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ON THE EXISTENCE OF PIECEWISE CONTINUOUS OPTIMAL CONTROLS

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Abstract

Existence of piecewise optimal control is proved when the cost function includes one or both of (*a*) a cost of sudden switching (discontinuity) of control variables, and (*b*) a cost associated with the maximum rate of variation of the control over segments of the path for which the control is continuous.

1. Introduction

Consider a system characterized by the equation

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(0) = x_0, \tag{1}$$

together with the associated cost functional

$$I(u) = J(x, u, \Phi) = \int_0^T f_0(t, x(t), u(t))\, dt + \Phi(u), \tag{2}$$

where $T \in (0, \infty)$ is prespecified. The vectors x and f will be n-dimensional. An admissible control u will be a piecewise-continuous r-vector-valued function on $[0, T]$ which may have values at time t in a non-empty compact set $M(t)$, with the set-valued function M piecewise-continuous in the Hausdorff topology [1]. We will assume the vector function f is continuous in all argument, is continuously differentiable with respect to x, and that the inner product inequality

$$(x, f(t, x, u)) \leqslant C[1 + |x|^2] \tag{3}$$

holds for some constant C and all $t \in [0, T]$, x, and $u \in M(t)$, where $(.,.)$ and $|.|$ are the usual inner product and norm in E_n, respectively. This condition prevents finite escape time of trajectories and could be replaced by any condition which allows us to restrict attention to a compact set of the (t, x) space. Using condition (3) we can assume that

$$|x(t)| \leqslant (1 + |x_0|^2 \exp(2CT) = C_1 \tag{4}$$

for any trajectory x of the system (1). The function $f_0(t, x, u)$ is assumed to be non-negative and continuous in all arguments for $t \in [0, T]$, $|x| \leqslant C_1$, $u \in M(t)$. The

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functional $\Phi(u)$ is defined by

$$\Phi(u) = \gamma_1(N(u) + \gamma_2 S(u), \tag{5}$$

where $\gamma_1 \geqslant 0$, $\gamma_2 \geqslant 0$, $N(u)$ is the number of discontinuities of u on $[0, T]$, and $S(u)$ is given by

$$S(u) = \sup\left\{\frac{|u(t)-u(t')|}{|t-t'|} : t \neq t' \text{ and } u \text{ is continuous on } [t, t']\right\}. \tag{6}$$

We shall denote by Ω the set of admissible controls with $S(u) < \infty$.

An important problem in optimal control theory is that of determining whether or not there exists a control function $u^* \in \Omega$ such that $I(u^*)$ equals the infimum of $I(u)$ as $u(.)$ varies over all admissible control functions. In the special case $\gamma_1 = 0 = \gamma_2$ we have the Pontryagin Policy for which existence proofs are available only under restrictive convexity assumptions imposed on the functions f, f_0. A sample of these results can be found in [2], [3]. Indeed, it is easy to construct quite simple examples for which one can prove by explicit calculations that no optimal control exists in Ω, [3].

In [5, pp. 276–281, Theorem 7] E. B. Lee and L. Markus have considered the case when the set of admissible controls consists of functions u of bounded variations such that $\|Du\| = |u(0) - u(0-)| + \text{var}\, u \leqslant E$ for a prescribed uniform bound $E \geqslant 0$. Moreover, the cost functional contains a non-integral cost term of the form $\gamma(\sup|X(t)|, \|Du\|)$, where γ is a continuous monotone nonincreasing in each of its two arguments. In this paper no uniform bound is prescribed on $\|Du\|$ and the functional $\Phi(u)$ has been chosen to satisfy two requirements:

(1) It should be possible to prove an existence theorem, to the effect that at least one piecewise continuous optimal control path exists; and

(2) It should be possible to find usable necessary conditions for the control path.

This paper is devoted to point 1. Point 2 is covered in a joint paper by the author and J. M. Blatt [4].

