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On the existence of global weak solutions to a generalized Keller Segel model with arbitrary growth and nonlinear signal production

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Abstract. In this work we present the mathematical analysis of a system able to describe the biological chemotaxis phenomena. The proposed model is a modification of the classical Keller Segel model and its subsequent developments, which, in many cases, have been developed to obtain models that prevent the non-physical blow up of solutions. We are concerned with the global existence in $L^2(\Omega)$ of weak global solutions to a class of parabolic-elliptic chemotaxis systems encompassing the prototype

$$u_t - \bigtriangledown.(\bigtriangledown u - \chi u \bigtriangledown v) = f(u), \quad x \in \Omega, t > 0,$$
$$-\Delta v + v = u^\gamma, \quad x \in \Omega, t > 0,$$

with nonnegative initial condition for u and no flux boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n \quad (n \geqslant 2)$, where $\chi > 0$, $0 < \gamma < 1$ and $f \in C^1(\mathbb{R})$ satisfying, $f(0) = 0$ and

$$f(s) \leqslant 0, \quad s \geqslant 0.$$

It is shown under those conditions that the problem admits weak solutions in $L^2(\Omega)$. In order to develop the mathematical analysis of our model, we define an approximating scheme with more regular initial conditions, then we make some estimations that will allow us to prove that the solution of the approximated system converge to the solution of our problem.

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1. Introduction

We talk about chemotaxis phenomenon when the movement of organisms (cells, bacteria) is affected or even directed by the presence of a chemical substance. This movement is characterized by both repulsion and attraction phenomena, and in the latter case, the chemical is called a chemoattractant. For example, cells may be attracted to nutrients or repelled in the presence of a substance which is toxic to them. A more interesting example is that of the amoebae Dyctyostelium discoideum which, in cases of lack of nutrients, start to secrete adenosine monophosphate cyclic (cAMP) that attracts other amoebae. Chemotaxis is revealed to be a powerful means of communication between amoebae that induces a collective movement. It has been observed aggregation phenomena where amoebae, initially monocellular, ultimately form a society, i.e. a multicellular organism. It can then move to get food or form like a stem at the end of which spores are created. These ones are then projected away in the hope of a more lenient environment, the cells forming the stem are sacrificing

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themselves for the survival of the society. To learn more about life social amoeba Dyctyostelium discoideum, we refer the reader to the article [3].

Keller and Segel [6] derived the first mathematical model describing the aggregation process of amoebae by chemotaxis and nowadays it is called Keller Segel model. Then several modifications of the original model have been done by various authors, with the aim of improving its consistency with the biological reality. The celebrated model has attracted applied mathematicians and has lead to many challenging problems; one can see [11, 12, 14, 15, 16, 18, 17]. The Keller Segel model, consists in two parabolic (some times one parabolic and one elliptic) partial differential equations for the cell density and chemo-attractant density.

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u - \chi div(u\nabla v) = f(u) & \text{in } Q_T =]0,T[\times\Omega \\ \tau\frac{\partial v}{\partial t} - \Delta v + v = g(u) & \text{in } Q_T \\ \frac{\partial u}{\partial v} = \frac{\partial v}{\partial v} = 0 & \text{in } \sum_T =]0,T[\times\partial\Omega \\ u(0,x) = u_0(x); v(0,x) = v_0(x) & \text{in } \Omega \end{cases} \tag{1.1}$$

with $\tau\epsilon\{0,1\}$, where $\Omega\subset\mathbb{R}^n$ $(n\geq\ 2)$ is a bounded domain with smooth boundary $\partial\Omega$ and $\frac{\partial}{\partial v}$ denote the derivative with respect to the outward normal vector ν of $\partial\Omega$. $u(x,t)$ denotes the cell density and $v(x,t)$ denotes the concentration of the chemoattractant. $\chi(>0)$ is referred to as the chemotactic sensitivity coefficient measuring the strength of chemotaxis. The kinetic term f describes cell proliferation and death and $g(u)$ accounts for the chemical secretion by cells. A diffusion hypothesis is made for both the cells and the chemical product. The flow of cells due to the chemoattractant is assumed proportional to the gradient of the concentration of chemoattractant. The system presents two time scales, which justify the possibility of taking $\tau=0$.

