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ON THE EXISTENCE OF COURNOT EQUILIBRIUM

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ABSTRACT

This paper examines the existence of n-firm Cournot
equilibrium in a market for a single homogeneous commodity. It proves
that if each firm's marginal revenue declines as the aggregate output
of other firms increases (which is implied by concave inverse demand)
then a Cournot equilibrium exists, without assuming that firms have
nondecreasing marginal cost or identical technologies. Also, if the
marginal revenue condition fails at a "potential optimal point," there
is a set of firms such that no Cournot equilibrium exists. The paper
also contains an example of nonexistence with two nonidentical firms,
each with constant returns to scale production.

ON THE EXISTENCE OF COURNOT EQUILIBRIUM*

by

William Novshek

1. Introduction

Cournot equilibrium is commonly used as a solution concept in oligopoly models, but the conditions under which a Cournot equilibrium can be expected to exist are not well understood. The nature of each firm's technology, whether all firms have identical technologies, and restrictions on the market inverse demand vary from model to model, and are all important for the existence of Cournot equilibrium. This paper examines the question of existence of (pure strategy) Cournot equilibrium in a single market for a homogeneous good. In this context there are two known types of existence theorems. The first type allows general (downward sloping) inverse demand and shows the existence of Cournot equilibrium when there are n identical firms with convex technologies (nondecreasing marginal cost and no avoidable fixed costs). See McManus [1962, 1964] and Roberts and Sonnenschein [1976]. The second type shows the existence of Cournot equilibrium in markets with n not necessarily identical firms when each firm's profit function is concave. Sometimes the concavity of profit functions is an explicit assumption (see Frank and Quandt [1963]), other times assumptions on the inverse demand and cost functions which imply concave profit functions are used (for example, inverse demand is assumed to be concave over the range where it is positive and all firms have convex cost functions in Szidarovszky and Yakowitz [1977]).[1]

The main result of the paper is a new existence theorem for n-firm Cournot equilibrium. With only minimal assumptions on cost functions, and without requiring identical firms or convex technologies we show that a commonly imposed assumption on inverse demand is sufficient to guarantee the existence of an n-firm Cournot equilibrium. The condition is equivalent to the condition that (throughout the relevant region) each firm's marginal revenue is declining in the aggregate output of other firms, and is commonly imposed in the industrial organization literature and in the literature concerning the comparative static properties of Cournot equilibrium (see for example Ruffin [1971] and Okuguchi [1973]). The new existence theorem shows that this literature can drop essentially all of the common assumptions imposed on the cost functions of the firms (for example, convexity of cost functions) and still obtain existence of equilibrium with only the marginal revenue condition. Assumptions on cost functions need only be introduced if needed in the subsequent comparative static analysis.

The marginal revenue condition is implied by concave inverse demand, another common assumption in this literature. Thus the new existence theorem shows it is possible to drop the explicit or implicit assumptions on the cost functions needed in the second, previous type of existence theorem (using concave profit functions).

We also provide two examples of nonexistence of Cournot

equilibrium to help delineate the conditions under which equilibrium can be expected to exist. The first example is of a well-known type, and it shows that general demand and identical firms with nonconvex technologies can lead to nonexistence of equilibrium. The second example does not seem to be well known. It shows that with general demand and convex technologies, if firms are not identical then equilibrium may not exist.

In a remark we also examine the extent to which the assumptions of the new existence theorem can be weakened. The only really substantial assumption, the marginal revenue condition, is not a necessary condition for existence of equilibrium since the condition may fail at an "irrelevant point." However, we show that if, for some inverse demand function, the condition fails at some point which is a "feasible optimal choice" then there exists an integer n, and n firms with cost functions satisfying the assumptions of the theorem, such that the market with these n firms and the given inverse demand function has no pure strategy Cournot equilibrium.

In Section 2 we introduce the basic definitions of the model and state versions of the previous existence theorems. In Section 3 we present our two examples of nonexistence. Section 4 contains the new existence theorem. Section 5 contains remarks on weakening the assumptions of the theorem, on extension of the existence theorem to endogenous n (i.e., the case of an unlimited number of potential firms), and on the use of the theorem to prove a very general version of the limit results in Novshek [1980].

2. Previous Existence Results

Consider the market for a single homogeneous good with inverse demand function $P(\cdot)$ and n firms. Firm $f \in \{1,2,\ldots,n\}$ has cost function $C_f(.)$.

Definition: $(y_1,y_2,\cdots,y_n) \in \mathbb{R}_+^n$ is a Cournot equilibrium if

$$P(\sum_{j=1}^{n} y_j)y_f - C_f(y_f) \geq P(\sum_{j=1}^{n} y_j - y_f + y)y - C_f(y)$$

for all $y \geq 0$, for all $f \in \{1,2,\ldots,n\}$.

Theorem 1: (McManus [1964]). Given a market for a single homogeneous good with inverse demand $P(\cdot)$ and n identical firms, each with cost function $C(\cdot)$, if

(1) $P:(0,\infty) \to \mathbb{R}_+$ is a nonincreasing, upper-semi-continuous function (i.e., for all $Y > 0$, for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $|X - Y| < \delta$ implies $P(X) < P(Y) + \varepsilon$), and total revenue, $YP(Y)$, is bounded, and

(2) $C:\mathbb{R}_+ \to \mathbb{R}_+$ is continuous and monotonically increasing, and the increase in cost for any given increase in output does not decrease with output (i.e., for any $y > y' \geq 0$ and $q > 0$, $C(y + q) - C(y) \geq C(y' + q) - C(y'))$,[2]

then an n-firm Cournot equilibrium exists.