2. Main results

LEMMA 1. *Let $\{u_n\}_{n=1}^{\infty}$ be a sequence in Ω such that $\Phi(u_n) \leqslant K$, $n = 1, 2, \ldots$. Then a subsequence $\{u_{n_i}\}$ and a function $u \in \Omega$ can be chosen with the following properties*:

(1) *$\Phi(u_{n_i})$ converges and $\Phi(u) \leqslant \lim_{i\to\infty} \Phi(u_{n_i})$.*

(2) *If $t_i, i = 1, 2, \ldots, N(u)$, $0 < t_1 < t_2 \ldots < t_{N(u)}$, are the points of discontinuities of u in $(0, T)$, then for any sufficiently small $\epsilon > 0$ the sequence $\{u_{n_i}\}$ converges uniformly to u on the intervals $[0, t_1 - \epsilon]$, $[t_1 + \epsilon, t_2 - \epsilon], \ldots, [t_{N(u)} + \epsilon, T]$.*

PROOF. By choosing a subsequence if necessary and using the hypothesis $\Phi(u_n) \leqslant K$, $n = 1, 2, \ldots$, we can assume that $\Phi(u_n)$ converges as $n \to \infty$ and $N(u_n) = N_K$ for some constant N_K independent of n. For simplicity of notations we shall make the following assumptions:

(1) The functions $u_n, n = 1, 2, \ldots$, are scalar valued.

(2) $N_K = 1$.

Both these assumptions will impose no loss of generality as will be seen from the nature of the proof. Let $0<t_n<T$ denote the point of discontinuity of u_n. By selecting a subsequence if necessary we can assume that t_n converges to $\hat{t}$ with $0\leqslant\hat{t}\leqslant T$. Assume $0<\hat{t}<T$.

For any given N let I_N denote the set $I_N=[0,\hat{t}-(1/N)]\cup[\hat{t}+(1/N),T]$. Choose $\xi>0$ large enough so that $\hat{t}-(1/\xi)>0$ and $\hat{t}+(1/\xi)<T$. We choose first a subsequence $\{u_n^\xi\}$ of $\{u_n\}$ and a function u^ξ which is defined and continuous on I_ξ.

Since $\lim_{n\to\infty}t_n=\hat{t}$, we can select a subsequence $\{u_{\tilde{n}}\}$ of $\{u_n\}$ such that $t_{\tilde{n}}\in(\hat{t}-(1/\xi),\ \hat{t}+(1/\xi))$ for all $\tilde{n}$. By the hypotheses on the set Ω, the sequence $\{u_{\tilde{n}}\}\subset C(I_\xi)=$ the set of continuous functions on I_ξ and $u_n(t)\in M(t)$ a compact subset of R. Furthermore, by the hypotheses $\Phi(u_n)\leqslant K$, $n=1,2,\ldots$, we have that $|u_n(t)-u_n(t')|\leqslant K|t-t'|$ for all t,t' in I_ξ. Using Ascoli Arzella's Theorem we can select a subsequence $\{u_n^\xi\}$ of $\{u_{\tilde{n}}\}$ which converges uniformly to a continuous function u^ξ on I_ξ.

We now proceed by induction to select subsequences $\{u_n^{\xi+j}\}$ of $\{u_n^\xi\}$ and continuous functions $u^{\xi+j}$ on $I_{\xi+j}$, $j=1,2,\ldots$, as follows. Suppose $\{u_n^{\xi+j}\}$ and $u^{\xi+j}$ has been chosen. To construct $\{u_n^{\xi+j+1}\}$ we choose first a subsequence $\{u_{\tilde{n}}^{\xi+j}\}$ of $\{u_n^{\xi+j}\}$ such that $t_{\tilde{n}}^{\xi+j}\in(\hat{t}-(\xi+j+1)^{-1},\ \hat{t}+(\xi+j+1)^{-1})$ for all $\tilde{n}$ where $t_{\tilde{n}}^{\xi+j}$ is the point of discontinuity of $u_{\tilde{n}}^{\xi+j}$. Applying Ascoli Arzella's Theorem as before, we can select a subsequence $\{u_n^{\xi+j+1}\}\subset\{u_{\tilde{n}}^{\xi+j}\}\subset\{u_n^{\xi+j}\}$ which converges to a continuous function $u^{\xi+j+1}$ on $I_{\xi+j+1}\supset I_{\xi+j}$.