As already mentioned the mathematical modelling of cell movement goes back to Patlak (1953), E. Keller and L. Segel (70s). This simplified system was first introduced for the case f(u) = 0 and g(u) = u (minimal model) and thereafter was studied by other authors in various contexts. It has been well-known that when $f(u)=0$ and $g(u)=u$, the minimal model possesses blow-up solutions in finite/infinite time in two or higher dimensions (see [20, 21, 22]). This limits the value of the model to explain the aggregation phenomena observed in experiment. The question for the system (1.1) is whether or not the appearance of growth source $f(u)$ can enforce the boundedness of solutions so that blow-up is inhibited.

Toward this end, many efforts have been made first for the linear chemical production and the logistic source:

$$f(u) = ru - \mu u^2 \text{ and } g(u) = u. \tag{1.2}$$

First, Osaki et al [5] showed that in the case n=2, the model (1.1) with $\tau=1$ and (1.2) has a classical uniform in time bounded solution for any $r\in\mathbb{R}$, $\mu>0$. Concerning higher dimensions ($n\geq 3$, Winkler [13] proved, under the logistic source:

$$f(u) = au - bu^2,\ f(0)\geq 0,\ a\geq 0,\ b>0,\ u\geq 0, \tag{1.3}$$

there exists a large positive number b_0 such that if $b>b_0$, then the chemotaxis-growth system (1.1) with $\tau=1$ and $g(u)=u$ has a classical uniform in time bounded solutions.

The existence of global weak solutions to (1.1) is newly known for $\mu>0$ in convex domains (see [4]). Some progress for (1.1) ($\tau=0$) has been made by Tello and Winkler (2007) wherein they showed that for $f(u)\leq a-bu^2$, $f(0)\geq 0$, $a\geq 0$, $b>0$, $u\geq 0$

and $g(u)=u$ and $b>b_0=(n-2)\chi/n$ the system admits globally bounded classical solutions.

This paper is devoted to the existence of weak solutions to the following chemotaxis system with nonlinear production of signal and growth source:

$$\begin{cases} \frac{\partial u}{\partial t}-\Delta u-\chi div(u\nabla v)=f(u) & \text{in } Q_T=]0,T[\times\Omega \\ -\Delta v+v=g(u) & \text{in } Q_T \\ \frac{\partial u}{\partial v}=\frac{\partial v}{\partial v}=0 & \text{in } \sum_T=]0,T[\times\partial\Omega \\ u(0,x)=u_0(x) & \text{in } \Omega \end{cases} \tag{1.4}$$

2. Mathematical analysis of the problem

2.1. Position problem. We suggest to consider the chemotaxis-growth model (1.4) with $0<\gamma<1$, more general conditions on $f(u)$ and the following less regular nonnegative initial data:

- $f:\mathbb{R}\longrightarrow\mathbb{R}$, $f\in C^1(\mathbb{R})$ with $f(0)=0$ and

$$f(u)\leqslant 0,\ \text{ for all } u\geqslant 0, \tag{2.1}$$

- $$u_0\in L^2(\Omega),\quad u_0\geq 0 \tag{2.2}$$

Before stating the main result of this paper, we have to clarify in which sense we want to solve problem (1.4).

Definition 2.1. (u,v) is a weak solution of (1.4) if and only if

$$\begin{cases} u\in C([0,T],L^2(\Omega))\cap L^2(0,T,H^1(\Omega)), v\in L^\infty\left(0,T,H^1(\Omega)\right), f(u)\in L^1(Q_T) \\ \bullet \text{ for every } \varphi\in C^1(Q_T) \text{ such that } \varphi(T,.)=0 \\ \int_{Q_T}(-u\frac{\partial\varphi}{\partial t}+\nabla u\nabla\varphi+\chi u\nabla v\nabla\varphi)=\int_\Omega u_0(x)\varphi(0,x)+\int_{Q_T}f(u)\varphi \\ \bullet \text{ for all } \psi\in H^1(\Omega) \text{ and a.e } 0<t<T \\ \int_\Omega\nabla v\nabla\psi+\int_\Omega v\psi=\int_\Omega u^\gamma\psi \end{cases} \tag{2.3}$$

2.2. Main result. The main result of this paper is the following theorem.

Theorem 2.1. *We suppose that the hypothesis (2.1) and (2.2) are satisfied, then the problem (1.4) admits a weak solution (u,v) satisfying $u\geq 0$ and $v\geq 0$, in Q_T.*

2.3. Proof of the main result. In order to develop the mathematical analysis of our model, we define an approximating scheme with a more regular initial condition in $C(\overline{\Omega})$, then we show the existence solutions for this approached problem. Finally by making some estimations we prove that the solution of the approximated problem converge to the solution of our problem.