McManus showed that all jumps in the reaction correspondence, $r(Y) := \{y \in \mathbb{R}_+ | P(Y + y)y - C(y) \geq P(Y + x)x - C(x)$ for all $x \in \mathbb{R}_+\}$, must be jumps up, so the line $y = Y/(n - 1)$ must intersect the graph of the reaction correspondence, yielding a symmetric n-firm

equilibrium.

As noted in the introduction, the second type of existence theorem takes various forms. For comparison we state it as follows:

Theorem 2: (Szidarovszky and Yakowitz [1977]). Given a market for a single homogeneous good with inverse demand P(·) and n firms with cost functions $C_1, C_2, \cdots, C_n$, if

(1) $P: \mathbb{R}_+ \to \mathbb{R}_+$ is nonincreasing and is twice continuously differentiable and concave on the interval where it has positive value (so $P(Y) > 0$ implies $P'(Y) \leq 0$ and $P''(Y) \leq 0$), and

(2) for all $f \in \{1, 2, \ldots, n\}$, $C_f: \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, twice continuously differentiable, and convex (so $C_f'(y) \geq 0$ and $C_f''(y) \geq 0$ for all y)

then there exists an n-firm Cournot equilibrium.

The proof of Theorem 2 follows from the observation that, in the relevant region, each firm's profit function is concave in its own output, so a standard existence theorem for concave games can be applied.

3. Examples of Nonexistence

Our first example of nonexistence has an inverse demand function which is not everywhere concave, and identical firms with nonconvex cost functions. The possibility of nonexistence under these conditions is well known.

Example 1: Inverse demand is

$$P(Y) = \begin{cases} 100 - 4Y & Y \in [0, .25] \\ 527 - 1712Y & Y \in (.25, .3] \\ 14 - 2Y & Y \in (.3, 7] \\ 0 & Y \in (7, \infty) \end{cases}$$

and all firms have identical cost functions with decreasing average cost:

$$C(Y) = \begin{cases} 0 & y = 0 \\ 10 + y & y > 0 \end{cases}$$

The reaction correspondence is then

$$r(Y) = \begin{cases} \{\frac{1}{4} - Y\} & Y \in [0, \frac{\sqrt{33516} - 183}{2}) \\ \{\frac{1}{4} - Y, \frac{13}{4} - \frac{Y}{2}\} & Y = \frac{\sqrt{33516} - 183}{2} \\ \{\frac{13}{4} - \frac{Y}{2}\} & Y \in (\frac{\sqrt{33516} - 183}{2}, \frac{13}{2} - 2\sqrt{5}) \\ \{\sqrt{5}, 0\} & Y = \frac{13}{2} - 2\sqrt{5} \\ \{0\} & Y \in (\frac{13}{2} - 2\sqrt{5}, \infty) \end{cases}.$$

For $n \geq 2$ there is no n-firm equilibrium in this example: each active firm produces more than all other firms combined (i.e., for all Y, all nonzero elements of r(Y) exceed Y) so at most one firm can be active. But if only one firm is active it produces the monopoly output, which is not viable with $n \geq 2$, since $0 \notin r(\frac{1}{4})$. This example can be easily modified to show nonexistence with U-shaped average cost.

We now turn to cases in which firms have different

technologies. Again inverse demand is not everywhere concave but the

two firms have convex cost functions (constant marginal cost with no

fixed cost) which are different. The possibility of nonexistence

under these conditions seems not to be well known.

Example 2:   Inverse demand is

$$P(Y) = \begin{cases} 2 - Y & Y \in [0 , .99] \\ \frac{8219}{8119} - \frac{19}{8119}Y & Y \in (.99 , \frac{100}{19}] \\ \frac{10019}{19} - 100Y & Y \in (\frac{100}{19} , \frac{100.19}{19}] \\ 0 & Y \in (\frac{100.19}{19} , \infty) \end{cases} .$$

There are two firms with constant marginal cost and no fixed costs,

but firm 1 has marginal cost 881/800 while firm 2 has marginal cost

381/400.

The first firm's reaction correspondence is

$$r_1(Y) = \begin{cases} \{\frac{719}{1600} - \frac{Y}{2}\} & Y \in [0 , \frac{719}{800}] \\ \{0\} & Y \in (\frac{719}{800} , \infty) \end{cases}$$

while the second firm's reaction correspondence is

$$r(Y) = \begin{cases} \{\frac{419}{800} - \frac{Y}{2}\} & Y \in [0 , \frac{21}{400}) \\ \{\frac{398}{300} , \frac{100}{19} - \frac{21}{400}\} & Y = \frac{21}{400} \\ \{\frac{100}{19} - Y\} & Y \in (\frac{21}{400} , \frac{3999639}{760000}] \\ \{\frac{8000722}{3040000} - \frac{Y}{2}\} & Y \in (\frac{3999639}{760000} , \frac{8000722}{1520000}] \\ \{0\} & Y \in (\frac{8000722}{1520000} , \infty) \end{cases} .$$

From Figure 1 we see there is no equilibrium. This example can be

easily modified to strictly increasing average cost, or strictly

decreasing average cost, or U-shaped average cost.

