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ON THE EXISTENCE OF EQUIVARIANT EMBEDDINGS OF PRINCIPAL BUNDLES INTO VECTOR BUNDLES

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ABSTRACT. Let G be a finite group and let X be, say, a connected CW-complex of dimension $k \geqslant 1$. Let $\pi\colon E \to X$ be a principal G-bundle and $p\colon V \to X$ an m-dimensional G-vector-bundle with trivial action of G on X. By an equivariant embedding of π into p we understand an equivariant embedding $h\colon E \to V$ commuting with projections. We prove a general embedding theorem, a main special case of which is the following

THEOREM. *If $k < m$ and if the action of G on V is free outside the zero section for p, then any principal G-bundle $\pi\colon E \to X$ can be embedded equivariantly into $p\colon V \to X$.*

1. Introduction. Throughout this paper G denotes a finite group. Let $\pi\colon E \to X$ be a principal G-bundle, and $p\colon V \to X$ a real G-vector-bundle with trivial action of G on X. The actions of G on E and V are taken as right actions. By an equivariant embedding of π into p we understand a map $h\colon E \to V$ which commutes with projections, maps E homeomorphically onto its image $h(E)$ in V, and satisfies $h(e \cdot g) = h(e) \cdot g$ for all $e \in E$ and $g \in G$:

$$\begin{array}{ccc} E & \xrightarrow{h} & V \\ & \pi\searrow \quad \swarrow p & \\ & X & \end{array}$$

We shall be concerned with the existence of such equivariant embeddings. As a corollary to our main theorem we get the following

THEOREM (COROLLARY 4.2). *Suppose that X is a connected, topological space with a nondegenerate base point and with the homotopy type of a CW-complex of dimension $k \geqslant 1$, and let $p\colon V \to X$ be an m-dimensional G-vector-bundle. If $k < m$ and if the action of G on V is free outside the zero section for p, then any principal G-bundle π: $E \to X$ can be embedded equivariantly into the G-vector-bundle $p\colon V \to X$.*

The study of equivariant embedding theorems is motivated by the embedding theorem for finite covering maps $\pi\colon E \to X$ into trivial vector bundles of dimension $m > k$ proved by the first author [**4**], and generalized to arbitrary vector bundles of dimension $m > k$ by Duvall and Husch [**2**]. For 2-fold coverings, embedding theorems into trivial vector bundles in the unstable dimensions $m \leqslant k$ have been obtained by Prevot [**6**].

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2. Preliminaries. Let $p\colon V \to X$ be a fixed real G-vector-bundle. Normally, a G-vector-bundle is just an ordinary locally trivial vector bundle $p\colon V \to X$ with G-actions on the total space V and the base space X, such that p is equivariant, and such that G acts linearly in the fibres of p. In addition, we shall assume throughout that the G-action on X is trivial, and that the action of G is effective in each fibre of $p\colon V \to X$. We note that p admits G-equivariant local trivializations. On request, Karsten Grove informs us that this can be proved with small amendments to the proof in his paper [**3**].

For each integer $k \geqslant 1$, the Whitney sum bundle

$$p^k\colon V^k = V \oplus \cdots \oplus V \to X$$

has an induced G-vector-bundle structure by coordinatewise action of G. Viewing p^k as a subbundle of p^{k+1} by putting the zero vector in the last coordinate and then taking the direct limit we obtain the infinite Whitney sum bundle $p^\infty\colon V^\infty \to X$. By definition, an element $v = (v_i)_{i=1}^{\infty}$ in V^∞ is a sequence of vectors in the same fibre of $p\colon V \to X$ all but a finite number of which are the zero vector. Also $p^\infty\colon V^\infty \to X$ has an induced G-vector-bundle structure by coordinatewise action of G.

For any G-space Z, and any element $g \neq 1$ in G, we denote by $Z^{[g]}$ the set of points in Z kept fixed by g. By Z_G we denote the subset of Z defined by

$$Z_G = Z \setminus \bigcup_{g \neq 1} Z^{[g]}.$$

Clearly, the action of G on Z induces a free G-action on Z_G.

We shall in particular consider the spaces V_G and V_G^∞. Since $p\colon V \to X$ is G-locally-trivial, it is easy to prove that the induced projections

$$p_G\colon V_G \to X \quad \text{and} \quad p_G^\infty\colon V_G^\infty \to X$$

are locally trivial fibrations with free G-actions on the fibres.