2.3.1. *Approximating scheme.* We associate to the function f the function f_m such that

$$f_m(r)=\frac{-r^2}{m}+\frac{f(r)}{1+\frac{|f(r)|}{m}}$$

Now, let's consider the following approximated system

$$\begin{cases} \frac{\partial}{\partial t}u_m - \Delta u_m - \chi div(u_m \nabla v_m) = f_m(u_m) & \text{in } Q_T \\ -\Delta v_m + v_m = u_m^\gamma & \text{in } Q_T \\ \frac{\partial u_m}{\partial v} = \frac{\partial v_m}{\partial v} = 0 & \text{in } \sum_T \\ u_m(0,x) = u_m^0(x) & \text{in } \Omega \end{cases} \tag{2.4}$$

where $u_m^0 \in C(\overline{\Omega})$, furthermore $u_m^0 \to u_0$ strongly in $L^2(\Omega)$.

The existence of (u_m, v_m) solution to the chemotaxis-growth system (2.4) is ensured by the work of Wang and Xiang [7] (one can see theorem 4.1), because $0 < \gamma < 1$ and the growth function f_m is defined such that there are $a_m \geqslant 0$ and $b_m > 0$ such that

$$f_m(r) \leqslant a_m - b_m r^2, \text{ for all } r \geqslant 0.$$

As $f_m(0) = 0$, the maximum principle ensure that both u_m and v_m are nonnegative, as shown in [19]. By integrating the equation on u_m in (2.4) and using (2.1), we have

$$\frac{d}{dt}\int_\Omega u_m = \int_\Omega f_m(u_m) \leqslant 0,$$

which yields that the L^1-norm of u_m is uniformly bounded.

2.3.2. *A priori estimates.*

Till the end of this paper we design by C every generic and positive constant. This constant can change its value in different situations, can depend on γ, $|u_0|_{L^2}(\Omega)$, and $|\Omega|$ but remains independent of m. In this part we give estimations concerning u_m, v_m in appropriate spaces. We start by proving in the following lemma, that $\sup\limits_{0\leq t\leq T} (\|u_m(t)\|_{L^2(\Omega)} + \|v_m(t)\|_{L^2(\Omega)} + \|\nabla v_m(t)\|_{L^2(\Omega)})$ is bounded independently of m.

Lemma 2.2. *There exist a constant $C = C(\|u_0\|_{L^2(\Omega)}, \gamma, |\Omega|)$ such that*

(i) $\underset{0\leq t\leq T}{sup} \|u_m(t)\|_{L^2(\Omega)} \leq \int_\Omega (u_0)^2$

(ii) $\int_0^T \int_\Omega |\nabla u_m|^2 \leq \frac{1}{2}\int_\Omega (u_0)^2$

(iii) $\underset{0\leq t\leq T}{sup} \|v_m(t)\|_{L^2(\Omega)} \leq C$

(iv) $\underset{0\leq t\leq T}{sup} \|\nabla v_m(t)\|_{L^2(\Omega)} \leq C$

Proof. (i) and (ii): Multiplying the u_m-equation in (2.4) by u_m and integrating over Ω by parts

$$\frac{1}{2}\frac{d}{dt}\int_0^t\int_\Omega u_m^2 + \int_0^t\int_\Omega |\nabla u_m|^2 + \chi\int_0^t\int_\Omega u_m\nabla(u_m)\nabla v_m = \int_0^t\int_\Omega f_m(u_m)u_m \leq 0$$

we end up with

$$\frac{1}{2}\frac{d}{dt}\int_0^t\int_\Omega u_m^2 + \int_0^t\int_\Omega |\nabla u_m|^2 + \chi\int_0^t\int_\Omega u_m\nabla(u_m)\nabla v_m \leq 0$$

which implies

$$\frac{1}{2}\frac{d}{dt}\int_0^t\int_\Omega u_m^2 + \int_0^t\int_\Omega |\nabla u_m|^2 \leq \chi\int_0^t\int_\Omega u_m^2\Delta v_m$$