4.   Existence Theorem

The new existence theorem improves Theorem 2 by removing the

requirement that firms have convex cost functions and weakening the

assumption that inverse demand be concave. The remaining assumption

on cost functions is quite minimal, and is needed to guarantee that

each firm's reaction correspondence is nonempty valued. The only

restrictive assumption is the (commonly used) requirement that a

firm's marginal revenue be everywhere (in the relevant region) a

declining function of the aggregate output of others; i.e., for all

nonnegative y and Y with $P(y + Y) > 0$, for revenue $yP(Y + y)$,

$\frac{\partial^2 \{yP(Y + y)\}}{\partial Y \partial y} = P'(Y + y) + yP''(Y + y) \leq 0$. This assumption is used

to establish the key to the proof, the fact that each firm's reaction

correspondence r_f, is nonincreasing in the sense that $Y' > Y$ implies

$\max r_f(Y') \leq \min r_f(Y)$. This implies that any jumps in r_f are jumps

down. McManus proved Theorem 1 by showing that with convex costs, all jumps in the reaction correspondence were jumps up, so with identical firms a symmetric equilibrium exists. Example 2 showed that the assumption of identical firms was necessary for his result. In the proof of Theorem 3 we use the fact that each firm's reaction correspondence is nonincreasing to show that even when firms are not identical, an n-firm Cournot equilibrium exists.

Theorem 3: Given a market for a single homogeneous good with inverse demand $P(\cdot)$ and n firms with cost functions $C_1, C_2, \ldots, C_n$, if

(1) $P: \mathbb{R}_+ \to \mathbb{R}_+$ is continuous,

(2) there exists $Z' < \infty$ such that $P(Z') = 0$ and P is twice continuously differentiable and strictly decreasing on $[0, Z')$,

(3) for all $Z \in [0, Z')$, $P'(Z) + ZP''(Z) \leq 0$, and

(4) for all $f \in \{1, 2, \ldots, n\}$, $C_f: \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing, lower-semi-continuous function (i.e., for all $y \geq 0$, for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - y| < \delta$ implies $C_f(x) > C_f(y) - \varepsilon$),

then there exists an n-firm Cournot equilibrium.

Note that given assumption (2), assumption (3) is equivalent to the assumption that for all nonnegative Y and y with $Y + y < Z'$, $P'(Y + y) + yP''(Y + y) \leq 0$, so each firm's marginal revenue is decreasing in the aggregate output of other firms.

Proof: Let the assumptions of Theorem 3 hold. First observe the properties of each firm's reaction correspondence. For any

$Y < Z'$, for any firm, all optimal responses to Y are less than $Z' - Y$ since the response $y = (Z' - Y)/2$ generates strictly positive revenue (compared to zero revenue for outputs greater than or equal to $Z' - Y$ because $P(Z') = 0$) at lower cost (because all cost functions are nondecreasing). For $Y < Z'$, the reaction correspondence is thus

$r_f(Y) := \{y \in [0, Z'] \mid P(Y + y)y - C_f(y) \geq P(Y + x)x - C_f(x)$
for all $x \in [0, Z']\}$.

For convenience we define $r_f(Z') := \{0\}$. Note zero is always an optimal response to Z', but other responses are also optimal if $C_f(y) = C_f(0)$ for some $y > 0$. These other responses are clearly not interesting. By our assumptions, each r_f is a nonempty valued, upper-hemi-continuous correspondence on $[0, Z']$. If $Y < Y' < Z'$ and $y < y' = \max r_f(Y)$ then

$$P(Y + y')y' - P(Y + y)y = \int_y^{y'} [P(Y + x) + xP'(Y + x)]dx$$
$$\geq \int_y^{y'} [P(Y' + x) + xP'(Y' + x)]dx = P(Y' + y')y' - P(Y' + y)y$$
$$\geq C_f(y') - C_f(y)$$

where the first weak inequality follows from assumptions (2) and (3) and our initial note, and the second weak inequality follows from $y' \in r_f(Y')$. Thus $y \notin r_f(Y)$, and r_f is nonincreasing in the sense that $Y' > Y$ implies $\max r_f(Y') \leq \min r_f(Y)$. Each r_f has at most countably many discontinuities, and all jumps are jumps down.

Next, for each f, use r_f to define the convex valued correspondence s_f by $s_f(Y) :=$ convex hull of $r_f(Y)$ for $Y \in [0, Z']$. Then s_f is also nonincreasing in the sense above, and for each

$y \in [0, \max r_f(0)]$, the set of Y with $y \in s_f(Y)$ is a nonempty, closed interval.

Now define the upper-hemi-continuous, possibly empty valued correspondence b_f by

$$b_f(Q) := \{q | \text{there exists } Y \in [0,Z'] \text{ such that } q \in r_f(Y) \text{ and } Y + q = Q\}$$

(see Novshek [1984] for a detailed discussion of this correspondence). Similarly define the correspondence h_f by

$$h_f(Q) := \{q | \text{there exists } Y \in [0,Z'] \text{ such that } q \in s_f(Y) \text{ and } Y + q = Q\}.$$

