Tohoku Math. J.
61 (2009), 241–252

ON THE EXISTENCE OF KÄHLER METRICS OF CONSTANT SCALAR CURVATURE

KENJI TSUBOI

(Received February 7, 2008, revised December 17, 2008)

Abstract. For certain compact complex Fano manifolds M with reductive Lie algebras of holomorphic vector fields, we determine the analytic subvariety of the second cohomology group of M consisting of Kähler classes whose Bando-Calabi-Futaki character vanishes. Then a Kähler class contains a Kähler metric of constant scalar curvature if and only if the Kähler class is contained in the analytic subvariety. On examination of the analytic subvariety, it is shown that M admits infinitely many nonhomothetic Kähler classes containing Kähler metrics of constant scalar curvature but does not admit any Kähler-Einstein metric.

1. Introduction. The question of whether a manifold admits a Riemannian metric of constant scalar curvature or not is a classical problem. For any real closed manifold M of dimension greater than two, Kazdan and Warner [10] proved that M admits at least a Riemannian metric of negative constant scalar curvature. On the other hand, there exists an obstruction to the existence of Kähler metrics of constant scalar curvature. Indeed, let M be an m-dimensional compact complex manifold. Denote by $\mathrm{Aut}(M)$ the complex Lie group consisting of all biholomorphic automorphisms of M and by $\mathfrak{h}(M)$ its Lie algebra consisting of all holomorphic vector fields on M. The Lie algebra $\mathfrak{h}(M)$ is called reductive if $\mathfrak{h}(M)$ is the complexification of the Lie algebra of a compact subgroup of $\mathrm{Aut}(M)$. In [14], Matsushima proved that $\mathfrak{h}(M)$ is the complexification of the real Lie algebra consisting of all infinitesimal isometries of M, and hence $\mathfrak{h}(M)$ is reductive, if M admits a Kähler-Einstein metric. Generalizing the result of Matsushima, Lichnerowicz proved in [12], [13] that $\mathfrak{h}(M)$ must satisfy a certain condition if M admits a Kähler metric of constant scalar curvature. (For details see also [11, Theorem 6.1].) When M is a compact simply connected Kähler manifold, the condition of Lichnerowicz is equivalent to that of Matsushima. For example, the one point blow-up of $\boldsymbol{CP}^2$ does not satisfy the condition (see [5, p. 100]) and hence does not admit any Kähler metric of constant scalar curvature. Thus the problem to solve is whether M with reductive $\mathfrak{h}(M)$ admits a Kähler metric of constant scalar curvature or not.

Generalizing the result of Futaki [3], Bando [1], Calabi [2] and Futaki [4] give an obstruction to the existence of a Kähler metric of constant scalar curvature whose Kähler form is contained in some particular Kähler class. Let Ω be a Kähler class, $\omega \in \Omega$ a Kähler form and s_ω the scalar curvature of ω. Let $c_1(M) \in H^2(M; \boldsymbol{Z})$ be the first Chern class of M and

---

2000 *Mathematics Subject Classification.* Primary 53C25; Secondary 53C55.
Key words and phrases. Kähler manifold, constant scalar curvature, Bando-Calabi-Futaki character.
The author is deeply grateful to the referee for valuable information.

set

$$\mu_\Omega = \frac{(\Omega^{m-1} \cup c_1(M))[M]}{\Omega^m[M]},$$

where $[M]$ denotes the fundamental cycle of M. Then there exists uniquely a smooth function h_ω up to constant such that

$$s_\omega - m\mu_\Omega = \triangle_\omega h_\omega,$$

and the integral

$$f_\Omega(X) = \int_M X h_\omega \omega^m$$

is defined for $X \in \mathfrak{h}(M)$. This integral $f_\Omega(X)$ is independent of the choice of Kähler forms $\omega \in \Omega$. Moreover, $f_\Omega : \mathfrak{h}(M) \to \boldsymbol{C}$ is a Lie algebra character and f_Ω vanishes if Ω contains a Kähler metric of constant scalar curvature. The character f_Ω is called the Bando-Calabi-Futaki character or the Futaki invariant.

When Ω is a Hodge class and a holomorphic line bundle L with $c_1(L) = \Omega$ admits a lifting of the Ω-preserving action of a subgroup G of Aut(M), in [16] Nakagawa gives a lifting of the Lie algebra character f_Ω to a group character $G \to \boldsymbol{C}/(\boldsymbol{Z} + \mu_\Omega \boldsymbol{Z})$ by using the results in [17] and [6].

Assume that there exists an inclusion $\iota : \mathrm{U}(1) \to \mathrm{Aut}(M)$ and that Ω is equal to the first Chern class of a holomorphic U(1)-line bundle L over M. For any integer $p \geq 2$ let Y denote the element $2\pi\sqrt{-1}$ of the Lie algebra of U(1) and set

$$X = \iota_* Y \in \mathfrak{h}(M), \quad X_p = \frac{1}{p} X \in \mathfrak{h}(M), \quad g_p = \exp X_p \in \mathrm{Aut}(M). \tag{1}$$

Then the order of g_p is p. We assume that the next assumption is satisfied. (See Assumption 2.2 and Lemma 2.3 in [7].)