we have $0 = \Delta v_m - v_m + u_m^\gamma$, so it follows

$$\frac{1}{2}\frac{d}{dt}\int_0^t\int_\Omega u_m^2 + \int_0^t\int_\Omega |\nabla u_m|^2 \le \chi(\int_0^t\int_\Omega u_m^2 v_m - \int_0^t\int_\Omega u_m^{2+\gamma})$$

upon a use of Young inequality, we get

$$u_m^2 v_m \le \frac{2}{2+\gamma}\left(u_m^2\right)^{\frac{2+\gamma}{2}} + \frac{\gamma}{2+\gamma} v_m^{\frac{2+\gamma}{\gamma}}$$

then,

$$\int_0^t\int_\Omega u_m^2 v_m \le \int_0^t\int_\Omega \frac{2}{2+\gamma}u_m^{2+\gamma} + \int_0^t\int_\Omega \frac{\gamma}{2+\gamma} v_m^{\frac{2+\gamma}{\gamma}}$$

then it follows that

$$\frac{1}{2}\frac{d}{dt}\int_0^t\int_\Omega u_m^2 + \int_0^t\int_\Omega |\nabla u_m|^2 \le \chi(\int_0^t\int_\Omega \frac{2}{2+\gamma}u_m^{2+\gamma} + \int_0^t\int_\Omega \frac{\gamma}{2+\gamma} v_m^{\frac{2+\gamma}{\gamma}} - \int_0^t\int_\Omega u_m^{2+\gamma})$$

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\textstyle\int_0^t\int_\Omega u_m^2 + \int_0^t\int_\Omega |\nabla u_m|^2 \quad &\le \chi(\textstyle\int_0^t\int_\Omega \tfrac{2}{2+\gamma}u_m^{2+\gamma} + \int_0^t\int_\Omega \tfrac{\gamma}{2+\gamma} v_m^{\frac{2+\gamma}{\gamma}} - \int_0^t\int_\Omega u_m^{2+\gamma}) \\ &\le \chi(\tfrac{2}{2+\gamma} - 1)\textstyle\int_0^t\int_\Omega u_m^{2+\gamma} + \chi\tfrac{\gamma}{2+\gamma}\int_0^t\int_\Omega v_m^{\frac{2+\gamma}{\gamma}}\end{aligned}$$

which immediately gives

$$\frac{1}{2}\frac{d}{dt}\int_0^t\int_\Omega u_m^2 + \int_0^t\int_\Omega |\nabla u_m|^2 \le \frac{-\gamma\chi}{2+\gamma}\int_0^t\int_\Omega u_m^{2+\gamma} + \frac{\gamma\chi}{2+\gamma}\int_0^t\int_\Omega v_m^{\frac{2+\gamma}{\gamma}} \tag{2.5}$$

Multiplying the v_m-equation in (2.4) by $v_m^{\frac{2}{\gamma}}$ and integrating over Ω by parts

$$\int_0^t\int_\Omega v_m^{\frac{2}{\gamma}+1} + \frac{2}{\gamma}\int_0^t\int_\Omega v_m^{\frac{2}{\gamma}-1}|\nabla v_m|^2 = \int_0^t\int_\Omega u_m^\gamma v_m^{\frac{2}{\gamma}}$$

using Young inequality yields

$$u_m^\gamma v_m^{\frac{2}{\gamma}} \le \frac{2}{2+\gamma} v_m^{\frac{2+\gamma}{\gamma}} + \frac{\gamma}{2+\gamma}u_m^{\gamma+2}$$

then

$$\int_0^t\int_\Omega v_m^{\frac{\gamma+2}{\gamma}} + \frac{8\gamma}{(\gamma+2)^2}\int_0^t\int_\Omega \left|\nabla v_m^{\frac{\gamma+2}{2\gamma}}\right|^2 \le \frac{2}{2+\gamma}\int_0^t\int_\Omega v_m^{\frac{2+\gamma}{\gamma}} + \frac{\gamma}{2+\gamma}\int_0^t\int_\Omega u_m^{\gamma+2}$$

which implies

$$\frac{\gamma}{2+\gamma}\int_0^t\int_\Omega v_m^{\frac{\gamma+2}{\gamma}} + \frac{8\gamma}{(\gamma+2)^2}\int_0^t\int_\Omega \left|\nabla v_m^{\frac{\gamma+2}{2\gamma}}\right|^2 \le \frac{\gamma}{2+\gamma}\int_0^t\int_\Omega u_m^{\gamma+2} \tag{2.6}$$