Consider the properties of b_f and h_f. First, note that $b_f(Z') = h_f(Z') = \{0\}$ for all f. The graph of h_f can be continuously parameterized as $(Q_f^*(t), q_f^*(t))$ for $t \in [0,1]$ such that $(Q_f^*(0), q_f^*(0)) = (Z',0)$, $q_f^*(t) \in h_f(Q_f^*(t))$ for all t, $q_f^*(t)$ is nondecreasing in t, and $(Q_f^*(1), q_f^*(1)) = (\max r_f(0), \max r_f(0))$. Thus larger q_f values are associated with larger t values. Vertical jumps in r_f correspond to "jumps" along a 45 degree line for b_f (if $y, y' \in r_f(Y)$ and $y \neq y'$ then $y \in b_f(Y + y)$ but $y' \in b_f(Y + y')$). The only difference between b_f and h_f is that these "45 degree jumps" have been filled in with a line segment in the graph of h_f. For each q, the set of Q with $q \in b_f(Q)$ is a closed interval, possibly empty. See Figure 2. Note the points Q_1 and Q_2 at which branches of b_f disappear as Q increases. By the definition of b_f, $(q_1,q_2,\cdots,q_n)$ is a Cournot equilibrium if and only if $q_f \in b_f(\sum_{j=1}^{n} q_j)$ for all f.

The points at which branches of the correspondence b_f disappear as Q increases (such as (Q_1,q_1) in Figure 2) play an important role in the proof. We first prove the result for the case in which the union over all f of these points is a finite set. Then we explain the modifications needed for the case in which this set is infinite.

Let T_f be the set of points at which a branch of b_f disappears as Q increases, let $T = \bigcup_{f=1}^{n} T_f$, and let T' be the set of Q values such that $(Q,q) \in T$ for some q. Until stated otherwise, we assume T is finite. Then T' is also finite. Let $T' = \{Q_1,\cdots,Q_k\}$ where $Q_1 > Q_2 > \cdots > Q_k$. Since $b_f(Z') = \{0\}$ for all f, $Z' > Q_1$. Starting at Z', we will decrease Q, assigning some $q_f(Q) \in b_f(Q)$ to each firm at each Q until we reach an equilibrium at $Q* = \sum_{f=1}^{n} q_f(Q*)$.

At $Q=Z'$, $b_f(Q) = \{0\}$ so $q_f(Z') = 0$ and $\sum_{f=1}^{n} q_f(Z') = 0 < Z'$. In (Q_1,Z'), for each f, $q_f(Q)$ is assigned so that it is continuous and nonincreasing on $(Q_1,Z']$. This is well defined because the graph of each h_f can be continuously parameterized as discussed earlier. If at any $Q \in (Q_1,Z']$, $\sum_{f=1}^{n} q_f(Q) = Q$ we are done. If not, for each f let

$$q_f = \lim_{\substack{Q \to Q_1 \\ Q > Q_1}} q_f(Q).$$ If $\sum_{f=1}^{n} q_f = Q_1$ we are done (since the b_f are upper-hemi-continuous). If not, let $F_1 = \{f | (Q_1,q) \in T_f \text{ for some } q\} \neq \emptyset$. Let $\{f_1,f_2,\cdots,f_m\} = F_1$ where $f_1 < f_2 < \cdots < f_m$. For $f \notin F_1$ set $q_f(Q_1) = q_f$, so $q_f(\cdot)$ is continuous from the right. We now

introduce discontinuities into $q_f(\cdot)$ for some $f \in F_1$ as follows. Let

$$q_{f_1}(Q_1) = \max \{q | q = q_{f_1} \text{ or both } (Q_1, q) \in T_{f_1} \text{ and } q_{f_1} \leq q \leq Q_1 -$$

$$\sum_{f \neq F_1} q_f(Q_1) - \sum_{j=2}^{m} q_{f_j}\}$$

i.e., $q_{f_1}(Q_1)$ jumps to the largest point of a discontinuity in b_{f_1} which does not lead to a sum of individual actions exceeding Q_1. If the sum equals Q_1 we are done. If not, repeat the process for f_2 (using $q_{f_1}(Q_1)$ as the value assigned to f_1), etc., until either the sum equals Q_1, in which case we are done, or $q_{f_j}(Q_1)$ is assigned for all $j = 1, 2, \ldots, m$, and $\sum_{f=1}^{n} q_f(Q_1) < Q_1$. In (Q_1, Q_2), for each f, $q_f(Q)$ is assigned so that it is continuous and nonincreasing on $(Q_2, Q_1]$. This is again well defined by the properties of b_f. At Q_2 we repeat the procedure used at Q_1.

Continue the process until we get an equilibrium or it cannot continue, as at (Q', q') in Figure 2. We cannot reach points such as this if all continuous branches of b_f are nonincreasing in Q as in Figure 3. (This is the case if no r_f ever has slope less than negative one, such as when $C_f''(y) \geq P'(Y + y)$ for all Y and y.) In Figure 3 we could not reach (Q', q') since $\sum_{i \neq f} q_i(Q') + q' < Q'$ implies

$$\sum_{i \neq f} q_i(Q_1) + q_1 \leq \sum_{i \neq f} q_i(Q') + q_1$$
$$= \sum_{i \neq f} q_i(Q') + q' + (q_1 - q')$$
$$= \sum_{i \neq f} q_i(Q') + q' + (Q_1 - Q') < Q_1$$

so firm f should have been moved to the branch ending at (Q_1, q_1) when

Q was Q_1.

