\varepsilon)$ that

$$\sum_{n=n_0}^{\infty} \sum_{x \in F_{n+1}(\varepsilon) \setminus T^{-1}F_n(\varepsilon)} b_n \exp[S_n f(T(x)) - ns]$$

$$\leq \sum_{n=n_0}^{\infty} \exp\left[-\frac{n}{2}\eta + n(P(\varepsilon) - s) + n(S_n f(T(x)) - d)\right]$$

$$\leq \sum_{n=n_0}^{\infty} \exp\left[-\frac{n}{2}\eta\right] < \infty.$$

Since $\lim_{s \downarrow P(\varepsilon)} M_s = \infty$, (3.6) follows.

Finally, consider a weakly convergent subsequence m_n $(n \geq 1)$ of the family $(m_\varepsilon : \varepsilon > 0)$, when $\varepsilon \downarrow 0$. Denote this limit by m. Then for every special set A satisfying $m(\partial A) = m(\partial TA) = 0$,

$$(3.15) \qquad m(TA) = \int_A \exp(P - f)\, dm.$$

By Lemma 2.4, (3.15) holds for any special set A not intersecting $\mathrm{Sing}(T)$. Hence, it is left to show that m has no atoms.

Let $x \in \mathrm{Sing}(T) \setminus \mathrm{Crit}(T) = X_0$. Choose decreasing closed neighbourhoods C_k of x, which are special and satisfy $m(\partial C_k) = 0$. Then

$$m(TC_k) \geq \limsup_{n \to \infty} m_n(TC_k)$$

$$= \lim_{n \to \infty} \int_{C_k} \exp(P - f)\, dm_n$$

$$= \int_{C_k} \exp(P - f)\, dm,$$

and the statement of Lemma 3.4 holds in the present situation also. Similarly, repeating the argument in the proof of Lemma 3.5 it follows that the statement of that lemma also holds for a point $x \in \mathrm{Crit}(T) \setminus X_0 = \mathrm{Crit}(T)$. By this and (3.15), it follows as in the proof of Proposition 3.11 that m cannot have any atom, and this finishes the proof in view of Corollary 2.7.

Remark 3.14. If T is expansive, the existence of one sequence $(F_n(\varepsilon) : n \geq 1)$ for some fixed expansive constant $\varepsilon > 0$ suffices in Theorem 3.13. Also note that in this case the above proof simplifies considerably.

4. EXPANDING MAPS ON S^1

The aim of this section is to compare equilibrium states for continuous functions f and certain Hausdorff measures. For this one would like to express the equilibrium measure of balls $B_n(x, \delta)$ in terms of partial sums of some associated function (cf. (4.12)). Except for very special cases (subshifts of finite type, for example) this is not possible. However, it can be done for $\exp(P(T, f) - f)$-conformal measures and the equilibrium state for f turns out to be absolutely continuous with respect to this conformal measure.

Throughout this section, T denotes an expanding map of the circle S^1 of class $C^{1+\varepsilon}$ and $X \subset S^1$ denotes an invariant (i.e., $TX \subset X$), closed subset. To begin with, let us make some assumptions and conventions concerning the notation. Denote by T_X the restriction of T to X and by $\mathscr{R}$ the family of all connected components of $S^1 \backslash X$. Define (see [20])

$$(4.1) \qquad \tau(X) = \{(a, b) \in \mathscr{R}: X \cap T(a, b) \neq \varnothing\}.$$

It is always assumed that $\tau(X)$ is a fintie set. Changing, if necessary, the Riemannian metric on S^1 it is also assumed that

$$(4.2) \qquad \lambda := \inf_{x \in S^1} |T'(x)| > 1.$$

Finally, $\delta > 0$ denotes an expansive constant, which is smaller than the diameters of sets from $\tau(X)$ and so small that on every arc in S^1 of diameter less than δ all inverse branches of T are defined.

Lemma 4.1. *There exists a sequence* $(E_n: n \geq 1)$ *of maximal* (n, δ)-*separated sets* $E_n \subset X$ *satisfying condition* (3.1) *and* $\sharp(E_{n+1} \backslash T_X^{-1} E_n) \leq k$ *for some* k *not depending on* δ *or* n.

Proof. The sets E_n will be defined recursively. Let E_1 be any maximal $(1, \delta)$-separated set. If E_{n-1} is chosen, extend $T_X^{-1} E_{n-1}$ to a maximal (n, δ)-separated set E_n. This is possible, since δ is an expansive constant, hence $T_X^{-1} E_{n-1}$ is already an (n, δ)-separated set.