combining (2.5) and (2.6) gives

$$\frac{1}{2}\frac{d}{dt}\int_0^t\int_\Omega u_m^2 + \int_0^t\int_\Omega |\nabla u_m|^2 \le \frac{-\gamma\chi}{2+\gamma}\int_0^t\int_\Omega u_m^{2+\gamma} - \frac{8\gamma\chi}{(\gamma+2)^2}\int_0^t\int_\Omega \left|\nabla v_m^{\frac{\gamma+2}{2\gamma}}\right|^2 + \frac{\gamma\chi}{2+\gamma}\int_0^t\int_\Omega u_m^{2+\gamma} \tag{2.7}$$

which implies

$$\frac{1}{2}\frac{d}{dt}\int_0^t\int_\Omega u_m^2 + \int_0^t\int_\Omega |\nabla u_m|^2 + \frac{8\gamma\chi}{(\gamma+2)^2}\int_0^t\int_\Omega \left|\nabla v_m^{\frac{\gamma+2}{2\gamma}}\right|^2 \le 0$$

Finally we conclude that

$$\begin{cases} \sup\limits_{0\leq t\leq T} \int_\Omega u_m^2(t) \leq \int_\Omega (u_0)^2 \\ \int_0^T \int_\Omega |\nabla u_m|^2 \leq \frac{1}{2}\int_\Omega (u_0)^2 \end{cases}$$

(iii) and (iv): Multiplying the v_m-equation in (2.4) by v_m and integrating over Ω by parts

$$-\int_\Omega \Delta v_m v_m + \int_\Omega v_m^2 = \int_\Omega u_m^\gamma v_m$$

which implies

$$\int_\Omega |\nabla v_m|^2 + \int_\Omega v_m^2 = \int_\Omega u_m^\gamma v_m$$

a simple use of Young's inequality, directly yields that

$$\int_\Omega u_m^\gamma v_m \leq \frac{\gamma}{2}\int_\Omega u_m^2 + \frac{2-\gamma}{2}\int_\Omega v_m^{\frac{2}{2-\gamma}}$$

we deduce

$$\int_\Omega |\nabla v_m|^2 + \int_\Omega v_m^2 \leq \frac{\gamma}{2}\int_\Omega u_m^2 + \frac{2-\gamma}{2}\int_\Omega v_m^{\frac{2}{2-\gamma}}.$$

using young inequality gives

$$\int_\Omega v_m^{\frac{2}{2-\gamma}} \leq \epsilon \int_\Omega v_m^2 + \epsilon^{\frac{1}{1-\gamma}} |\Omega|$$

which implies

$$\int_\Omega |\nabla v_m|^2 + \int_\Omega v_m^2 \leq \frac{\gamma}{2}\int_\Omega u_m^2 + \frac{\epsilon(2-\gamma)}{2}\int_\Omega v_m^2 + \frac{\epsilon^{(\frac{1}{1-\gamma})}(2-\gamma)}{2}|\Omega|$$

By choosing $\epsilon < \frac{2}{2-\gamma}$, we get

$$\int_\Omega |\nabla v_m|^2 + \int_\Omega v_m^2 \leq C(\|u_0\|_{L^2(\Omega)}, \gamma, |\Omega|).$$

Finally

$$\begin{cases} \sup\limits_{0\leq t\leq T} \|v_m(t)\|_{L^2(\Omega)} \leq C \\ \sup\limits_{0\leq t\leq T} \|\nabla v_m(t)\|_{L^2(\Omega)} \leq C \end{cases}$$

□

Concerning the term $f_m(u_m)$, we have the following result.

Lemma 2.3. (i) *There exist a constant $C = C(\|u_0\|_{L^1(\Omega)})$ independent on m, such that*

$$\|f_m(u_m)\|_{L^1(Q_T)} \leq C$$

(ii) *There exist a constant $C = C(\|u_0\|_{L^2(\Omega)}, \gamma, |\Omega|)$ independent on m, such that*

$$\sup_{0\leq t\leq T} \|u_m(t) \bigtriangledown v_m(t)\|_{L^1(\Omega)} \leq C$$

Proof. (i) Let's consider the equation satisfied by u_m, we have

$$\frac{\partial u_m}{\partial t} - \Delta u_m - \chi div(u_m \nabla v_m) = f_m(u_m)$$