ASSUMPTION 1.1. Assume that the fixed point set of g_p^k for $1 \leq k \leq p-1$ is independent of k and that the connected components $N_1, \ldots, N_n$ of the fixed point set, which are compact complex submanifolds of M, have cell decompositions with no codimension one cells.

Let α_p denote the primitive p-th root of unity defined by

$$\alpha_p = e^{2\pi\sqrt{-1}/p}$$

hereafter. Suppose that g_p^k acts on $K_M^{-1}|_{N_i}$ via multiplication by $\alpha_p^{kr_i}$ and acts on $L|_{N_i}$ via multiplication by $\alpha_p^{k\kappa_i}$. Suppose moreover that the normal bundle $\nu(N_i, M)$ is decomposed into the direct sum of subbundles

$$\nu(N_i, M) = \bigoplus_j \nu(N_i, \theta_j),$$

where g_p^k acts on $\nu(N_i, \theta_j)$ via multiplication by $e^{\sqrt{-1}\theta_j}$. Then a cohomology class $\Phi(\nu(N_i, M))$ is defined by

$$\Phi(\nu(N_i, M)) = \prod_j \prod_{k=1}^{R_j} \frac{1}{1 - e^{-x_k - \sqrt{-1}\theta_j}} \in H^*(N_i; \boldsymbol{C}) \quad (R_j = \mathrm{rank}_{\boldsymbol{C}}(\nu(N_i, \theta_j))),$$

where $\prod_k(1 + x_k)$ is equal to the total Chern class of $\nu(N_i, \theta_j)$. For $1 \leq k \leq p-1$, $\varepsilon = -1,\ 0,\ +1$ and an integer ζ, we define numbers $T_i(k, \varepsilon, \zeta)$ and $S_\varepsilon(\zeta)$ by

$$T_i(k, \varepsilon, \zeta) = \frac{1}{1 - \alpha_p^k}(\alpha_p^{k(-\varepsilon r_i + \zeta\kappa_i)} e^{-\varepsilon c_1(K_M^{-1}|_{N_i}) + \zeta c_1(L|_{N_i})} - 1)^{m+1} \mathrm{Td}(TN_i)\Phi(\nu(N_i, M))[N_i],$$

$$S_\varepsilon(\zeta) = \frac{1}{p}\sum_{i=1}^{n}\sum_{k=1}^{p-1} T_i(k, \varepsilon, \zeta),$$

where $\mathrm{Td}(TN_i)$ is the Todd class of TN_i. Then $F_L(g_p)$ is defined by

$$\begin{aligned} F_L(g_p) = (m+1)&\sum_{i=0}^{m}(-1)^i\binom{m}{i}(S_{-1}(m-2i) - S_{+1}(m-2i)) \\ &- m\mu_\Omega \sum_{i=0}^{m+1}(-1)^i\binom{m+1}{i} S_0(m+1-2i). \end{aligned}$$

The lifting of the character f_Ω given by Nakagawa is expressed by a Simons character of a certain foliation. In [7], we gives a localization formula for the Simons character under Assumption 1.1. The next theorem follows from [16, Theorem 4.7] and [7, Theorem 2.5].

THEOREM 1.2. *There exists a non-zero constant $A(m, n)$ determined only by m, n such that $F_L(g_p) \equiv A(m, n) f_\Omega(X_p)$ $(\mathrm{mod}\ \boldsymbol{Z} + \mu_\Omega \boldsymbol{Z})$.*

2. Main result. For $m,\ n \geq 1$, let $H_m,\ H_n$ be the hyperplane bundles over the complex projective spaces $\boldsymbol{CP}^m$, $\boldsymbol{CP}^n$ respectively, and

$$\pi_1 : H_m \to \boldsymbol{CP}^m, \quad \pi_2 : H_n \to \boldsymbol{CP}^n$$

their projections. Let $E = \pi_1^* H_m \oplus \pi_2^* H_n$ be the rank 2 vector bundle over $\boldsymbol{CP}^m \times \boldsymbol{CP}^n$. Let M be the total space of the projective bundle of E and J_M the tautological bundle of M. Then M is an $(m+n+1)$-dimensional simply-connected compact Kähler manifold and the same argument as in [3, Proposition 3.1] shows that M is a Fano manifold (see also [5, Proposition 4.2.1]) and the identity component of $\mathrm{Aut}(M)$ coincides with the factor group $(\mathrm{GL}(m+1, \boldsymbol{C}) \times \mathrm{GL}(n+1, \boldsymbol{C}))/\boldsymbol{C}^*$, where $\boldsymbol{C}^*$ is the center of $\mathrm{GL}(m+n+2, \boldsymbol{C})$. Hence the Lie algebra $\mathfrak{h}(M)$ is isomorphic to

$$\{(A, B) \in \mathfrak{gl}(m+1, \boldsymbol{C}) \oplus \mathfrak{gl}(n+1, \boldsymbol{C})\,;\ \mathrm{Tr}\, A + \mathrm{Tr}\, B = 0\},$$

which satisfies the condition of Matsushima.