The lemma will be proved by showing that there exists an injective map σ from the set $E_{n+1} \backslash T_X^{-1} E_n$ into the set G of endpoints of arcs from $\tau(X)$.

Let $z \in E_{n+1} \backslash T_X^{-1} E_n$. Since E_n is a maximal separated set, $T(z) \in B_n(z', \delta)$ for some $z' \in E_n$. Denote by $g: B(T(z), \delta) \to S^1$ the branch of T^{-1} determined by the condition $g(T(z)) = z$. Then $g(z') \notin E_{n+1}$, since $|z - g(z')| < \delta$ (by the expanding property (4.2)), $z \in E_{n+1}$ and $z \neq g(z')$. Consequently, $g(z') \notin T^{-1} E_n$ and hence there exists an interval $(a, b) \in \tau(X)$ with $g(z') \in (a, b)$. Therefore the arc $B_{n+1}(g(z'), \delta)$ contains exactly one of the endpoints a or b. This endpoint will be denoted by $\sigma(z)$.

It is left to show that σ is injective. Assume that $\sigma(z_1) = \sigma(z_2) := c$, where $z_1, z_2 \in E_{n+1} \backslash T_X^{-1} E_n$. By the above construction of σ there exist $\overline{z}_1, \overline{z}_2 \in T^{-1} E_n$ such that $\overline{z}_i \in B_{n+1}(z_i, \delta)$ and c separates the arcs $(z_i, \overline{z}_i)$ $(i = 1, 2)$. Moreover, c does not separate the arc (z_1, z_2) and hence either

$$z_1 \in (z_2, \overline{z}_2) \subset B_{n+1}(z_2, \delta) \quad \text{or} \quad z_2 \in (z_1, \overline{z}_1) \subset B_{n+1}(z_1, \delta).$$

In either case, $z_1 \in B_{n+1}(z_2, \delta)$, and so $z_1 = z_2$, since E_{n+1} is maximal separated.

Theorem 4.2. *Let the continuous function* $f: X \to \mathbf{R}$ *satisfy*

$$P(T, f) > \sup_{x \in X} f(x).$$

Then there exists an atom-free $\exp(P(T, f) - f)$-*conformal measure* m_f *on* $(X, \mathscr{F})$.

Proof. The theorem follows from Theorem 3.13 (in fact Corollary 3.14) and Lemma 4.1. Since there are no critical points, it suffices to show that there are only finitely many points in X_0, where X_0 is defined after Definition 2.3.

The proof of Proposition 1 in [20] shows that the set G of endpoints of intervals from $\tau(X)$ can be decomposed into pairs (a_i, b_i), $(1 \le i \le k)$ such that $X \subset \bigcup_{1 \le i \le k}[a_i, b_i]$ and $(T_X)_{|[a_i, b_i] \cap X}$ has the Darboux property $\bmod X$, that is: every interval $I \subset [a_i, b_i] \cap X$ is mapped by T_X onto an interval of X. Therefore, every singular point z is either isolated or has an open neighbourhood of the form $[z, a)$ resp. $(a, z]$. In the first case, there are two intervals from the family $\mathscr{R}$ adjacent to z, and at least one of them contains preimages of points in X. So, at least one of the intervals belongs to $\tau(X)$, and hence the number of these singular points is finite, since $\tau(X)$ is. In the second case, there is an interval from the family $\mathscr{R}$ adjacent to z, which must contain a preimage of some point from X. Again, the number of these singular points is finite.

Theorem 4.3. *Let* $f\colon X \to \mathbf{R}$ *be a Lipschitz continuous function satisfying* $P(T, f) - \sup_{x \in X} f(x) > 0$, *and let* m_f *be an atom-free,* $\exp(P(T, f) - f)$-*conformal measure on* $(X, \mathscr{F})$. *Then there exists an ergodic,* T_X-*invariant probability measure* μ_f *on* $(X, \mathscr{F})$, *which is absolutely continuous with respect to* m_f. *Moreover,* μ_f *is an equilibrium state for* f, *and there exists an integer* $k \ge 1$, *such that for any function* g *of bounded variation the process* $(g \circ T^{kn}\colon n \ge 1)$ *is a functional of an absolutely regular process with exponentially fast decreasing mixing coefficients* (*in the sense of* [8], *see also* [6] *and* [7]).

