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On the existence of fundamental and total bounded biorthogonal systems in Banach spaces

by

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Abstract. Every separable Banach space admits (for any $\varepsilon > 0$) a biorthogonal system $(x_n; x_n^*)$ with $\|x_n\|\,\|x_n^*\| \leqslant 1+\varepsilon$ which may be selected either so that (x_n) is fundamental or so that (x_n^*) is total. The first part of this result extends to certain non-separable spaces (in particular $m(\varkappa)$): If X has a weakly compactly generated quotient with the same density character as X, then X has a bounded biorthogonal system $(x_a; x_a^*)$ with (x_a) fundamental.

I. Introduction and notation. It is known (cf., e.g. [2], p. 238 or [12]) that if X is a finite dimensional Banach space (say, $\dim X = m$) then X admits a biorthogonal sequence $(x_n, x_n^*)_{n=1}^m$ with $\|x_n\| = \|x_n^*\| = 1$ for $n = 1, \ldots, m$. In Section II we prove two infinite dimensional versions of this result. We show that, for each $\varepsilon > 0$, every separable Banach space admits a fundamental biorthogonal sequence bounded by $1+\varepsilon$ and a total biorthogonal sequence bounded by $1+\varepsilon$. The first result answers in the affirmative a question of Singer's ([8], p. 169); still unsolved is Banach's problem [2]: Does every separable Banach space admit a fundamental, total bounded biorthogonal sequence?

Our techniques also yield some information in the non-separable case. Theorem 2 shows that if X is a non-separable Banach space which has a weakly compactly generated quotient with the same density character as the density character of X, then X admits a fundamental bounded biorthogonal system.

Henceforth X, Y, and Z will refer to infinite dimensional Banach spaces over either the real or complex numbers. "Subspace" means "closed, infinite dimensional linear subspace". For $A \subset X$, $A^\perp$ is the annihilator of A in X^*. For $A \subset X^*$, $A^\top$ is the annihilator of A in X. If Y is a subspace of X, the dual of the quotient space X/Y is identified with $Y^\perp$ in the canonical way. The real restriction of the Banach space X is the real Banach space obtained from X by allowing multiplication by real scalars only.

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X is *weakly compactly generated* provided X contains a weakly compact subset whose closed linear span is X. The density character of X (written $\operatorname{dens} X$) is the smallest cardinal, $\varkappa$, for which X has a dense subset of cardinality $\varkappa$. We identify the cardinal $\varkappa$ with the set of ordinals less than $\varkappa$. N denotes the set of positive integers.

$[x_\alpha]$ is the closed linear span of the indexed family (x_α). A family (x_α, x_α^*) with $(x_\alpha) \subset X$, $(x_\alpha^*) \subset X^*$ is called *biorthogonal* provided $x_\alpha^*(x_\beta) = \delta_{\alpha\beta}$. (x_α, x_α^*) is: *fundamental* if $[x_\alpha] = X$; *total* if $(x_\alpha^*)^\top = \{0\}$; *bounded* provided (x_α) and (x_α^*) are both bounded; *bounded by* λ (where $\lambda \geqslant 1$) provided (x_α, x_α^*) is bounded and $\|x_\alpha\|\,\|x_\alpha^*\| \leqslant \lambda$ for every α.

A sequence $(x_n) \subset X$ is called *basic* provided that for each $x \in [x_n]$, there exists a unique sequence $(x_n^*(x))$ of scalars with $x = \sum x_n^*(x) x_n$. It is well known that each x_n^* is linear and continuous, and that (x_n, x_n^*) is biorthogonal. For $\lambda \geqslant 1$, the basic sequence (x_n) is said to be λ-equivalent to the basic sequence (y_n) provided that the mapping taking x_n to y_n extends to a linear homeomorphism T of $[x_n]$ onto $[y_n]$ with $\|T\|\,\|T^{-1}\| \leqslant \lambda$.