Applying the Gysin exact sequence to the fibration

$$F = \boldsymbol{CP}^1 \to M \xrightarrow{p} B = \boldsymbol{CP}^m \times \boldsymbol{CP}^n,$$

we have the split exact sequence

$$\begin{aligned} H^{-1}(B;\boldsymbol{Z})=0 \;\to\; & H^2(B;\boldsymbol{Z})\simeq H^2(\boldsymbol{CP}^m;\boldsymbol{Z})\oplus H^2(\boldsymbol{CP}^n;\boldsymbol{Z})\\ &\xrightarrow{p^*} H^2(M;\boldsymbol{Z})\xrightarrow{f} H^0(B;\boldsymbol{Z})\simeq\boldsymbol{Z}\;\to\; H^3(B;\boldsymbol{Z})=0\,,\end{aligned}$$

where f is the integration along the fiber. Then $H_m,\ H_n$ are naturally regarded as vector bundles over $\boldsymbol{CP}^m\times\boldsymbol{CP}^n$, and since $f(c_1(J_M^*))=1$, it follows that

$$H^2(M;\boldsymbol{Z})=\{\lambda\tilde{u}+\mu\tilde{v}+\nu\tilde{w}\,;\ \lambda,\ \mu,\ \nu\in\boldsymbol{Z}\}\simeq\boldsymbol{Z}^3\,,$$

where $\tilde{u}=c_1(p^*H_m),\ \tilde{v}=c_1(p^*H_n)$ and $\tilde{w}=c_1(J_M^*)$.

REMARK 2.1. Let $\hat{u},\ \hat{v}$ be the first Chern forms of $H_m,\ H_n$, respectively. Then $x\hat{u}+y\hat{v}$ is a Kähler form on $\boldsymbol{CP}^m\times\boldsymbol{CP}^n$ for $x,\ y>0$, and hence $x\tilde{u}+y\tilde{v}+z\tilde{w}$ is a Kähler class of M for $x,\ y>0$ and sufficiently small $z>0$. Therefore the set of Kähler classes of M is contained in the subset $\{x\tilde{u}+y\tilde{v}+z\tilde{w}\,;\ x,\ y,\ z>0\}$ of $H^2(M;\boldsymbol{R})\simeq\boldsymbol{R}^3$.

Now, let $F(x,y,z)$ be an integral homogeneous polynomial of degree $m+n+4$ defined by

$$F(x,y,z)=-(m(m+2)yz+n(n+2)xz+2xy)g(x,y,z)+xyz\,h(x,y,z)\,,$$

where

$$\begin{aligned} g(x,y,z)=&\sum_{s=0}^{m+n}\sum_{q=0}^{m}\binom{m+n+2}{s}\binom{s}{m-q}\binom{m+n-s}{q}(-1)^{m+n+s+q+1}\\ &((x-z)^{m-q}y^{n+q+2}-x^{m-q}(y-z)^{n+q+2})\,,\\ h(x,y,z)=&\sum_{s=0}^{m+n}\sum_{q=0}^{m}\binom{m+n+2}{s}\binom{s}{m-q}\binom{m+n-s}{q}(-1)^{m+n+s+q+1}\\ &\begin{pmatrix}\{(m+n+2-s)+(n+2)(s-m+q)\}(x-z)^{m-q}y^{n+q+1}\\ \quad+m(m-q)(x-z)^{m-q-1}y^{n+q+2}\\ +\{(m+n+2-s)-n(s-m+q)\}x^{m-q}(y-z)^{n+q+1}\\ \quad-(m+2)(m-q)x^{m-q-1}(y-z)^{n+q+2}\end{pmatrix}.\end{aligned}$$

For example, if $(m,n)=(1,2)$, we have

$$\begin{aligned} F(x,y,z)=&\,120x^2y^3z^2-420x^2y^2z^3+390x^2yz^4-120x^2z^5+60xy^4z^2-90xy^3z^3\\ &+150xy^2z^4-99xyz^5+24xz^6-90y^4z^3+90y^3z^4-45y^2z^5+9yz^6\,.\end{aligned}$$

Our main result is the next theorem.

THEOREM 2.2. *The character f_Ω for $\Omega=x\tilde{u}+y\tilde{v}+z\tilde{w}$ vanishes if and only if $F(x,y,z)=0$. Hence the open subset of $H^2(M;\boldsymbol{R})\simeq\boldsymbol{R}^3$ defined by $F(x,y,z)\neq0$ does not contain any Kähler metric of constant scalar curvature.* (*See Remark* 3.2.)

REMARK 2.3. The group $\mathrm{Aut}(M)$ contains an $(m+n+1)$-dimensional algebraic torus. Hence M is toric and the character can be calculated also by the formula of Nakagawa [15].

3. Proof of the Theorem. Let $q \in M$, $q_m \in p^*H_m$, $q_n \in p^*H_n$ and $q_J \in J_M^*$ be points. Then the point q and the set (q_m, q_n, q_J) are expressed as follows:

$$\begin{aligned} q &= [(z_0, \dots, z_m), (w_0, \dots, w_n), (\eta_0, \eta_1)] \\ &= [(az_0, \dots, az_m), (bw_0, \dots, bw_n), (ca\eta_0, cb\eta_1)], \\ (q_m, q_n, q_J) &= [[(z_0, \dots, z_m), (w_0, \dots, w_n), (\eta_0, \eta_1)], h_m, h_n, \xi] \\ &= [[(az_0, \dots, az_m), (bw_0, \dots, bw_n), (ca\eta_0, cb\eta_1)], ah_m, bh_n, c\xi] \end{aligned}$$

for $a,\ b,\ c \in \boldsymbol{C}^*$.