Then we integrate on Q_T

$$\int_0^T \int_\Omega |f_m(u_m)| = -\int_{Q_T} \frac{\partial u_m}{\partial t} = \int_\Omega u_m^0(x) - \int_\Omega u_m(T,x) \leq \int_\Omega u_m^0(x)$$

because we know that

$$u_m(T,x) \geq 0$$

we conclude that

$$\|f_m(u_m)\|_{L^1(Q_T)} \leq C = C(\|u_0\|_{L^1(\Omega)})$$

(ii) Using young inequality yields

$$\int_\Omega |u_m \nabla v_m| \leq \frac{1}{2}\int_\Omega u_m^2 + \frac{1}{2}\int_\Omega |\nabla v_m|^2 \leq C(\|u_0\|_{L^2(\Omega)}, \gamma, |\Omega|)$$

□

The following lemma gives estimation on $u_m f_m(u_m)$ in $L^1(Q_T)$. That estimation will be very important to fulfill the proof of the main result.

Lemma 2.4. *There exists a constant C such that:*

$$\|u_m f_m(u_m)\|_{L^1(Q_T)} \leq C$$

Proof. Multiplying the u_m-equation in (2.4) by u_m and integrating over Ω by parts

$$\frac{1}{2}\frac{d}{dt}\int_0^t \int_\Omega u_m^2 + \int_0^t \int_\Omega |\nabla u_m|^2 + \chi \int_0^t \int_\Omega u_m \nabla(u_m)\nabla v_m = \int_0^t \int_\Omega f(u_m)u_m$$

we end up with

$$\frac{1}{2}\int_0^t \int_\Omega u_m^2 - \frac{1}{2}\int_0^t \int_\Omega (u_m^0)^2 + \int_0^t \int_\Omega |\nabla u_m|^2 + \chi \int_0^t \int_\Omega u_m \nabla(u_m)\nabla v_m = \int_0^t \int_\Omega f(u_m)u_m$$

then

$$\int_0^t \int_\Omega |f(u_m)u_m| = \frac{1}{2}\int_0^t \int_\Omega (u_m^0)^2 - \frac{1}{2}\int_0^t \int_\Omega u_m^2 - \int_0^t \int_\Omega |\nabla u_m|^2 - \chi \int_0^t \int_\Omega u_m \nabla(u_m)\nabla v_m$$

which implies

$$\int_0^t \int_\Omega |f(u_m)u_m| = \frac{1}{2}\int_0^t \int_\Omega (u_m^0)^2 - \chi \int_0^t \int_\Omega u_m \nabla(u_m)\nabla v_m \tag{2.8}$$

Multiplying the v_m-equation in (2.4) by u_m^2 and integrating over Ω by parts

$$-\int_\Omega \Delta v_m u_m^2 + \int_\Omega v_m u_m^2 = \int_\Omega u_m^{\gamma+2}$$

which implies

$$2\int_0^t \int_\Omega u_m \nabla(u_m)\nabla v_m + \int_0^t \int_\Omega v_m u_m^2 = \int_0^t \int_\Omega u_m^{\gamma+2}$$

then

$$-\int_0^t \int_\Omega u_m \nabla(u_m)\nabla v_m = \frac{1}{2}\int_0^t \int_\Omega v_m u_m^2 - \frac{1}{2}\int_0^t \int_\Omega u_m^{\gamma+2}$$

a simple use of Young's inequality, directly yields that

$$\int_0^t \int_\Omega u_m^2 v_m \leq \frac{2}{\gamma+2} \int_0^t \int_\Omega u_m^{\gamma+2} + \frac{\gamma}{\gamma+2} \int_0^t \int_\Omega v_m^{\frac{\gamma+2}{\gamma}}$$

we deduce

$$-\int_0^t \int_\Omega u_m \nabla(u_m) \nabla v_m \leq -\frac{1}{2} \int_0^t \int_\Omega u_m^{\gamma+2} + \frac{1}{\gamma+2} \int_0^t \int_\Omega u_m^{\gamma+2} + \frac{\gamma}{2(\gamma+2)} \int_0^t \int_\Omega v_m^{\frac{\gamma+2}{\gamma}}$$

then

$$-\int_0^t \int_\Omega u_m \nabla(u_m) \nabla v_m \leq \frac{-\gamma}{2(\gamma+2)} \int_0^t \int_\Omega u_m^{\gamma+2} + \frac{\gamma}{2(\gamma+2)} \int_0^t \int_\Omega v_m^{\frac{\gamma+2}{\gamma}} \tag{2.9}$$