II. The existence theorems. Our first lemma generalizes a result of Day's [3] (and uses Day's technique). In the proof we make use of a consequence of the Borsuk antipodal mapping theorem observed by Day [3]: *If F and G are subspaces of the real restriction of the same Banach space and $\dim F < \dim G \leqslant \infty$, then there is a unit vector g in G whose distance $d(g, F)$ from F is one.*

LEMMA 1. *Suppose that X is separable and set $n_k = \dfrac{k(k+1)}{2}$ for $k = 0, 1, \ldots$ X admits a biorthogonal sequence (x_n, x_n^*) satisfying*

(i) $\|x_n\| = \|x_n^*\| = x_n^*(x_n) = 1$ *for* $n = 1, 2, \ldots$

(ii) *For each* $x \in [rx_n]$, $x = \lim\limits_{k\to\infty} \sum\limits_{i=1}^{n_k} x_i^*(x) x_i$.

(iii) *In the real restriction of X, $(x_i)_{i=n_k+1}^{n_{k+1}}$ is $\left(1+\dfrac{1}{k+1}\right)$-equivalent to an orthogonal basis in the $k+1$ dimensional real Euclidean space l_2^{k+1} for $k = 0, 1, 2, \ldots$*

(iv) $(x_n^*)^\top + [x_n]$ *is dense in* X.

Proof. Let (d_n) be a dense sequence in X with $d_0 = 0$. It is sufficient to define sequences $(x_n) \subset X$, $(x_n^*) \subset X^*$ and finite sets $\varphi = F_0 \subset F_1 \subset F_2 \subset \ldots$ of unit vectors in X^* to satisfy (i), (iii) and

(v) $x_{n_k+j} \in \left(F_k \cup (x_i^*)_{i=1}^{n_k+j-1}\right)^\top$ *for each* $k = 0, 1, \ldots$ *and* $j = 1, \ldots, k+1$.

(vi) $x_{n_k+j}^* \in \left((d_i)_{i=0}^{k} \cup (x_i)_{i=1}^{n_k+j-1}\right)^\perp$ *for each* $k = 0, 1, \ldots$ *and* $j = 1, \ldots, k+1$.

(vii) *for each* $k = 0, 1, \ldots$ *and* $x \in [(x_i)_{i=1}^{n_{k+1}}]$ *there is* $f \in F_{k+1}$ *such that* $\|x\| \leqslant \left(1+\dfrac{1}{k+1}\right)|f(x)|$.

For then (x_n, x_n^*) is biorthogonal by (i), (v), and (vi). From (vii) and (v) it follows that, for any scalars (a_i),

$$\Big\|\sum_{i=1}^{n_k} a_i x_i\Big\| \leqslant \Big(1+\frac{1}{k}\Big) \max_{f\in F_k} \Big|f\Big(\sum_{i=1}^{n_k} a_i x_i\Big)\Big| = \Big(1+\frac{1}{k}\Big) \max_{f\in F_k} \Big|f\Big(\sum_{i=1}^{\infty} a_i x_i\Big)\Big| \leqslant \Big(1+\frac{1}{k}\Big)\Big\|\sum_{i=1}^{\infty} a_i x_i\Big\|;$$

(ii) is an easy consequence of this inequality. Finally, from (vi) we have $d_k \in \left((x_i^*)_{i=n_k+1}^{\infty}\right)^\top$, hence $d_k - \sum\limits_{i=1}^{n_k} x_i^*(d_k) x_i \in (x_n^*)^\top$, whence $d_k \in [x_n] + (x_n^*)^\top$, so that (iv) holds.

Pick x_1 and x_1^* to satisfy (i). Suppose that $(x_i, x_i^*)_{i=1}^{n_k}$ and $(F_i)_{i=1}^{k}$ have been defined. Set $m = 2(n_{k+1}+3k)$ and use the Dvoretzky theorem [4] to get an isomorphism T from a real m dimensional subspace Z of the real restriction of $\left((x_i^*)_{i=1}^{n_k} \cup F_k\right)^\top$ onto l_2^m with $\|T\| \leqslant 1+\dfrac{1}{k}$, $\|T^{-1}\| = 1$. We select $(x_i)_{i=n_k+1}^{n_{k+1}} \subset Z$ and $(x_i^*)_{i=n_k+1}^{n_{k+1}}$ to satisfy (i), (v), (vi) and