REMARK 3.1. Since f_Ω vanishes on $[\mathfrak{h}(M), \mathfrak{h}(M)]$ and $\mathfrak{h}(M)/[\mathfrak{h}(M), \mathfrak{h}(M)]$ is represented by the vector field along the fiber $\boldsymbol{CP}^1$, the character f_Ω vanishes if and only if $f_\Omega(X) = 0$ for the vector field X along the fiber.

Now we assume that p is an odd prime number hereafter. Then an action of $\boldsymbol{Z}_p = \langle g_p \rangle \subset (\mathrm{GL}(m+1, \boldsymbol{C}) \times \mathrm{GL}(n+1, \boldsymbol{C}))/\boldsymbol{C}^*$ on M is defined by

$$\begin{aligned} &g_p \cdot [(z_0, \dots, z_m), (w_0, \dots, w_n), (\eta_0, \eta_1)] \\ &\qquad = [(z_0, \dots, z_m), (\alpha_p w_0, \dots, \alpha_p w_n), (\eta_0, \eta_1)]. \end{aligned} \tag{2}$$

This action naturally extends to an inclusion $\iota : \mathrm{U}(1) \to \mathrm{Aut}(M)$, which defines vector fields $X,\ X_p \in \mathfrak{h}(M)$ along the fiber as in (1) and we have $g_p = \exp(X_p)$. The fixed point set of g_p^k has the following two connected components

$$N_1 = [(z_0, \dots, z_m), (w_0, \dots, w_n), (1, 0)], \quad N_2 = [(z_0, \dots, z_m), (w_0, \dots, w_n), (0, 1)]$$

for $1 \le k \le p-1$, which are isomorphic to $\boldsymbol{CP}^m \times \boldsymbol{CP}^n$ and have cell decompositions with no codimension one cells. Let $\nu(N_i, M)$ be the normal bundle of N_i $(i = 1,\ 2)$ in M. Then, since

$$\begin{aligned} [(z_0, \dots, z_m), (w_0, \dots, w_n), (1, \tau)] &= [(az_0, \dots, az_m), (bw_0, \dots, bw_n), (1, a^{-1}b\tau)], \\ g_p \cdot [(z_0, \dots, z_m), (w_0, \dots, w_n), (1, \tau)] &= [(z_0, \dots, z_m), (w_0, \dots, w_n), (1, \alpha_p^{-1}\tau)], \end{aligned}$$

we have

$$\nu(N_1, M) \simeq H_m^{-1} \otimes H_n, \quad g_p|\nu(N_1, M) = g_p|(K_M^{-1}|_{N_1}) = \alpha_p^{-1}.$$

The same argument shows that

$$\nu(N_2, M) \simeq H_m \otimes H_n^{-1}, \quad g_p|\nu(N_2, M) = g_p|(K_M^{-1}|_{N_2}) = \alpha_p.$$

Hence it follows from the equality $c_1(K_M^{-1}|_{N_i}) = c_1(M)|_{N_i} = c_1(TN_i) + c_1(\nu(N_i, M))$ that

$$\begin{aligned} c_1(\nu(N_1, M)) = -u + v, &\quad c_1(\nu(N_2, M)) = u - v, \\ c_1(K_M^{-1}|_{N_1}) = mu + (n+2)v, &\quad c_1(K_M^{-1}|_{N_2}) = (m+2)u + nv, \end{aligned}$$

where $u = c_1(H_m),\ v = c_1(H_n)$. It is obvious that $\tilde{u}|_{N_i} = u$, $\tilde{v}|_{N_i} = v$ for $i = 1,\ 2$. Also, since

$$\begin{aligned} &[[(z_0, \dots, z_m), (w_0, \dots, w_n), (1, 0)], \xi] \\ &\qquad = [[(az_0, \dots, az_m), (bw_0, \dots, bw_n), (1, 0)], a^{-1}\xi], \end{aligned}$$

it follows that $\tilde{w}|_{N_1} = -u$. The same argument shows that $\tilde{w}|_{N_2} = -v$. Using the equalities above, we see that

$$c_1(M) = (m+2)\tilde{u} + (n+2)\tilde{v} + 2\tilde{w}\,,$$

and hence for $\Omega = x\tilde{u} + y\tilde{v} + z\tilde{w}$ it follows that

$$\mu_\Omega = \frac{m(m+2)yz + n(n+2)xz + 2xy}{(m+n+1)xyz}\,. \tag{3}$$

Let $\lambda,\ \mu,\ \nu$ be integers. Then $\Omega = \lambda\tilde{u} + \mu\tilde{v} + \nu\tilde{w}$ coincides with the first Chern class of the complex line bundle L defined by

$$L = p^* H_m^\lambda \otimes p^* H_n^\mu \otimes (J_M^*)^\nu\,.$$

The action (2) lifts to actions on $p^* H_m,\ p^* H_n,\ J_M^*$ as follows:

$$\begin{aligned} &g_p \cdot [[(z_0, \dots, z_m), (w_0, \dots, w_n), (\eta_0, \eta_1)], h_m, h_n, \xi] \\ &\quad = [[(z_0, \dots, z_m), (\alpha_p w_0, \dots, \alpha_p w_n), (\eta_0, \eta_1)], h_m, h_n, \xi]\,. \end{aligned}$$