Proof. By Proposition 2.2, m_f is a fixed point of the Perron-Frobenius operator. Thus the results in [6] (see also [7]) can be applied directly to obtain the theorem.

Remark 4.4. (1) After this paper was written, G. Keller and the authors made the statement of Theorem 4.3 more precise, using Theorem 3.13. Theorem 1 in [4] shows that every equilibrium state is absolutely continuous with respect to some $\exp(P(T, f) - f)$-conformal measure. Hence the existence of μ_f in the previous theorem follows already from expansiveness, and, moreover, every equilibrium state has the property given in Theorem 4.3 for some conformal measure. Therefore, the following results apply to all equilibrium states for f.

(2) The preceding theorem generalizes and strengthens the result in [7] in the sense that it provides a larger class of functions, for which the existence of fixed points of the Perron-Frobenius operator can be achieved. It is clear from the present discussion, how the above arguments can be applied in other situations of expansive maps, e.g. subshifts.

(3) The last statement in the preceding theorem has an important conse-

quence. It implies that the process $(g \circ T^{kn} : n \geq 1)$ satisfies the central limit theorem and the a.s. invariance principle if

$$(4.3) \qquad \sigma^2 = \sigma^2(g) = \int_X (g - \mu_f(g))^2 \, d\mu_f$$

$$+ 2 \sum_{n=1}^{\infty} (g - \mu_f(g))(g \circ T_X^{kn} - \mu_f(g)) \, d\mu_f$$

is strictly positive. The latter theorem means that one can redefine the process $(g \circ T_X^{kn} : n \geq 1)$ on some probability space on which there is defined a standard Brownian motion $(B(t) : t \geq 0)$ such that for some $\beta > 0$

$$\sum_{0 \leq j \leq t} [g \circ T_X^{kj} - \mu_f(g)] - B(\sigma^2 t) \ll t^{1/2 - \beta} \quad \mu_f \text{ a.e.}$$

(4) Let $h : [1, \infty) \to \mathbf{R}$ be a positive nondecreasing function. h is said to belong to the lower class if

$$(4.4) \qquad \int_1^{\infty} \frac{h(t)}{t} \exp\left(-\frac{1}{2} h(t)^2\right) < \infty$$

and to the upper class if

$$(4.5) \qquad \int_1^{\infty} \frac{h(t)}{t} \exp\left(-\frac{1}{2} h(t)^2\right) = \infty.$$

From the third remark and well known results on Brownian motion it follows that (for a function g as in (3))

$$\mu_f\left(\left\{x \in X : \sum_{j=0}^{n-1} [g(T_X^{kj}(x)) - \mu_f(g)] > \sigma h(n)\sqrt{n}\right.\right.$$

$$(4.6) \qquad\qquad\qquad \left.\left. \text{for infinitely many } n \in \mathbf{N}\right\}\right)$$

$$= \begin{cases} 0 & \text{if } h \text{ belongs to the lower class,} \\ 1 & \text{if } h \text{ belongs to the upper class,} \end{cases}$$

where σ^2 is defined by (4.3) and strictly positive.

Let K be the topological support of μ_f. Since μ_f has no atoms, no isolated point belongs to K. Hence $B_X(x, \tau) := B(x, r) \cap X$ has a positive diameter for $x \in K$, and so, for an open interval U intersecting K, consider integers $n(U)$ satisfying the following two conditions (4.7) and (4.8):

$$(4.7) \qquad T_X^{kn(U)}(U \cap X) \supset B_X(x, \delta)$$

for some $x \in K$ and

$$(4.8) \qquad T^{kn(U)} : U \to T^{kn(U)} U$$

is injective. Note that integers with these properties always exist.

Finally, let $d(U)$ denote the diameter of the set $U \cap X$, where U is an open interval.