(viii) $(Tx_i)_{i=n_k+1}^{n_{k+1}}$ *is orthogonal.*

Indeed, having defined $(x_i, x_i^*)_{i=n_k+1}^{n_k+j-1}$ for some j, $1 \leqslant j \leqslant k+1$, we let W be the orthogonal complement in l_2^m to $(Tx_i)_{i=n_k+1}^{n_k+j-1}$ and, using Day's lemma, select a unit vector $x_{n_k+j} \in (T^{-1}W) \cap \left((x_i^*)_{i=n_k+1}^{n_k+j-1}\right)^\top$ so that $d\left(x_{n_k+j}, [(d_i)_{i=1}^{k} \cup (x_i)_{i=1}^{n_k+j-1}]\right) = 1$. (Note that Day's lemma applies, because if we set $G = (T^{-1}W) \cap \left((x_i^*)_{i=n_k+1}^{n_k+j-1}\right)^\top$ and $F = [(d_i)_{i=1}^{k} \cup (x_i)_{i=1}^{n_k+j-1}]$, then in the real restriction of X, $\dim F \leqslant 2k+2(n_k+j-1) < 2k+2n_{k+1}$, while $\dim G \geqslant m-(j-1)-2(j-1) \geqslant m-3k = 2n_{k+1}+3k$.) Now we use the Hahn–Banach theorem to get $x_{n_k+j}^*$ to satisfy (i) and (vi).

Finally, using the compactness of the unit ball of the finite dimensional space $[(x_i)_{i=1}^{n_{k+1}}]$ and the Hahn–Banach theorem, pick a finite set $F_{k+1} \supset F_k$ of unit vectors to satisfy (vii).

Clearly (x_n, x_n^*) and (F_n) satisfy (i) and (v)–(viii), while (iii) follows from (viii). ∎

Remark 1. By using the techniques in [6] and a bit more care in the above proof of Lemma 1, (x_n, x_n^*) may be chosen so that (x_n) is basic and (x_n^*) is w^*-basic in the sense of [6].

THEOREM 1. *Suppose X is separable and let $\varepsilon > 0$.* (a) *X admits a fundamental biorthogonal sequence bounded by $1+\varepsilon$.* (b) *X admits a total biorthogonal sequence bounded by $1+\varepsilon$.*

Proof. Let (x_n, x_n^*) be a biorthogonal sequence for X satisfying (i)–(iv) of Lemma 1. Let $p\colon N\times N\to N$ be a bijection such that for each n, $p(n,1)<p(n,2)<\ldots$, and for each n and k there exists j so that, in real restriction of X, $(x_{p(n,i)})_{i=j+1}^{j+k}$ is 2-equivalent to the usual basis for l_2^k. It follows that for each n, $(x_{p(n,i)})_{i=1}^{\infty}$ is a basic sequence in the real restriction of X not equivalent to the usual basis for l_1 (the space of absolutely summable real sequences), so there is a sequence $(a_i^n)_{i=1}^{\infty}$ of real numbers with $\sum_{i=1}^{\infty} a_i^n x_{p(n,i)}$ convergent and $\sum_{i=1}^{\infty}|a_i^n| = \infty$.

Let (y_n) be dense in the unit ball of $(x_n^*)^{\top}$, and set, for each n and i,

$$w_i^n = -\varepsilon \operatorname{sign} a_i^n y_n + x_{p(n,i)}.$$

Obviously $(w_i^n, x_{p(n,i)}^*)_{i,n=1}^{\infty}$ is biorthogonal and $\|w_i^n\|\,\|x_{p(n,i)}^*\|\leqslant 1+\varepsilon$, so we can complete the proof of (a) by showing that $(w_i^n)^{\perp} = \{0\}$.

Suppose $x^*\in (w_i^n)^{\perp}$. Then for each n and k,

$$x^*\Big(\sum_{i=1}^{k} a_i^n x_{p(n,i)}\Big) = \varepsilon \sum_{i=1}^{k} |a_i^n|\, x^*(y_n).$$

For each fixed n the left side of the preceding equation is bounded in k, so $x^*(y_n) = 0$, from which it follows that $x^*(x_{p(n,i)}) = 0$ for $i = 1, 2, \ldots$ Thus x^* vanishes on $(x_n^*)^{\top}+[x_n]$ whence, by (iv), $x^* = 0$.

To prove (b), note that (iv) implies that, for each n, $x_{p(n,i)}^*$ converges weak* to 0 as $i\to\infty$.

Let (z_n) be a weak* dense sequence in the unit ball of $(x_n)^{\perp}$ and set, for each n and i,

$$b_i^n = -\varepsilon z_n + x_{p(n,i)}^*.$$

Clearly $(x_{p(n,i)}, b_i^n)$ is biorthogonal and bounded by $1+\varepsilon$; we complete the proof by showing (b_i^n) is total.