From the above construction we notice that

(1) The subsequences $\{u_n^{\xi+j}\}$, $j=0,1,2,\ldots$, are subsequences of $\{u_n\}$, and $\{u_n^{\xi+j+1}\}\subset\{u_n^{\xi+j}\}$.

(2) $u^{\xi+j+1}(t)=u^{\xi+j}(t)$ for $t\in I_{\xi+j}\subset I_{\xi+j+1}$.

Define $u(t)=u^{\xi+j}(t)$ if $t\in I_{\xi+j}$, $j=0,1,2,\ldots$. Since $\bigcup_{j=1}^\infty I_{\xi+j}=[0,\hat{t})\cup(\hat{t},T]$, the function $u(t)$ is defined and continuous on $[0,T]-\{\hat{t}\}$. Let $\{u_{n_i}\}$ denote the subsequence defined by

$$u_{n_i}=u_i^{\xi+i},\quad i=1,2,\ldots.$$

We show now that the sequence $\{u_{n_i}\}$ and the function u defined above satisfy the required properties (1) and (2) of Lemma 1.

First notice that $\hat{t}$ is the only possible point of discontinuity of u. Hence $N(u)\leqslant N_K$. For any given $\epsilon>0$ there is $\eta=\eta(\epsilon)$ such that $I_{\xi+\eta}\supset[0,\hat{t}-\epsilon]\cup[\hat{t}+\epsilon,T]$ and $\{u_{n_j}:j\geqslant\eta\}$ converges uniformly to u on $I_{\xi+\eta}$. This proves property (2).

We notice that $\lim_{t\to\hat{t}-}u(t)$ and $\lim_{t\to\hat{t}+}u(t)$ both exist. This follows from the inequality

$$\begin{aligned}|u(t)-u(t')|&=\lim|u_{n_i}(t)-u_{n_i}(t')|\\&\leqslant K|t-t'|\end{aligned}$$

for $t,t'\in[0,\hat{t})$ or $t,t'\in(\hat{t},T]$. If t is a point of continuity of u then $\{u_{n_i}(t)\}$ converges to $u(t)$. Hence $u(t)\in M(t)$.

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To prove property (1) we notice first that $\Phi(u) = \gamma_1 N(u) + \gamma_2 S(u) \leqslant \gamma_1 N_K + \gamma_2 S(u)$ and $\lim_{i\to\infty} \Phi(u_{n_i}) = \gamma_1 N_K + \gamma_2 \lim S(u_{n_i})$. Hence it is enough to show that

$$S(u) \leqslant \lim_{i\to\infty} S(u_{n_i}) = S^* \tag{*}$$

To prove (*) we consider the following two cases:

(I) $\hat{t}$ is a point of discontinuity of $u(t)$: Let $t \neq t'$ and t, t' belong to either $[0, \hat{t})$ or $(\hat{t}, T]$. Then

$$\begin{aligned} |u(t) - u(t')| &\leqslant |u(t) - u_{n_i}(t)| + |u_{n_i}(t) - u_{n_i}(t')| + |u_{n_i}(t') - u(t')| \\ &\leqslant \lim_{i\to\infty} S(u_{n_i}) |t - t'|. \end{aligned} \tag{7}$$

(II) $u(t)$ is continuous on $[0, T]$: Assume $t < \hat{t} < t'$ and $t, t' \in [0, T]$. Let $\epsilon > 0$ and choose $\delta > 0$ such that

$$|u(\hat{t} - \delta) - u(\hat{t} + \delta)| < \epsilon. \tag{8}$$

Using (7) and (8) we obtain

$$\begin{aligned} |u(t) - u(t')| &\leqslant |u(t) - u(\hat{t} - \delta)| + \epsilon + |u(\hat{t} + \delta) - u(t')| \\ &\leqslant S^*(t - t') + \epsilon \end{aligned} \tag{9}$$

for any $\epsilon > 0$. Hence

$$|u(t) - u(t')| \leqslant S^*(t - t'). \tag{10}$$

Using (7) and (10) we obtain (*). This completes the proof of Lemma 1.