Lemma 4.5. *There exists a constant $\alpha > 0$ such that for every interval $U \subset S^1$*

$$(4.9) \qquad\qquad m_f(U) \leq (\operatorname{diam} U)^\alpha.$$

Proof. Since m_f is positive on open sets of K, no matter how small δ is chosen,

$$(4.10) \qquad\qquad A := A(\delta) = \inf_{x \in K} m_f(B(x, \delta)) > 0.$$

By the distortion theorem (see, for example, Lemma 1 in [17]), there exists a constant $C_1 > 0$ such that for $x \in K$ and U containing x

$$(4.11) \qquad C_1^{-1} |(T^{kn(U)})'(x)|^{-1} \leq d(U) \leq C_1 |(T^{kn(U)})'(x)|^{-1}.$$

Since $T^{kn(U)}$ is injective on U and since f is Lipschitz continuous, there exists a constant C_2 such that

$$|S_{kn(U)} f(x) - S_{kn(U)} f(y)| \leq C_2$$

for all $x \in K$, all U containing x, and $y \in U$. Therefore, by conformality and (4.10), there exists a constant C_3 such that for $x \in K$ and U containing x

$$(4.12) \qquad C_3^{-1} \exp(S_{kn(U)}(f - P(T, f))(x))$$
$$\leq m_f(U) \leq C_3 \exp(S_{kn(U)}(f - P(T, f))(x)).$$

Since $P(T, f) > \sup_{x \in X} f(x)$, there exists an $\alpha > 0$ satisfying

$$\inf_{x \in X}(P(T, f) - f(x)) > \alpha \sup_{x \in X} \log |T'(x)|.$$

From this, (4.11), and (4.12), one easily deduces the lemma. Note that the full strength of the estimates (4.11) and (4.12) is not needed here, but they will be used later.

Lemma 4.6. *For all $\varepsilon > 0$ and $n \in \mathbf{N}$, the expansive constant δ can be chosen so small that*

$$\mu_f(\{x \in K \colon \exists U \text{ with } n(U) = n \text{ and } x \in U\}) \geq 1 - \varepsilon.$$

Proof. Denote by G the set of endpoints of intervals from $\tau(X)$ and choose $1 < \eta^{-1} < \lambda^k$. For $n \geq 1$ define

$$G_n = \{x \in K \colon \operatorname{dist}(T^{kj}(x), G) \geq \eta^{n-j} \ (\forall 0 \leq j \leq n)\}.$$

By Lemma 4.5,

$$m_f(G_n) \geq 1 - \sharp G \sum_{j=0}^{n} (\delta \eta^{n-j})^\alpha \geq 1 - \frac{\delta^\alpha}{1 - \eta^\alpha} \sharp G.$$

By this estimate and since $\mu_f \ll m_f$, δ may be chosen so small that $\mu_f(G_n) \geq 1 - \varepsilon$. For $x \in G_n$ let $U = gB(T^{kn}(x), \delta)$, where g denotes the branch of

T^{-kn} determined by $g(T^{kn}(x)) = x$. Then n may serve as some $n(U)$ as in (4.7) and (4.8), and $x \in U$. It follows that

$$G_n \subset \{x \in K: \exists U \text{ with } n(U) = n \text{ and } x \in U\},$$

and the lemma is proved.

Lemma 4.7. *For every constant h the expansive constant δ can be chosen so small, such that for every function g of bounded variation*

$$\mu_f\left(\left\{x \in X: \sum_{j=0}^{n-1}[g(T_X^{kj}(x)) - \mu_f(g)] > \sigma h\sqrt{n} \text{ for infinitely many}\right.\right.$$

$$\left.\left. n \in \mathbf{N} \text{ satisfying } n = n(U) \text{ for some } U \text{ with } x \in U\right\}\right) = 1,$$

provided σ^2 defined in (4.3) is larger than zero.

Proof. Let $0 < 2\varepsilon < (2\pi)^{-1/2} \int_h^\infty \exp(-u^2/2)\,du$ and choose δ as in Lemma 4.6. Define

$$Z_n' = \{x \in K: \exists U \text{ with } n(U) = n \text{ and } x \in U\}.$$

By Lemma 4.6 and the central limit theorem (see Remark 4.4.3)), the set

$$Z_n = Z_n' \cap \left\{x \in X: \frac{1}{\sqrt{\sigma^2 n}}\sum_{j=0}^{n-1}(g(T^{kj}(x)) - \mu_f(g)) > h\right\}$$

has measure $> \varepsilon$ for all sufficiently large n. The set $Z = \bigcap_{n \in \mathbf{N}}\bigcup_{l>n} Z_l$ is invariant under T and has positive measure, whence by ergodicity, $\mu_f(Z) = 1$. Now, $x \in Z$ belongs to infinitely many Z_n, consequently for such n

$$\sum_{j=0}^{n-1}(g(T^{kj}(x)) - \mu_f(g)) \geq h\sigma\sqrt{n}$$

and $n(U) = n$ for some U containing x. This proves the lemma.