Suppose $x\in (b_i^n)^{\top}$. Then for each n and i, $\varepsilon z_n(x) = x_{p(n,i)}^*(x)$. Letting $i\to\infty$, we have that $z_n(x) = 0$ for each n, hence also $x_{p(n,i)}^*(x) = 0$ for each n and i. But then $x\in (x_n^*)^{\top}\cap[x_n]$ and thus, by (ii), $x = 0$. ■

Remark 2. The perturbation technique used in the above proof (and in the proof of Theorem 2 below) was suggested by Singer's proof of Proposition 1 in [9]; however, Singer's construction there produced unbounded biorthogonal sequences. Singer [11] has also modified his technique of [9] to give a proof of 1 (b) with "$1+\varepsilon$" replaced by "$2+\varepsilon$".

Lemma 2. *Suppose that X is weakly compactly generated and* dens $X = \varkappa > \varkappa_0$. *Then X has a quotient Y which admits a bounded fundamental biorthogonal system $(y_n^a, g_n^a)_{a\in\varkappa, n\in N}$ such that for each a, 0 is a weak cluster point of $(y_n^a)_{n=1}^{\infty}$.*

Proof. It follows from the results of Amir and Lindenstrauss [1] that there is a family $\{P_a\colon a\in\varkappa\cup\{\varkappa\}\}$ of norm one projections on X satisfying

(a) $P_a P_\beta = P_{\min(a,\beta)}$ for all a, β.

(b) $[P_{a+1}-P_a]X$ is infinite dimensional for each $a\in\varkappa$.

(c) $P_\varkappa$ is the identity, and for each limit ordinal $\beta\leqslant\varkappa$, $\{P_a\colon a<\beta\}$ tends strongly to P_β.

For each $a\in\varkappa$, write $a = m_a+n_a$, where m_a is a limit ordinal (or zero), n_a is a non-negative integer, and "+" denotes ordinal addition. As in the proof of Lemma 1, for each a we can choose a biorthogonal sequence $(x_i^a, \tilde f_i^a)_{i=1}^{n_a+1}$ in $[P_{a+1}-P_a]X$ with $\|x_i^a\| = \|\tilde f_i^a\| = 1$ so that, in the real restriction of X, $(x_i^a)_{i=1}^{n_a+1}$ is 2-equivalent to the usual basis for $l_2^{n_a+1}$.

Set $f_i^a = \tilde f_i^a(P_{a+1}-P_a)$. The system $(x_i^a, f_i^a)_{a\in\varkappa, i\leqslant n_a+1}$ is biorthogonal by (a). Now for each $a\in\varkappa$, $[P_{a+1}-P_a]X$ is the direct sum of $[(x_i^a)_{i=1}^{n_a+1}]$ and $((f_i^a)_{i=1}^{n_a+1})^{\top}$. From this and (c) it follows that $[x_i^a]+(f_i^a)^{\top}$ is dense in X. Thus by reindexing (x_i^a, f_i^a) we have that X admits a bounded biorthogonal system $(\tilde y_i^a, g_i^a)_{a\in\varkappa, i\in N}$ satisfying

(i) $[\tilde y_i^a]+(g_i^a)^{\top}$ is dense in X.

(ii) for each $a\in\varkappa$ and $n = 1, 2, \ldots$, there exists k such that in the real restriction of X, $(\tilde y_i^a)_{i=k+1}^{k+n}$ is 2-equivalent to the usual basis for l_2^n.

Let $Y = X/(g_i^a)^{\top}$, let $T\colon X\to Y$ be the quotient map, and set $y_i^a = T\tilde y_i^a$. Clearly (y_i^a, g_i^a) is a bounded biorthogonal system for Y and it is fundamental by (i). From (ii) it follows that, for each $a\in\varkappa$, 0 is a weak cluster point of $(\tilde y_i^a)_{i=1}^{\infty}$, hence also 0 is a weak cluster point of $(y_i^a)_{i=1}^{\infty}$. ■

Theorem 2. *Suppose that* dens $X = \varkappa > \varkappa_0$ *and X has a weakly compactly generated quotient whose density character is $\varkappa$. Then X admits a fundamental bounded biorthogonal system.*