Thus to reach an end of the process before reaching an equilibrium some firm must have a branch of b_f which increases as Q increases. Pick one firm, say j, for which the process of decreasing Q cannot continue. Now <u>increase</u> Q maintaining all other firms on their previously determined paths, $q_f(Q)$. Recall $q_f(Q)$ is nonincreasing in Q (it is nonincreasing where continuous and all jumps are jumps down). Firm j follows a new continuous <u>nondecreasing</u> path $q_j^*(Q)$ which may require following a "45 degree jump" which is part of h_j but not b_j (such as the increasing dotted segment beginning at (Q', q') in Figure 2). However, as in the discussion of Figure 3, $Q - (\sum_{i \neq j} q_f(Q) + q_j^*(Q))$ cannot change sign along this "45 degree jump." But this process cannot continue until this nondecreasing branch of h_j ends (such as at (Q_2, q_2) in Figure 2) since $q_j(Q)$ did not jump up to that point in the previous process of defining $q_j(\cdot)$. At the starting point of the new procedure, $Q - (\sum_{i \neq j} q_f(Q) + q_j^*(Q))$ was positive. As Q increases, all the jumps in the $q_f(\cdot)$ are jumps down, while q_j^* is continuous. At the end of the continuous branch of $q_j^*(\cdot)$, $Q - (\sum_{i \neq j} q_f(Q) + q_j^*(Q))$ is negative. Thus we must reach an n-firm Cournot equilibrium in this process.

Now consider the case in which T is infinite. Two possibilities must be dealt with: there may be some Q such that for infinitely many q, $(Q, q) \in T$, or there may be infinitely many Q in T'. In the first case, it may be necessary to replace "maximum" with "supremum" in the step used to define $q_{f_j}(Q)$ at a Q value

corresponding to the end of infinitely many branches of b_{f_j}, but this creates no problem since b_{f_j} is upper-hemi-continuous. In the second case, we must explain how to continue the process of defining the q_f when Q is a limit point of T'. If q_f has been defined on $[Q',Z']$ for all f, then either $(q_1(Q'),\ldots,q_n(Q'))$ is an equilibrium, or q_f need not be defined for any $Q < Q'$ since we have reached a point from which Q should be increased with only one q_f being replaced with a continuous, nondecreasing q_f^* as in the last step of the proof for T finite, or there is a "branch" of b_f which is "continuous from the left" at $(Q',q_f(Q'))$. In the case of the continuous "branch", the "branch" may consist of infinitely many actual branches, but the jumps between branches become arbitrarily small as $(Q',q_f(Q'))$ is approached. If $\sum_{f=1}^{n} q_f(Q') < Q'$ then for some $\varepsilon > 0$, for $Q \in [Q' - \varepsilon, Q']$, q_f can be defined as the maximum of the b_f values in the "branch". Then q_f may have infinitely many jumps in $[Q' - \varepsilon, Q']$, but is continuous at Q', and $\sum_{f=1}^{n} q_f(Q) < Q$ for Q near Q'. Thus the process of defining q_f can be continued at limit points of T' when necessary. This completes the extension of the result to T infinite.

Q.E.D.

5. Remarks

5.1 Can the assumptions of Theorem 3 be significantly weakened? The first thing to note is that the marginal revenue condition is the only really substantial assumption. The assumptions about cost functions are either basic economic assumptions (nondecreasing cost)

or necessary to guarantee that each firm's reaction correspondence is nonempty valued (lower-semi-continuous cost). The other assumptions on inverse demand can't be weakened significantly without being inconsistent with the marginal revenue condition. To be consistent with the marginal revenue condition (or its nondifferentiable analog) P cannot be increasing or have jumps up, and any jumps down must be jumps to zero (Theorem 3 can easily be extended to the case of a single jump to zero for the inverse demand). Inverse demand is assumed twice continuously differentiable to use the differentiable version of the marginal revenue condition. Finally, if we assume that monopoly revenue is not maximized at infinite output ($\lim_{Z \to \infty} ZP(Z) < \sup_{Z} ZP(Z)$) then the marginal revenue condition implies that there is some Z' such that $P(Z') = 0$. Thus the only significant assumption is the marginal revenue condition.

The marginal revenue condition is not a necessary condition for the existence of Cournot equilibrium because it can fail at an irrelevant point. However, if it fails at a "potential optimal point" then a counterexample can be constructed. A "potential optimal point" y (in response to aggregate output Y by other firms) is a point at which total revenue exceeds the total revenue at all smaller outputs (in response to Y). Since all cost functions are nondecreasing, any point failing this condition could not be an optimal response for a firm with any cost function. Counterexamples to a general existence theorem when the marginal revenue condition fails at a "potential optimal point" are constructed in the proof of the following result.

Theorem 4: Given an inverse demand function P such that

(1) $P: \mathbb{R}_+ \to \mathbb{R}_+$ is continuous,

(2) there exists $Z' < \infty$ such that $P(Z') = 0$ and P is twice

continuously differentiable and strictly decreasing on $[0,Z')$,

and

(3) there exist nonnegative y and Y such that $y + Y < Z'$,

$P'(Y + y) + yP''(Y + y) > 0$, and $\int_{y'}^{y} [P(Y + z) + zP'(Y + z)]dz > 0$

for all $y' \in [0,y)$

there exists an integer n and n cost functions $C_1, \ldots, C_n$, satisfying

(4) for all $f \in \{1,2,\ldots,n\}$, $C_f: \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing,

lower-semi-continuous function (i.e., for all $y \geq 0$, for all

$\varepsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta$ implies

$C_f(x) > C_f(y) - \varepsilon$),

such that the market with inverse demand P and n firms with cost

functions $C_1, \ldots, C_n$ does not have a Cournot equilibrium.