Some more notations are used in the sequel. Denote the Lyapunov characteristic exponent by $\chi = \chi_f = \int_X \log|(T^k)'|\,d\mu_f$ and the Hausdorff dimensin of μ_f by $\kappa = HD(\mu_f) = \inf\{HD(Y): Y \subset X, \mu_f(Y) = 1\}$. Note that $\kappa = h_{\mu_f}(T_X^k)/\chi$ (see [24], [19] or [20]).

Lemma 4.8. *Let the assumptions of Theorem 4.3 be satisfied. Suppose that σ^2, defined in (4.3) for the function $g = \sum_{0 \leq j < k}(f - P(T, f) + \kappa\log|T'|)\circ T^j$, is strictly positive. If h belongs to the lower class (i.e. satisfies (4.4)), then for μ_f a.e. $x \in X$,*

$$(4.13) \qquad \limsup_{U \downarrow x} \frac{m_f(U)}{d(U)^\kappa \exp(\sigma\chi^{-1/2}h(-\log d(U))\sqrt{-\log d(U)})} = 0.$$

If h is constant, then

$$(4.14) \qquad \limsup_{U \downarrow x} \frac{m_f(U)}{d(U)^\kappa \exp(\sigma\chi^{-1/2}h(-\log d(U))\sqrt{-\log d(U)})} = \infty.$$

Moreover, if for μ_f a.e. $x \in X$, the set of all integers n, for which there exists an interval U containing x and satisfying $n(U) = n$ for some $n(U)$ defined by (4.7) and (4.8), has bounded gaps, then (4.14) continues to hold for all functions h belonging to the upper class.

Remark. The sequence $n(U)$ has μ_f a.e. bounded gaps, if, for example, the system (X, T_X) admits a Markov partition.

Proof. Use formulas (4.11) and (4.12) to show that for $x \in K$ and U containing x,

$$(C_1 C_3)^{-1} \exp \left(S_{kn(U)}(f - P(T, f) + \kappa \log |T'|)(x) \right.$$
$$\left. - \frac{\sigma}{\sqrt{\chi}} h(-\log d(U)) \sqrt{-\log d(U)} \right)$$
$$\leq \frac{m_f(U)}{d(U)^\kappa \exp(\sigma \chi^{-1/2} h(-\log d(U)) \sqrt{-\log d(U)})}$$
$$\leq C_1 C_3 \exp \left(S_{kn(U)}(f - P(T, f) + \kappa \log |T'|)(x) \right.$$
$$\left. - \frac{\sigma}{\sqrt{\chi}} h(-\log d(U)) \sqrt{-\log d(U)} \right).$$

By the ergodic theorem, the law of the iterated logarithm and (4.11),

$$|\log d(U)) + n(U)\chi| \leq 2 \sqrt{\sigma^2 (\log |(T^k)'|) n(U) \log \log n(U)}.$$

If h belongs to the lower class, Remark 4.4(4) shows that

$$\limsup_{U \downarrow x} \exp \left(S_{kn(U)}(f - P(T, f) + \kappa \log |T'|)(x) - \sigma h(\chi n(U)) \sqrt{n(U)} \right) \leq 1$$

for μ_f a.e. x. Replacing h by $h_0(t) = h(t/\chi) + a n^{-1/4}$ for some $a < 0$ proves the lemma in case of lower class functions.

Finally, if h is constant or belongs to the upper class, apply Lemma 4.7 or Remark 4.4(4) to obtain the other statements of the lemma in the same way.

The results about the relation between Hausdorff measures and equilibrium states are contained in the following two theorems. For a function $h: [1, \infty) \rightarrow \mathbf{R}_+$ define for sufficiently small $t > 0$

$$\tilde{h}(t) = t^\kappa \exp \left(\frac{\sigma}{\sqrt{\chi}} h(-\log t) \sqrt{-\log t} \right).$$

Theorem 4.9. *Let the assumptions of Theorem 4.3 be satisfied, and suppose $\sigma^2 > 0$, where σ^2 is defined by (4.3) for the function*

$$g = \sum_{0 \leq j < k} (f - P(T, f) + \kappa \log |T'|) \circ T^j.$$

Then

(4.15) $\qquad\qquad \mu_f \ll H_{\tilde{h}}$ *if* h *belongs to the lower class*

(*cf.* [15] *for a definition of* $H_{\tilde{h}}$) *and*

(4.16) $\qquad\qquad \mu_f \perp H_{\tilde{h}}$ *if* h *is constant.*

Moreover, if for μ_f *a.e.* x, *the sequence*

$$\{n \in \mathbf{N}: \exists U \text{ such that } x \in U, \ n(U) = n\}$$

has bounded gaps, then (4.16) *holds for all functions* h *belonging to the upper class.*

Remark. Taking $h \equiv 0$ it follows from (4.16) that the measure μ_f is orthogonal to the κ-dimensional Hausdorff measure on K.