LEMMA 2.1. *The fibration $p_G^\infty\colon V_G^\infty \to X$ has contractible fibres.*

PROOF. A fibre of p_G^∞ is homeomorphic to a space

$$R_G^\infty = \varinjlim_k (R^m)_G^k,$$

where R^m denotes euclidean m-space equipped with a certain effective G-action by linear isomorphisms.

Since the G-action on $(R^m)^k$ is coordinatewise, we get an identification

$$(R^m)_G^k = (R^m)^k \setminus \bigcup_{g \neq 1} (\operatorname{Fix}(g))^k,$$

where $\operatorname{Fix}(g)$ denotes the fixpoint set for the isomorphism $g\colon R^m \to R^m$ defined by the element $g \in G$.

Since $g \neq 1$ and the action of G on R^m is effective, $\mathrm{Fix}(g)$ is a subspace of codimension $\geqslant 1$ in R^m. Hence $(\mathrm{Fix}(g))^k$ has codimension $\geqslant k$ in $(R^m)^k = R^{mk}$ for each $g \neq 1$.

A simple transversality argument shows now that $(R^m)^k_G$ is $(k-2)$-connected. Hence the direct limit of these spaces, R^∞_G, is contractible, since it has the homotopy type of a CW-complex. This proves the lemma. □

3. Equivariant embeddings into p^∞_G. Let $\omega\colon EG \to BG$ be the universal numerable principal G-bundle constructed by Milnor. We follow the exposition in Husemoller [**5**]. By definition the elements of the total space EG are equivalence classes of sequences

$$\langle g, t \rangle = (t_0 g_0, t_1 g_1, t_2 g_2, \ldots),$$

where $g_j \in G$ and $t_j \in I = [0,1]$ such that only a finite number of $t_j \neq 0$ and $\Sigma^\infty_{j=0} t_j = 1$.

PROPOSITION 3.1. *There exists a continuous map*

$$\mu\colon V^\infty_G \times EG \to V^\infty_G$$

with the following properties:

(i) $p^\infty_G \circ \mu(v, \langle g, t \rangle) = p^\infty_G(v)$,

(ii) $\mu(v, \langle g, t \rangle \cdot g') = \mu(v, \langle g, t \rangle) \cdot g'$, *for* $v \in V^\infty_G$, $\langle g, t \rangle \in EG$ *and* $g' \in G$.

We think of μ *as a fibrewise action of* EG *on* V^∞_G.

PROOF. Define $\mu\colon V^\infty_G \times EG \to V^\infty_G$ by

$$\mu(v, \langle g, t \rangle) = \left((t_{k-i} v_i \cdot g_{k-i})^k_{i=1} \right)^\infty_{k=1},$$

and check that it has the properties (i) and (ii). □

We are now ready to prove our first main result.

THEOREM 3.2. *Any principal* G*-bundle* $\pi\colon E \to X$ *embeds equivariantly into* $p^\infty_G\colon V^\infty_G \to X$ *through a fibrewise map.*

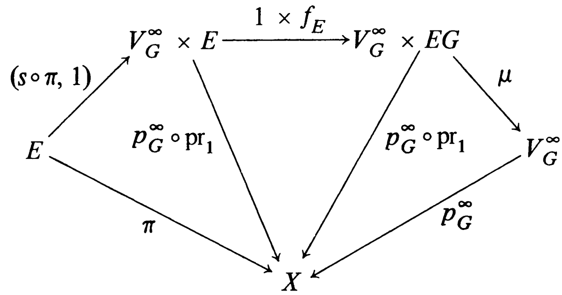
PROOF. Let $s\colon X \to V^\infty_G$ be a section of $p^\infty_G\colon V^\infty_G \to X$. Such a section exists by Dold [**1**], since the fibres of p^∞_G are contractible by Lemma 2.1. Let $f_E\colon E \to EG$ be the map between total spaces in the classifying map for π:

$$\begin{array}{ccc} E & \xrightarrow{f_E} & EG \\ \pi \downarrow & & \downarrow \omega \\ X & \xrightarrow[f_B]{} & BG \end{array}$$

An explicit equivariant fibrewise embedding as required can then be constructed as the composition

$$h = \mu \circ (1 \times f_E) \circ (s \circ \pi, 1)$$

of maps in the diagram



where pr_1 denotes projection onto the first factor. Here, G acts on $V_G^\infty \times E$ and on $V_G^\infty \times EG$ via the second factor. That h is G-equivariant now follows from Proposition 3.1(ii). □

4. Equivariant embeddings into p. Suppose that X is a connected, topological space with a nondegenerate base point and with the homotopy type of a CW-complex of dimension $k \geqslant 1$. Let p: $V \to X$ be an m-dimensional G-vector-bundle in which G acts effectively on each fibre of p.