Conditions (1), (2), and (4) of Theorems 3 and 4 are

identical. Condition (3) of Theorem 4 requires that Condition (3) of

Theorem 3 fail at a "potential optimal point". Theorem 3 shows that

if an inverse demand function satisfies conditions (1), (2), and (3)

of that theorem then for any integer n, and any n cost functions

satisfying condition (4), an n-firm Cournot equilibrium exists. In

the proof of Theorem 4 we will construct an example to show that if an

inverse demand function satisfies conditions (1) and (2) of Theorem 3

but fails condition (3) of Theorem 3 at a "potential optimal point"

then there exists an integer n and n cost functions satisfying

condition (4) of Theorem 3 such that there is no n-firm Cournot

equilibrium.

Proof: Assume the conditions of the Theorem hold. Then we

can find strictly positive y* and Y* such that $P(Y^* + y^*) > 0$,

$P'(Y^* + y^*) + y^*P''(Y^* + y^*) > 0$, $P(Y^* + y^*) + y^*P'(Y^* + y^*) > 0$, and

$\int_{y'}^{y^*} [P(Y^* + z) + zP'(Y^* + z)]dz > 0$ for all $y' \in [0,y^*)$. We will

sketch the construction of the required example.

The first firm has no set up cost, and marginal cost which is

constant and near zero on $[0, y^* - \delta]$, continuous and linear on

$[y^* - \delta, y^* + \delta)$ with $C_1'(y^*) = P(Y^* + y^*) + y^*P'(Y^* + y^*)$, and very

large and nondecreasing on $[y^* + \delta, \infty)$. By choice of sufficiently

small δ, this generates a firm with a corresponding $b_1(\cdot)$ which, in a

neighborhood of $Y^* + y^*$, is single valued and strictly increasing with

slope very small and strictly positive.

All other firms are identical, with marginal cost constant and

near zero on $[0,\varepsilon)$ and extremely large on (ε, ∞). These firms have a

set up cost so that there are two optimal responses to aggregate

output $Y^* + y^* - \varepsilon$ by other firms, zero and ε. By choice of ε

sufficiently small, these firms all have corresponding b_f which are

$\{\varepsilon\}$ on $[0, Y^* + y^* - \varepsilon)$, $\{0,\varepsilon\}$ on $[Y^* + y^* - \varepsilon, Y^* + y^*]$, and $\{0\}$ on

$(Y^* + y^*, \infty)$.

With $\delta < Y^*$, $\sum_{f=1}^{n} b_f(Q) = b_1(Q) \leq y^* + \delta < Q$ for

$Q \in (Y^* + y^*, \infty)$. Choosing n very large,

$\sum_{f=1}^{n} b_f(Q) \geq (n-1)\varepsilon > Y^* + y^* > Q$ for $Q \in [0, Y^* + y^* - \varepsilon)$. Thus if an

equilibrium exists, Q^* must be in $[Y^* + y^* - \varepsilon, Y^* + y^*]$. Let $[Y^*/\varepsilon]$
be the greatest integer less or equal to Y^*/ε. If ε is chosen very
small and such that the fractional part of Y^*/ε (i.e., $Y^*/\varepsilon - [Y^*/\varepsilon]$)
is very near one (relative to the slope of b_1 near $Y^* + y^*$) then
$b_1(Q) + ([Y^*/\varepsilon] + 1)\varepsilon > Q$ for all $Q \in [Y^* + y^* - \varepsilon, Y^* + y^*]$ while
$b_1(Q) + [Y^*/\varepsilon]\varepsilon < Q$ for all $Q \in [Y^* + y^* - \varepsilon, Y^* + y^*]$. Thus for
appropriately chosen small δ and ε there is no equilibrium.

<div align="right">Q.E.D.</div>

5.2   We now consider the extension of the theorem to exogenous
determination of n. For some types of cost functions it is already
the case that in equilibrium some firms will be inactive (when n is
sufficiently large), so n is endogenous in those cases. However,
Theorem 3 cannot be extended to $n = \infty$ without additional conditions.
First, note that equilibrium does not exist for $n = \infty$ under the
conditions of Theorem 1—identical firms with convex cost, whether or
not inverse demand is concave. This is easily seen using the
"backward mapping" b_j. In this case, for large Q, $b_j(Q) = \{0\}$ but as
Q declines b_j continuously increases (at least initially). If Q' is
the smallest output at which $b_j(Q) = \{0\}$ then $\sum_{j=1}^{\infty} b_j(Q') = 0 < Q'$ but
for all $\varepsilon > 0$, $\sum_{j=1}^{\infty} b_j(Q' - \varepsilon) = \infty > Q'$. Because average cost is
minimized at infinitesimal outputs, if any firm is active then <u>all</u>
firms are active. For any exogenous n this creates no problem.

However, for $n = \infty$ there is no equilibrium. The continuity of b_j near
Q' prevents the extension of the existence result to endogenous n.
This is in contrast to the case of the U-shaped average cost (and
additional conditions, see Novshek [1980]), in which discontinuities
in b_j were used to show existence of equilibrium for $n = \infty$. (In
equilibrium all but a finite number of firms are inactive.)

There are other technical issues beyond continuity. However,
these other issues have minimal economic content. Thus Theorem 3 can
be extended to endogenous determination of the number of active firms
except for the case of identical firms with convex cost functions (and
some other technicalities).