Definition. Let

$$\Omega_K = \{u \in \Omega : \Phi(u) \leqslant K\}. \tag{11}$$

Lemma 2. *Let* $\{u_n\}_{n=1}^{\infty}$ *be a sequence in* Ω_K. *Then there exists a subsequence* $\{u_{n_i}\} \subset \{u_n\}$ *and a function* $u \in \Omega_K$ *such that* $\Phi(u_{n_i})$ *converges as* $i \to \infty$, *and*

$$\lim_{i\to\infty} I(u_{n_i}) = \int_0^T f_0(t, x(t), u(t))\, dt + \lim_{i\to\infty} \Phi(u_{n_i}). \tag{12}$$

Proof. By Lemma 1 we can choose a subsequence $\{u_{n_i}\}$ and an element $u \in \Omega_K$ satisfying properties (1) and (2) of Lemma 1.

Let $x_{n_i}(t)$, $x(t)$ denote the trajectories of the initial value problem (1) corresponding to u_{n_i}, u, respectively. Since $M(t)$ is continuous in the Hausdorff topology for $t \in [0, T]$ there is a constant L such that $|u_{n_i}(t)| \leqslant L$, $|u(t)| \leqslant L$, $t \in [0, T]$. Also, f is assumed continuously differentiable with respect to x. Let C be the maximum of the absolute value of all partial derivatives $(\partial/\partial x_j)\ (f_i(t, x, u))$ for $t \in [0, T]$, $|x| \leqslant C_1$,

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$|u| \leqslant L$. Then

$$
\begin{aligned}
|x_{n_i}(t) - x(t)| &\leqslant \int_0^t |f(x_{n_i}(t), u_{n_i}(t), t) - f(x(t), u(t), t)|\, dt \\
&\leqslant \int_0^t |f(x_{n_i}(t), u_{n_i}(t), t) - f(x(t), u_{n_i}(t), t)|\, dt \\
&\quad + \int_0^T |f(x(t), u_{n_i}(t), t) - f(x(t), u(t), t)|\, dt \\
&\leqslant C \int_0^t |x_{n_i}(t) - x(t)|\, dt + \alpha_i, \qquad (13)
\end{aligned}
$$

where

$$\alpha_i = \int_0^T |f(x(t), u_{n_i}(t), t) - f(x(t), u(t), t)|\, dt.$$

An application of Gronwall inequality [1] to (13) gives

$$|x_{n_i}(t) - x(t)| \leqslant \alpha_i e^{CT}. \qquad (14)$$

Let $g_i(t) = |f(t, x(t), u_{n_i}(t)) - f(t, x(t), u(t))|$. Since $u_{n_i}(t)$ converges to $u(t)$ for each $t \neq t_i$, $i = 1, 2, \ldots N(u)$, and $f(t, x, u)$ is assumed continuous in all arguments for $t \in [0, T]$, $|x| \leqslant C_1$, $|u| \leqslant L$, $g_i(t)$ converges to 0 for $t \neq t_i$, $i = 1, 2, \ldots, N(u)$. Also, $g_i(t) \leqslant 2L$. Hence the Dominated Convergence Theorem implies that $\lim_{i \to \infty} \alpha_i = 0$. We then conclude from (14) that x_{n_i} converges uniformly to x on $[0, T]$. We obtain by a similar application of the Dominated Convergence Theorem that

$$\lim_{i \to \infty} \int_0^T f_0(t, x_{n_i}(t), u_{n_i}(t))\, dt = \int_0^T f_0(t, x(t), u(t))\, dt$$

and the desired conclusion (12) follows.

THEOREM 3 (*Existence of an optimal control*): *There exists a function* $u^* \in \Omega$ *such that* $I(u^*) = \inf\{I(u) \colon u \in \Omega\}$.