Proof. (1) *Proof of* (4.15): Let h satisfy (4.4). By Lemma 4.8 (4.13), for a set E with $\mu_f(E) > 0$, there exist a set $E' \subset E$ satisfying $\mu_f(E') > \mu_f(E)/2$ and a $\theta > 0$ such that for every $x \in E'$ and all intervals U of diameter less than θ containing x, $m_f(U) < \tilde{h}(d(U))$. Therefore $H_{\tilde{h}}(E) \geq H_{\tilde{h}}(E') \geq m_f(E')$. Since $\mu_f \ll m_f$, $m_f(E') > 0$, whence $H_{\tilde{h}}(E) > 0$ and μ_f is absolutely continuous with respect to $H_{\tilde{h}}$.

(2) *Proof of* (4.16): The additional statement concerning the formula (4.16) is shown exactly in the same way as it is proved for constant functions. It is only necessary to apply the corresponding statements from Lemma 4.8. Hence it suffices to show (4.16) for constant functions h.

For $n \geq 1$ and $\varepsilon > 0$, by Lemma 4.8 (4.14), there exists a set $E_n \in \mathscr{F} \cap K$ such that $\mu_f(E_n) > 1 - \varepsilon 2^{-n}$ and such that for $x \in E_n$ and some suitable interval U_x of diameter $< 1/n$ containing x, $m_f(U_x) > 2n\tilde{h}(d(U_x))$. From the cover $(U_x: x \in E_n)$ of E_n choose a subcover $(U_l: l \geq 1)$ of multiplicity ≤ 2. Since $d(U) < 1/n$,

$$H_{\tilde{h}}\left(E_n, \frac{1}{n}\right) \leq \frac{1}{2n} \sum_{l=1}^{\infty} m_f(U_l) \leq \frac{1}{n} m_f(K) \leq \frac{1}{n}.$$

Setting $F_\varepsilon = \bigcap_{n \geq 1} E_n$ it follows that $H_{\tilde{h}}(F_\varepsilon) = 0$ and $\mu_f(F_\varepsilon) \geq 1 - \varepsilon$. Finally the set $F = \bigcup_{l \geq 1} F_{1/l}$ satisfies $H_{\tilde{h}}(F) = 0$ and $\mu_f(F) = 1$, proving (4.16).

Theorem 4.10. *Let the assumptions of Theorem 4.3 be satisfied, and suppose* $\sigma^2 = 0$, *where* σ^2 *is defined by* (4.3) *for the function*

$$g = \sum_{0 \leq j < k} (f - P(T, f) + \kappa \log |T'|) \circ T^j.$$

Then $HD(\mu_f) = HD(K) = \kappa$ *and* μ_f *is equivalent to the* κ-*dimensional Hausdorff measure on* $K = \mathrm{supp}(\mu_f)$.

Proof. Since $\sigma^2 = 0$ it is well known that g is homologous to a constant function, and since $\int_X g \, d\mu_f = 0$ this constant must be zero. In particular,

$P(T_X^k, -\kappa \log |(T^k)'|) = 0$. From this formula it follows that $HD(\mu_f) = \kappa$ (see [10] and also [2], [14] or [19]).

The functions $\sum_{0 \leq j < k} (f - P(T, f)) \circ T^j$ and $\sum_{0 \leq j < k} (\kappa \log |T'|) \circ T^j$ are homologous by a Hölder continuous function. Consequently, rewriting (4.12), there exists a constant $C > 0$ such that for $x \in K$ and $r > 0$

$$C^{-1} |(T^{kn(x,r)})'(x)|^{-\kappa} \leq m_f(B_X(x,r)) \leq C |(T^{kn(x,r)})'(x)|^{-\kappa},$$

where $n(x, r) = n(B(x, r))$. It is well known (and easy to show) that this formula, together with (4.11), imply that m_f and H_{f^κ} are equivalent on K (even with bounded Radon-Nikodym derivatives). But μ_f and m_f are equivalent on K, whence the theorem.

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