5.3   If we consider a sequence of Markets M_k which converge to a
perfectly competitive limit market M with infinitesimal firms, we can
ask (1) for k large does M_k have a Cournot equilibrium and (2) how do
the Cournot equilibria of M_k compare to the competitive equilibria of
M? For a special case in which firms were identical within each M_k,
and the markets were related in a very strong way, Novshek [1980]
shows that with downward sloping inverse demand and U-shaped average
cost, (1) for large k, M_k has a Cournot equilibrium and (2) the
Cournot equilibria of M_k converge to the competitive equilibra of M.
Using Theorem 3 this result can be considerably generalized: for
downward sloping inverse demand, as long as the markets M_k converge to
M in an appropriate sense, (1) for large k, M_k has a Cournot
equilibrium and (2) the Cournot equilibria converge to the competitive
equilibrium of M. (When firms are identical and have convex cost

functions, this result requires that the measure of available firms in M be finite. As discussed above, this need not rule out endogenous determination of the number of active firms in other cases.) General results of this type are contained in Novshek [1983]. The existence question, (1), is also addressed in Bamon and Fraysse [1983]. In their paper, Bamon and Fraysse independently prove a fixed point theorem which is similar to, but weaker than, Theorem 3. Their result directly assumes that reaction correspondences have at most one jump, which is down, and have slope everywhere greater than negative one. These assumptions are natural consequences of the assumptions they place on cost functions in their sequence of markets framework, though they may not hold in a single market.