This action defines a lift of the action (2) to L and we can show that

$$\begin{gathered} g_p|(p^* H_m|_{N_i}) = 1\,, \quad g_p|(p^* H_n|_{N_i}) = \alpha_p^{-1} \quad (i = 1,\ 2) \\ g_p|(J_M^*|_{N_1}) = 1\,, \quad g_p|(J_M^*|_{N_2}) = \alpha_p\,, \end{gathered}$$

and hence that

$$g_p|(L|_{N_1}) = \alpha_p^{-\mu}\,, \quad g_p|(L|_{N_2}) = \alpha_p^{-\mu+\nu}\,. \tag{4}$$

Using the results above, we have

$$\begin{aligned} T_i(k, \varepsilon, \zeta) = u^m v^n\text{-coeff. of} \\ &\frac{1}{1-\alpha_p^k}\left(\alpha_p^{k(-\varepsilon r + \zeta\kappa)} e^{-\varepsilon(au+bv)+\zeta(\rho u + \tau v)} - 1\right)^{m+n+2} \\ &\left(\frac{u}{1-e^{-u}}\right)^{m+1} \left(\frac{v}{1-e^{-v}}\right)^{n+1} \frac{1}{1-\alpha_p^{-k\delta} e^{-\delta(u-v)}}\,, \end{aligned}$$

where $r,\ \kappa,\ a,\ b,\ \rho,\ \tau,\ \delta$ are numbers determined by i as follows:

	r	κ	a	b	ρ	τ	δ
$i=1$	-1	$-\mu$	m	$n+2$	$\lambda-\nu$	μ	-1
$i=2$	1	$-\mu+\nu$	$m+2$	n	λ	$\mu-\nu$	1

Then, using the substitution $x = e^u - 1,\ y = e^v - 1$, we have

$$
\begin{aligned}
T_i(k, \varepsilon, \zeta) &= u^{-1}v^{-1}\text{-coeff. of} \\
&\qquad \frac{1}{1-\alpha_p^k}(\alpha_p^{k(-\varepsilon r+\zeta\kappa)} e^{u(\zeta\rho-\varepsilon a)} e^{v(\zeta\tau-\varepsilon b)} - 1)^{m+n+2} \\
&\qquad \left(\frac{e^u}{e^u-1}\right)^{m+1}\left(\frac{e^v}{e^v-1}\right)^{n+1}\frac{1}{1-\alpha_p^{-k\delta}e^{-\delta u}e^{\delta v}} \\
&= \left(\frac{1}{2\pi i}\right)^2 \oint_{C(u)}\oint_{C(v)} \frac{1}{1-\alpha_p^k}(\alpha_p^{k(-\varepsilon r+\zeta\kappa)} e^{u(\zeta\rho-\varepsilon a)} e^{v(\zeta\tau-\varepsilon b)} - 1)^{m+n+2} \\
&\qquad \frac{(e^u)^m}{(e^u-1)^{m+1}}\frac{(e^v)^n}{(e^v-1)^{n+1}}\frac{1}{1-\alpha_p^{-k\delta}e^{-\delta u}e^{\delta v}} e^u e^v\, dv du
\end{aligned}
$$

(where $C(u),\ C(v)$ are sufficiently small counterclockwise loops around the origin)

$$
\begin{aligned}
&= \left(\frac{1}{2\pi i}\right)^2 \oint_{C(x)}\oint_{C(y)} \frac{1}{1-\alpha_p^k}(\alpha_p^{k(-\varepsilon r+\zeta\kappa)}(1+x)^{\zeta\rho-\varepsilon a}(1+y)^{\zeta\tau-\varepsilon b} - 1)^{m+n+2} \\
&\qquad \frac{(1+x)^m}{x^{m+1}}\frac{(1+y)^n}{y^{n+1}}\frac{1}{1-\alpha_p^{-k\delta}(1+x)^{-\delta}(1+y)^{\delta}}\, dy dx
\end{aligned}
$$

(where $C(x),\ C(y)$ are sufficiently small counterclockwise loops around the origin).

Here we set $\beta = \zeta\rho - \varepsilon a, \gamma = \zeta\tau - \varepsilon b$ and

$$
\begin{aligned}
\Phi &= (1+x)^{-\delta}(1+y)^{\delta} - 1 = -\delta x + \delta y + Q(x, y)\,, \\
\Psi &= (1+x)^{\beta}(1+y)^{\gamma} - 1 = \beta x + \gamma y + R(x, y)\,,
\end{aligned}
$$