Proof. From Lemma 2 it follows that X admits a bounded biorthogonal system $(x_n^a, f_n^a)_{a\in\varkappa, n\in N}$ with $[x_n^a]+(f_n^a)^{\top}$ dense in X and, letting $T\colon X\to X/(f_n^a)^{\top}$ denote the quotient map, 0 is a weak cluster point of $(Tx_n^a)_{n=1}^{\infty}$ for each $a\in\varkappa$. Let $(y_a)_{a\in\varkappa}$ be dense in the unit ball of $(f_n^a)^{\top}$ and, for each $a\in\varkappa$, define

$$w_n^a = -y_a + x_n^a - x_{n+1}^a \quad \text{for } n = 1, 2, \ldots,$$
$$g_1^a = f_1^a,$$
$$g_n^a = f_{n-1}^a + f_n^a \quad \text{for } n = 2, 3, \ldots$$

Then (w_n^a, g_n^a) is a bounded biorthogonal system. We complete the proof by showing that $(w_n^a)^{\perp} = \{0\}$.

Suppose $x^*\in (w_n^a)^{\perp}$. Then for each $a\in\varkappa$ and $n = 1, 2, \ldots$, $nx^*(y_a) = x^*(x_1^a) - x^*(x_{n+1}^a)$, hence by the boundedness of (x_n^a), $x^*\in (y_a)^{\perp}$

$= (X/(f_n^a)^\top)^*$. But the nfor each $a \in \varkappa$, $x^*(x_1^a) = x^*(x_2^a) = x^*(x_3^a) = \ldots$ and, since $(Tx_n^a)_{n=1}^\infty$ has 0 as a weak cluster point, we have $x^* \in (x_n^a)^\perp$. Thus $x^* = 0$ by the denseness of $[x_n^a]+(f_n^a)^\top$. ∎

Remark 3. Of course it is a particular case of the theorems that every reflexive Banach space admits a fundamental bounded biorthogonal system. It follows by duality that every reflexive space also admits a total bounded biorthogonal system. A more general result than this latter one follows easily from a recent argument of Singer's: a trivial modification of Singer's proof of Theorem 1 in [11] shows that the Banach space Z admits a bounded total biorthogonal system of cardinality dens $Z = \varkappa > \varkappa_0$ provided Z has a subspace Y with dens $Y = \varkappa$ and Y admits a total, fundamental, bounded biorthogonal system. Now if Z contains a weakly compactly generated subspace X with dens $X = \varkappa$, then such a subspace Y exists. Indeed, letting $\{P_a\colon a \leqslant \varkappa\}$ be a "long sequence" of projections on X satisfying (a), (b), and (c) of the proof of Lemma 2 above; selecting unit vectors $y_a \in [P_{a+1}-P_a]X$; and setting $Y = [y_a]$; we have that the functionals (y_a^*) on Y^* biorthogonal to (y_a) are total over Y and $\|y_a^*\| \leqslant \|P_{a+1}-P_a\| \leqslant 2$.

Remark 4. Since $m(\varkappa)$ (the space of bounded scalar valued functions on the infinite cardinal $\varkappa$) has a quotient isomorphic to a Hilbert space of orthogonal dimension $2^\varkappa$ (cf. [7], p. 203), $m(\varkappa)$ admits a fundamental bounded biorthogonal system. Obviously $m(\varkappa)$ also admits a total bounded biorthogonal system; however, $m(\varkappa)$ does not admit a total, fundamental biorthogonal system [5].

Remark 5. The fact that the construction in Theorem 2 produces fundamental biorthogonal systems (x_a, x_a^*) with $X/(x_a^*)^\top$ weakly compactly generated is not purely accidental: the argument of [5] shows that if (x_a, x_a^*) is a fundamental biorthogonal system for a Grothendieck space X (i.e., weak* convergent sequences in X^* are weakly convergent) then $[x_a^*]$ — and, consequently, also $X/(x_a^*)^\top$ — is reflexive. Thus if X is a Grothendieck space, the following are equivalent: (a) X admits a fundamental bounded biorthogonal system; (b) X admits a fundamental biorthogonal system; (c) X has a reflexive quotient with density character dens X.

PROBLEM. *Does every Banach space have a (bounded) fundamental biorthogonal system?*

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