Using (2.5) in (2.8) gives

$$-\int_0^t \int_\Omega u_m \nabla(u_m) \nabla v_m \leq \frac{-4\gamma}{(\gamma+2)^2} \int_0^t \int_\Omega \left| \nabla v_m^{\frac{\gamma+2}{2\gamma}} \right|^2$$

Finally

$$\int_0^t \int_\Omega |f(u_m) u_m| \leq \frac{1}{2} \int_0^t \int_\Omega (u_0)^2$$

□

2.3.3. *Convergence.* The point is to show that (u_m, v_m) solution of the problem (2.4) converge to (u, v) solution of (1.4).
Considering the v_m-equation, we already know that $\sup\limits_{0\leq t\leq T} \int_\Omega u_m^\gamma \leq C$ (this can be obtained by testing the v_m-equation by 1) , then by using the compactness theorem [2] we can deduce, up to extracting subsequence if necessary, the following convergences for all $t \in (0, T)$

$$\begin{cases} v_m(t) \to v(t) & \text{in } L^1(\Omega) \text{ and a.e. in } Q_T. \\ \nabla v_m(t) \to \nabla v(t) & \text{in } L^1(\Omega) \text{ and a.e. in } \Omega. \end{cases}$$

Furthermore, we have $\Delta u_m \in L^1(0, T, (H^1(\Omega))')$, $\nabla(u_m \nabla v_m) \in L^1(0, T, (H^1(\Omega))')$ and $f_m(u_m)$ bounded in $L^1(Q_T)$, which yields from Aubin-Simon compactness [1] $\partial_t u_m$ is bounded in $L^1(0, T, (H^1(\Omega))') + L^1(Q_T)$.
Consequently,up to a subsequence also denoted by u_m

$$u_m \to u \text{ in } L^2(Q_T) \text{ strongly, and a.e.}$$

Then,

$$\partial_t u_m - \Delta u_m \to \partial_t u - \Delta u \quad \text{in } D'(Q_T).$$

As ∇v_m is bounded in $L^2(Q_T)$, which is a reflexive space, then $(\nabla v_m)_m$ converges weakly in $L^2(Q_T)$. then,

$$\nabla v_m \to \nabla v \text{ weakly in } L^2(Q_T)$$

Consequently,

$$u_m \nabla v_m \to u \nabla v \text{ weakly in } L^2(Q_T)$$

Then

$$\nabla(u_m \nabla v_m) \to \nabla(u \nabla v) \text{ in } D'(Q_T)$$

Consequently

$$u_m - \Delta u_m - \nabla(u_m \nabla v_m) \to u - \Delta u - \nabla(u \nabla v) \text{ in } D'(Q_T)$$

Thanks to vitali theorem, to prove that $f_m(u_m)$ converge to $f(u)$ in $L^1(Q_T)$ is equivalent to prove that $f_m(u_m)$ is equi-integrable in $L^1(Q_T)$. We have the following lemma:

Lemma 2.5. *$f_m(u_m)$ is equi-integrable in $L^1(Q_T)$.*

Proof. Let be E a measurable set of Q_T. We have

$$\int_E |f_m(u_m)| \leq \int_{E\cap[u_m\leq k]} |f_m(u_m)| + \frac{1}{k}\int_{E\cap[u_m>k]} u_m\,|f_m(u_m)|$$

However

$$\begin{aligned}\int_{E\cap[u_m\leq k]} |f_m(u_m)| &\leq \max_{0\leq|r|\leq k} |f(r)|\,.\,|E| \\ ... &\leq C(k)\,|E|\end{aligned}$$

according to Lemma 2.3

$$\frac{1}{k}\int_{E\cap[u_m>k]} u_m\,|f_m^n(u_m)| \leq \frac{C(T)}{k}$$

by choosing k sufficiently large, we deduce

$$\int_{E\cap[u_m\leq k]} |f_m(u_m)| \leq \frac{\varepsilon}{2} \text{ and } \frac{1}{k}\int_{E\cap[u_m>k]} u_m\,|f_m(u_m)| \leq \frac{\varepsilon}{2}$$

consequently, $f_m(u_m)$ is equi-integrable in $L^1(Q_T)$. □

Furthermore we have

$$-\Delta v_m + v_m = u_m^\gamma \to -\Delta v + v = u^\gamma \quad \text{in} \quad D'(Q_T)$$

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