where the total degrees of $Q(x, y),\ R(x, y)$ are greater than 1. Then we have

$$
\begin{aligned}
&T_i(k, \varepsilon, \zeta) \\
&\quad = x^m y^n\text{-coeff. of} \\
&\qquad \frac{1}{1-\alpha_p^k}(\alpha_p^{k(\zeta\kappa-\varepsilon r)} - 1 + \alpha_p^{k(\zeta\kappa-\varepsilon r)}\Psi)^{m+n+2}(1+x)^m(1+y)^n(1-\alpha_p^{-k\delta} - \alpha_p^{-k\delta}\Phi)^{-1} \\
&\quad = x^m y^n\text{-coeff. of} \\
&\qquad \frac{1}{1-\alpha_p^k}\sum_{s=0}^{m+n}\binom{m+n+2}{s}(\alpha_p^{k(\zeta\kappa-\varepsilon r)} - 1)^{m+n+2-s}\alpha_p^{ks(\zeta\kappa-\varepsilon r)}\Psi^s(1+x)^m(1+y)^n \\
&\qquad\qquad \sum_{j=0}^{m+n}\frac{\alpha_p^{-kj\delta}\Phi^j}{(1-\alpha_p^{-k\delta})^{j+1}} \\
&\quad = x^m y^n\text{-coeff. of} \\
&\qquad \sum_{s=0}^{m+n}\sum_{j=0}^{m+n-s}\binom{m+n+2}{s}(-1)\Lambda_j(\alpha_p^k)(1+x)^m(1+y)^n\Phi^j\Psi^s\,,
\end{aligned}
$$

where $\Lambda_j(t)$ is an element of $\mathbf{Z}[t, t^{-1}]$ defined by

$$\Lambda_j(t) = \frac{t^{s(\zeta\kappa-\varepsilon r)+\delta}(t^{\zeta\kappa-\varepsilon r}-1)^{m+n+2-s}}{(t-1)(t^{\delta}-1)^{j+1}} .$$

Here, since

$$\sum_{k=1}^{p-1} \alpha_p^{kl} \equiv -1 \pmod{p}$$

for any integer l, we have

$$(-1)\sum_{k=1}^{p-1} \Lambda_j(\alpha_p^k) \equiv \Lambda_j(1) \pmod{p}$$
$$= \begin{cases} 0 & \text{if } j < m+n-s \\ \delta^{m+n-s+1}(\zeta\kappa-\varepsilon r)^{m+n+2-s} & \text{if } j = m+n-s \end{cases} .$$

Therefore we have

$$\begin{aligned}
&\sum_{k=1}^{p-1} T_i(k,\varepsilon,\zeta) \\
&\quad \equiv x^m y^n\text{-coeff. of} \\
&\qquad \sum_{s=0}^{m+n} \binom{m+n+2}{s} \delta^{m+n-s+1}(\zeta\kappa-\varepsilon r)^{m+n+2-s}(-\delta(x-y))^{m+n-s}(\beta x+\gamma y)^s \\
&\qquad\qquad \pmod{p} \\
&\quad = x^m y^n\text{-coeff. of} \\
&\qquad \sum_{s=0}^{m+n} \binom{m+n+2}{s} \delta^{m+n-s+1}(\zeta\kappa-\varepsilon r)^{m+n+2-s}(-\delta)^{m+n-s} \\
&\qquad \sum_{h=0}^{s} \binom{s}{h} \beta^h x^h \gamma^{s-h} y^{s-h} \sum_{q=0}^{m} \binom{m+n-s}{q} x^q (-y)^{m+n-s-q} \\
&\quad = \sum_{s=0}^{m+n} \sum_{q=0}^{m} \binom{m+n+2}{s}\binom{s}{m-q}\binom{m+n-s}{q}(-1)^q \\
&\qquad\qquad \delta(\kappa\zeta-r\varepsilon)^{m+n+2-s}(\rho\zeta-a\varepsilon)^{m-q}(\tau\zeta-b\varepsilon)^{s-m+q} ,
\end{aligned}$$

and hence it follows that

$$\begin{aligned}
S_\varepsilon(\zeta) \equiv{}& \frac{1}{p}\sum_{s=0}^{m+n}\sum_{q=0}^{m}\binom{m+n+2}{s}\binom{s}{m-q}\binom{m+n-s}{q}(-1)^q \\
&\begin{pmatrix} (-1)^{m+n+s+1}(\mu\zeta-\varepsilon)^{m+n+2-s}((\lambda-\nu)\zeta-m\varepsilon)^{m-q}(\mu\zeta-(n+2)\varepsilon)^{s-m+q} \\ +((-\mu+\nu)\zeta-\varepsilon)^{m+n+2-s}(\lambda\zeta-(m+2)\varepsilon)^{m-q}((\mu-\nu)\zeta-n\varepsilon)^{s-m+q} \end{pmatrix} \\
&\hfill (\mathrm{mod}\ \boldsymbol{Z}) \\
={}& \frac{1}{p}g(\lambda,\mu,\nu)\zeta^{m+n+2}-\varepsilon\frac{1}{p}h(\lambda,\mu,\nu)\zeta^{m+n+1}+\varphi(\zeta)\,,
\end{aligned}$$

where the degree of $\varphi(\zeta)$ is less than $m+n+1$.