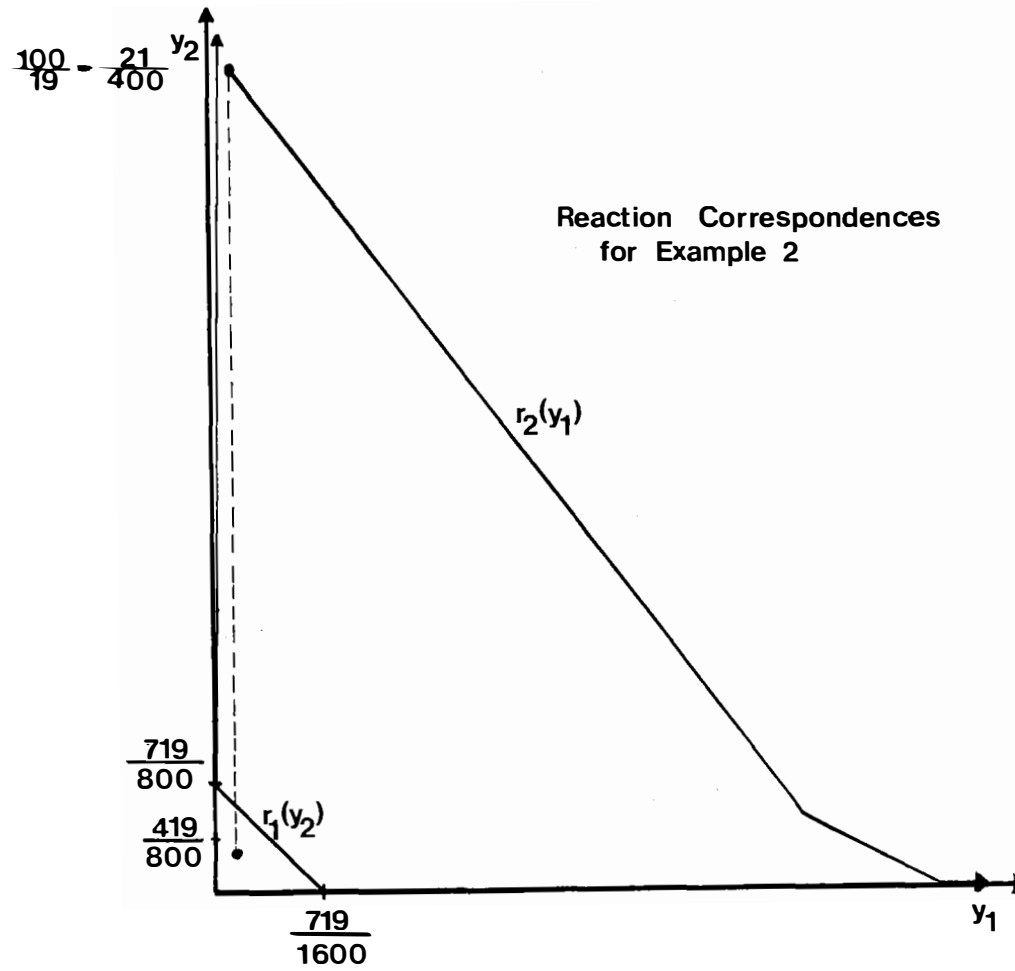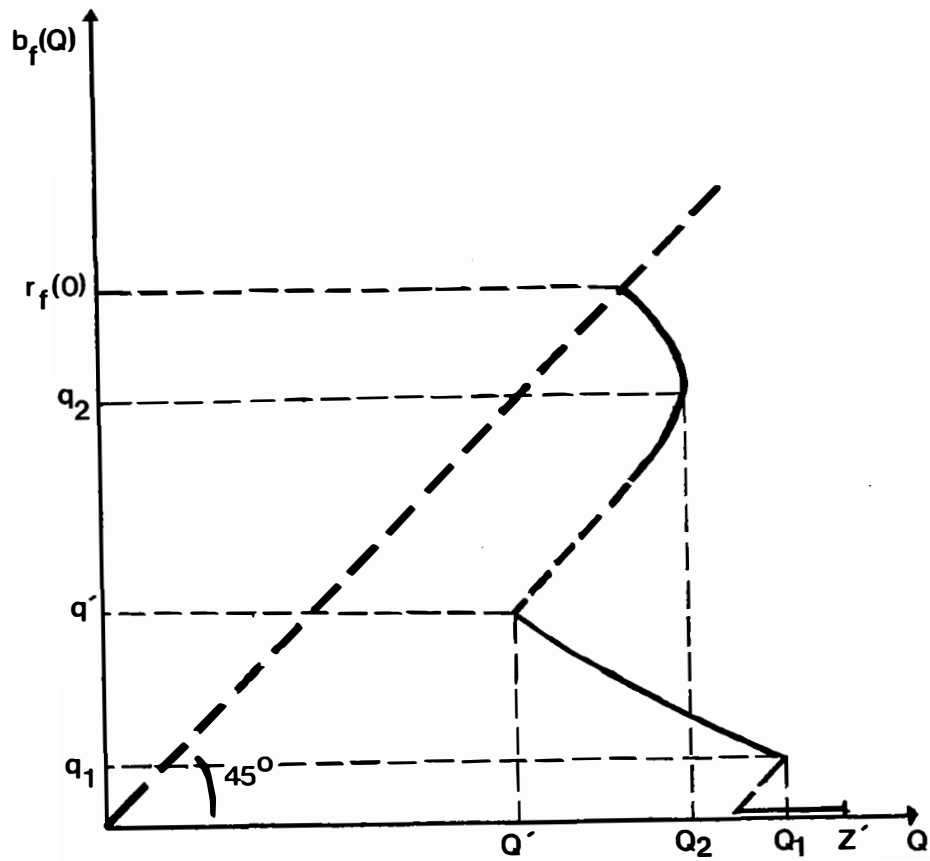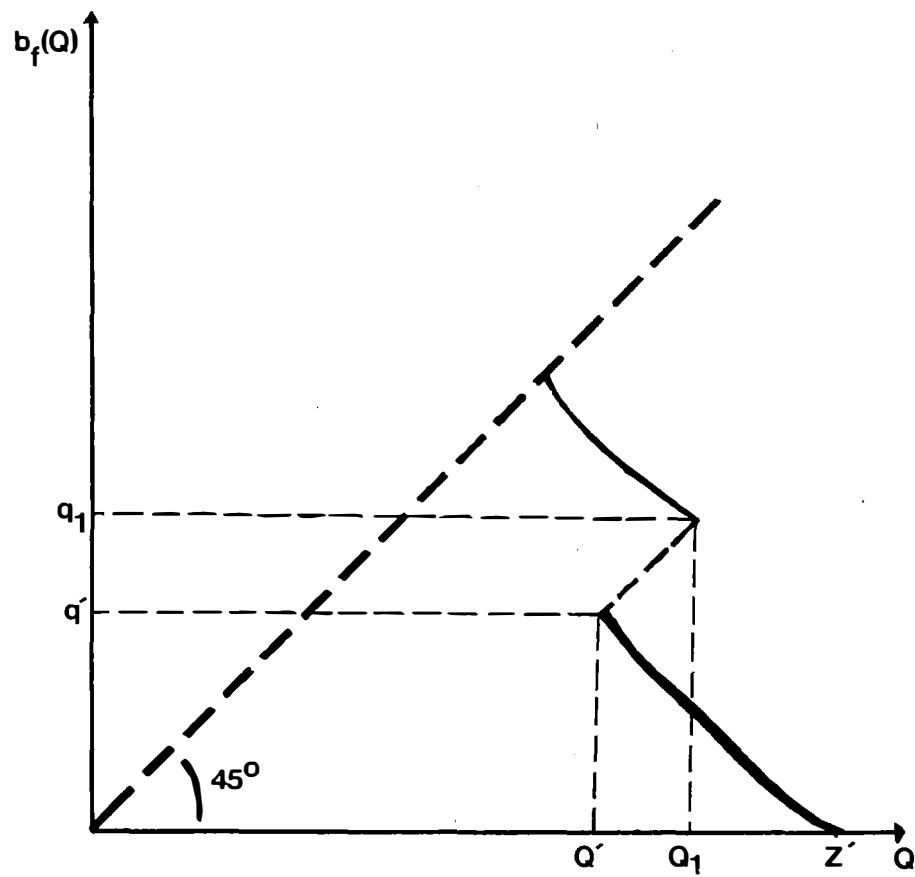
FIGURE 1

Reaction Correspondences for Example 2

FIGURE 2

FIGURE 3

Footnotes

1. Kim Border brought to my attention a paper by Nishimura and Friedman [1981] in which they prove a third type of existence theorem for Cournot equilibrium. They have an assumption on the derived reaction correspondence which does not have a natural counterpart in terms of the primitive inverse demand and cost functions. Their assumption requires that for any $(y_1, y_2, \ldots, y_n)$ which is not an equilibrium, for at least one firm j, either all optimal responses to Y by firm j are strictly greater than y_j for all Y sufficiently near $\sum_{i \neq j} y_i$, or they are all strictly less than y_j. In this paper we are concerned with assumptions on the basic elements of the model, inverse demand and cost functions, and their relationship to the existence question.

2. This implies marginal cost is nondecreasing where defined. Average cost could be U-shaped in the case treated by McManus, but the cost function could not be discontinuous at zero. Thus $C(0)$ was a sunk cost, and the relevant cost function, $C*(y) = C(y) - C(0)$, has a non-decreasing average cost.

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