Denote by

$$m(p, G) = \min_{g \neq 1} \operatorname{codim} \operatorname{Fix}(g),$$

the minimum of the codimensions of the fixpoint sets $\operatorname{Fix}(g)$ for the isomorphisms g: $R^m \to R^m$ defined in an arbitrary fibre R^m of p by the elements $g \in G$, $g \neq 1$.

Since the action of G in each fibre of p is effective we have $1 \leqslant m(p, G) \leqslant m$.

We are now ready to prove our main theorem.

THEOREM 4.1. *With notation as above suppose that $1 \leqslant k < m(p, G)$. Then any principal G-bundle π: $E \to X$ can be embedded equivariantly into the G-vector-bundle p: $V \to X$.*

As a corollary to Theorem 4.1 we get immediately the theorem stated in the introduction.

COROLLARY 4.2. *With notation as above suppose that $1 \leqslant k < m$. If the action of G is free outside the zero section for p, then any principal G-bundle π: $E \to X$ can be embedded equivariantly into the G-vector-bundle p: $V \to X$.*

For the proof of Theorem 4.1 we need two lemmas. The first lemma can be proved along the same lines as G. W. Whitehead [7, Chapter II, §3, Lemma 3.1, p. 70].

LEMMA 4.3. *Let $F' \xrightarrow{i'} E' \xrightarrow{p'} B$ be a subfibration of the fibration $F \xrightarrow{i} E \xrightarrow{p} B$ over the same base space B. Suppose that the pair (F, F') is n-connected and that B is a CW-complex of dimension $\leqslant n$. Then any section s in p is vertical homotopic to a section s' in p'.*

LEMMA 4.4. *Suppose that* $2 \leqslant m(p, G)$. *Consider* p_G: $V_G \to X$ *as a subfibration of* p_G^∞: $V_G^\infty \to X$ *by inclusion on the first coordinate. Then the pair of fibres for the pair of fibrations* (p_G^∞, p_G) *is* $(m(p, G) - 1)$*-connected.*

PROOF OF LEMMA 4.4. Following the notation from the proof of Lemma 2.1, a fibre in P_G: $V_G \to X$ can be identified with

$$R^m \setminus \bigcup_{g \neq 1} \mathrm{Fix}(g).$$

By assumption $\operatorname{codim} \mathrm{Fix}(g) \geqslant m(p, G) \geqslant 2$ for each $g \in G$, $g \neq 1$. Hence by a simple transversality argument the fibres in p_G are $(m(p, G) - 2)$-connected.

Since by Lemma 2.1 the fibres of p_G^∞ are contractible, the pair of fibres for the pair of fibrations (p_G^∞, p_G) is clearly $(m(p, G) - 1)$-connected. □

PROOF OF THEOREM 4.1. Let π: $E \to X$ be an arbitrary principal G-bundle. We shall apply the usual technique for transforming problems of bundle maps into section problems. Let therefore $\mathrm{Emb}(\pi, p_G)$, respectively $\mathrm{Emb}(\pi, p_G^\infty)$, denote the fibration over X for which the sections are the equivariant fibrewise embeddings of π into p_G, respectively p_G^∞. In the obvious way, we consider $\mathrm{Emb}(\pi, p_G)$ as a subfibration of $\mathrm{Emb}(\pi, p_G^\infty)$. Since an equivariant embedding of a principal G-bundle is completely determined by its values on a single element in each fibre, the pair of fibres for the pair of fibrations $(\mathrm{Emb}(\pi, p_G^\infty), \mathrm{Emb}(\pi, p_G))$ can be identified with the pair of fibres for the pair of fibrations (p_G^∞, p_G), and is therefore $(m(p, G) - 1)$-connected by Lemma 4.4. By Lemma 4.3, an equivariant embedding of π into p_G^∞ represented by a section in $\mathrm{Emb}(\pi, p_G^\infty)$ can therefore be deformed into an equivariant embedding of π into p_G represented by a section in $\mathrm{Emb}(\pi, p_G)$. In particular we obtain an equivariant embedding of π into p, and Theorem 4.1 is proved. □

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