Here for $f(x)=(\sinh x)^k$ we have

$$f(x)=\frac{1}{2^k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}e^{(k-2i)x}\,,\qquad f(x)=x^k+\frac{k}{6}x^{k+2}+\text{higher order terms}$$

and hence it follows that

$$2^k f^{(l)}(0)=\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-2i)^l=\begin{cases}0 & \text{if } 0\le l<k \text{ or } l=k+1\\ 2^k k! & \text{if } l=k\end{cases}.$$

Therefore it follows from (3) that

$$\begin{aligned}
&\lambda\mu\nu F_L(g_p) \\
&= (m+n+2)\lambda\mu\nu \\
&\quad \sum_{i=0}^{m+n+1}(-1)^i\binom{m+n+1}{i}(S_{-1}(m+n+1-2i)-S_{+1}(m+n+1-2i)) \\
&\quad -(m(m+2)\mu\nu+n(n+2)\lambda\nu+2\lambda\mu)\sum_{i=0}^{m+n+2}(-1)^i\binom{m+n+2}{i}S_0(m+n+2-2i) \\
&\equiv \frac{2^{m+n+2}(m+n+2)!}{p}F(\lambda,\mu,\nu)\quad (\mathrm{mod}\ \boldsymbol{Z})\,.
\end{aligned}$$

Hence, for any odd prime number p, it follows from Theorem 1.2 that

$$\begin{aligned}
\frac{1}{p}A(m,n)\lambda\mu\nu f_{\Omega(\lambda,\mu,\nu)}(X) &= A(m,n)\lambda\mu\nu f_{\Omega(\lambda,\mu,\nu)}(X_p) \\
&\equiv \frac{1}{p}2^{m+n+2}(m+n+2)!F(\lambda,\mu,\nu)\quad (\mathrm{mod}\ \boldsymbol{Z})\,,
\end{aligned}$$

where $\Omega(\lambda,\mu,\nu)=\lambda\tilde{u}+\mu\tilde{v}+\nu\tilde{w}$, which implies that

$$A(m,n)\lambda\mu\nu f_{\Omega(\lambda,\mu,\nu)}(X)=2^{m+n+2}(m+n+2)!F(\lambda,\mu,\nu)\,. \tag{5}$$

Now, since $\triangle_{k\omega}=k^{-1}\triangle_\omega$, it follows that $xyzf_{\Omega(x,y,z)}(X)$ is a homogeneous function in $x,\ y,\ z$ of degree $m+n+4$ as well as $F(x,y,z)$. Moreover, since the set

$$\{(x,y,z)\in\boldsymbol{R}^3\,;\ (rx,ry,rz)\in\boldsymbol{Z}^3 \text{ for some } r>0\}$$

is dense in $\boldsymbol{R}^3$, the equality (5) implies that for any $(x, y, z) \in \boldsymbol{R}^3$

$$A(m,n)xyzf_{\Omega(x,y,z)}(X) = 2^{m+n+2}(m+n+2)!F(x,y,z)\,.$$

The result in Theorem 2.2 follows immediately from the equality above.

REMARK 3.2. Let $G = (U(m+1) \times U(n+1))/U(1)$ be the maximal compact subgroup of $\mathrm{Aut}(M)$ and $q = [(z_0, \dots, z_m), (w_0, \dots, w_n), (\eta_0, \eta_1)]$ a point in M. Then we can see that the real dimension of the isotropy subgroup of G at q is equal to m^2+n^2 if $\eta_0\eta_1 \neq 0$ and is equal to m^2+n^2+1 if $\eta_0\eta_1 = 0$, which implies that the real codimension of the principal orbit of G in M is one. Hence it follows from Corollary 1.1 in [8] that each Kähler class of M contains an extremal metric, and therefore it follows from [2, Theorem 4] (see also [5, Theorem 3.3.1]) that a Kähler class contains a Kähler metric of constant scalar curvature if the character for the Kähler class vanishes. Hence a Kähler class $\Omega = x\tilde{u} + y\tilde{v} + z\tilde{w}$ contains a Kähler metric of constant scalar curvature if and only if $F(x,y,z) = 0$. Moreover we can see that the $\mathrm{Aut}(M)$-orbit of q with $\eta_0\eta_1 \neq 0$ coincides with the open subset $M \setminus (N_1 \cup N_2)$ of M. Hence M is an almost-homogeneous manifold (see [9]) and therefore it follows from [8, Theorem 4] that M admits a Kähler metric of constant scalar curvature.

4. Examples. In this section, we consider the cases $1 \leq m < n \leq 10$. Since $F(x,y,z)$ is a homogeneous polynomial, $F(x,y,z)$ for $x,\ y,\ z > 0$ is determined by its restriction to the face f of a regular octahedron defined by

$$f = \{(x,y,z)\,;\ x+y+z = 1\,,\ x,\ y,\ z > 0\}\,.$$

Let C be a point in f defined by

$$C = \frac{1}{m+n+6}(m+2, n+2, 2)$$

and set $A = (1,0,0),\ B = (0,1,0)$. Then, since C is homothetic to $c_1(M) > 0$, C is a Kähler class and hence the interior of the triangle ABC is contained in the set of Kähler classes of M (see Remark 2.1). Let $l_1,\ l_2$ be lines in f defined by

$$\begin{aligned} l_1(t) &= (x_1(t), y_1(t), z_1(t)) = (1-t)A + tC\ , \\ l_2(t) &= (x_2(t), y_2(t), z_2(t)) = (1-t)\left(\frac{1}{2}, \frac{1}{2}, 0\right) + t(0,0,1) \end{aligned}$$

for $0 < t < 1$. Then we have

$$
\begin{aligned}
&\lim_{t\to+0} F(l_1(t))/y_1(t)^{n+3}\\
&\quad = \lim_{t\to+0}\sum_{s=0}^{m+n}\sum_{q=0}^{m}\binom{m+n+2}{s}\binom{s}{m-q}\binom{m+n-s}{q}(-1)^{m+n+s+q+1}t^q\\
&\qquad\qquad 2(n+2)^{-n-2}(m+n+6)^{-q}\\
&\qquad\qquad \{((n+1)s+(n+2)q-m-mn-2n-n^2)(n+2)^{n+1+q}\\
&\qquad\qquad\quad -((n+1)s+nq-m-mn-2n-n^2-2)n^{n+1+q}\}\\
&\quad = \sum_{s=0}^{m+n}\binom{m+n+2}{s}\binom{s}{m}(-1)^{m+n+s+1}2(n+2)^{-n-2}\\
&\qquad\quad \{((n+1)s-m-mn-2n-n^2)(n+2)^{n+1}\\
&\qquad\quad -((n+1)s-m-mn-2n-n^2-2)n^{n+1}\},\\
&\lim_{t\to+0} F(l_2(t))/z_2(t)^2\\
&= \sum_{s=0}^{m+n}\sum_{q=0}^{m}\binom{m+n+2}{s}\binom{s}{m-q}\binom{m+n-s}{q}(-1)^{m+n+s+q+1}2^{-m-n-2}\\
&\qquad \{2(n-m)q^2+(2(n+1)s+(m^2-4mn-n^2-7m-3n-2))q\\
&\qquad +(-mn+n^2-m+2n+1)s+3m^2+m^2n-n^3-mn-4n^2-2m-4n\}.
\end{aligned}
$$

Direct computation using the equalities above shows that

$$\lim_{t\to+0} F(l_1(t))/y_1(t)^{n+3}<0\,,\quad \lim_{t\to+0} F(l_2(t))/z_2(t)^2>0\,,$$

which imply that there exist points P_1, P_2 in the interior of the triangle ABC such that $F(P_1)<0$ and $F(P_2)>0$. Therefore there exist infinitely many Kähler classes Ω such that f_Ω vanishes and hence that Ω contains a Kähler metric of constant scalar curvature (see Remark 3.2).

On the other hand, direct computation also shows that

$$F(m+2,n+2,2)\neq 0\,,$$

which implies that $c_1(M)$ does not contain any Kähler metric of constant scalar curvature. This result shows that M does not admit any Kähler-Einstein metric. (See [3].)

REFERENCES

[1] S. BANDO, An obstruction for Chern class forms to be harmonic, Kodai Math. J. 29 (2006), 337–345.

[2] E. CALABI, Extremal Kähler metrics II, Differential geometry and complex analysis, (I. Chavel and H. M. Farkas eds.), 95–114, Springer-Verlag, Berline-Heidelberg-New York, 1985.

[3] A. FUTAKI, An obstruction to the existence of Einstein-Kähler metrics, Invent. Math. 73 (1983), 437–443.

[4] A. FUTAKI, On compact Kähler manifold of constant scalar curvature, Proc. Japan Acad. Ser. A 59 (1983), 401–402.

[5] A. FUTAKI, Kähler-Einstein metrics and integral invariants, Lecture Notes in Math. 1314, Springer-Verlag, Berlin, 1988.

[6] A. FUTAKI AND S. MORITA, Invariant polynomials of the automorphism group of a compact complex manifold, J. Differential Geom. 21 (1985), 135–142.

[7] A. FUTAKI AND K. TSUBOI, Fixed point formula for characters of automorphism groups associated with Kähler classes, Math. Res. Lett. 8 (2001), 495–507.

[8] A. D. HWANG, On existence of Kähler metrics with constant scalar curvature, Osaka J. Math. 31 (1994), 561–595.

[9] A. T. HUCKLEBERRY AND D. M. SNOW, Almost-homogeneous Kähler manifolds with hypersurface orbits, Osaka J. Math. 19 (1982), 763–786.

[10] J. KAZDAN AND F. WARNER, Prescribing curvatures, Proc. Sympos. Pure Math. 27 (1975), 309–319.

[11] S. KOBAYASHI, Transformation groups in differential geometry, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[12] A. LICHNEROWICZ, Sur les transformations analytiques d'une variété Kählerienne compacte, 1959, Colloque Geom. Diff. Global (Bruxelles, 1958), 11–26, Centre Belge Rech. Math., Louvain.

[13] A. LICHNEROWICZ, Isométrie et transformations analytiques d'une variété Kählerienne compacte, Bull. Soc. Math. France 87 (1959), 427–437.

[14] Y. MATSUSHIMA, Sur la structure du groupe d'homéomorphismes d'une certaine variété Kaehlérienne, Nagoya Math. J. 11 (1957), 145–150.

[15] Y. NAKAGAWA, Bando-Calabi-Futaki character of compact toric manifolds, Tohoku Math. J. 53 (2001), 479–490.

[16] Y. NAKAGAWA, The Bando-Calabi-Futaki character and its lifting to a group character, Math. Ann. 325 (2003), 31–53.

[17] G. TIAN, Kähler-Einstein metrics on algebraic manifolds, in: Proc. C.I.M.E. conference on Transcendental methods in algebraic geometry, Lecture Notes in Math. 1646, Springer-Verlag, Berlin-Heidelberg-New York, 1996, 143–185.

TOKYO UNIVERSITY OF MARINE SCIENCE AND TECHNOLOGY
4–5–7 KOUNAN, MINATO-KU
TOKYO 108–8477
JAPAN

E-mail address: tsubois@kaiyodai.ac.jp