PROOF: Let $K = I(0)$. Then we claim that

$$\inf\{I(u) \colon u \in \Omega\} = \inf\{I(u) \colon u \in \Omega, \phi(u) \leqslant K\}. \qquad (15)$$

To prove (15) it is enough to show that

$$\inf\{I(u) \colon u \in \Omega\} \geqslant \inf\{I(u) \colon u \in \Omega, \Phi(u) \leqslant K\}. \qquad (16)$$

So assume to the contrary that there is $u \in \Omega$ with $\Phi(u) > K$ such that

$$I(u) < \inf\{I(u) \colon u \in \Omega, \Phi(u) \leqslant K\} \qquad (17)$$

But $I(u) \geqslant \Phi(u) > K$. From (17) we then obtain

$$K < \Phi(u) \leqslant I(u) < I(0) = K$$

which is a contradiction. This proves (15).

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Let $\alpha_K = \inf\{I(u): u \in \Omega, \Phi(u) \leqslant K\}$. Choose $\{u_n\} \subset \Omega$ with $\Phi(u_n) \leqslant K$ such that

$$\alpha_K = \lim_{n\to\infty} I(u_n). \tag{18}$$

By choosing a subsequence if necessary we can assume that $\Phi(u_n)$ converges. Let

$$\Phi_K^* = \lim_{n\to\infty} \Phi(u_n). \tag{19}$$

By Lemma 1 we can select a subsequence $\{u_{n_i}\} \subset \{u_n\}$ and a function $u^* \subset \Omega$ satisfying properties (1) and (2) of Lemma 1. By Lemma 2 we have

$$\lim_{i\to\infty} I(u_{n_i}) = \int_0^T f_0(t, x(t), u^*(t))\,dt + \lim_{i\to\infty} \Phi(u_{n_i}). \tag{20}$$

By (18), (19) and (20) we have

$$\alpha_K = \lim_{i\to\infty} I(u_{n_i}) = \int_0^T f_0(t, x(t), u^*(t))\,dt + \Phi_K^*. \tag{21}$$

But from Lemma 1 we have

$$\Phi(u^*) \leqslant \Phi_K^*. \tag{22}$$

Using (21) and (22) we obtain

$$I(u^*) = \int_0^T f_0(t, x(t), u^*(t))\,dt + \Phi(u^*) \leqslant \alpha_K. \tag{23}$$

Using (15), (18), (22) and (23) we obtain

$$I(u^*) = \inf\{I(u): u \in \Omega\}.$$

This proves Theorem 3.

Remark 1. The hypotheses on the function f can be relaxed. Instead of requiring continuous differentiability of f with respect to x, it is enough to require the existence of a non-negative, measurable and locally Lebesgue integrable function L defined on $[0, T]$ so that

$$|f(t,x,u) - f(t,y,u)| \leqslant L(t)|x-y| \quad \text{almost everywhere on } [0,T] \text{ for all } u \in \Omega$$

Remark 2. Other control problems [Lagrange problems, free problems ($m = u$, $f = u$)] can be considered. These will appear elsewhere.

Remark 3. Other choices of the term $\Phi(u)$, representing the cost of "variation" in the control u, are possible and lead to interesting problems. Let us write Ω_Φ to denote the class of admissible controls associated with the cost functional (2). Then the following problems admit existence of optimal controls:

I. $\Phi = \gamma_1 N(u)$, $\gamma_1 > 0$, $\Omega_\Phi = \{u \in \Omega: S(u) \leqslant K_0\}$, where K_0 is some prescribed constant. In this case it is clear that Theorem 3 still holds. An optimal control will depend on K_0.

II. $\Phi = \gamma_2 S(u)$, $\Omega_\Phi = \{u \in C[0,T]: u(t) \in M(t)\}$. In this case one can easily prove that an optimal control exists by a simple application of Ascoli Arzella's Theorem.

III. $\Phi = \gamma T[u]$, where $T[u]$ is the total variation of u on $[0,T]$. $\Omega_\Phi = \{u$: $u(t)$ is of bounded variation on $[0,T]$ and $u(t) \in M$ a compact subset of $E_m\}$.

Obviously there are several other choices of Φ and Ω_Φ. However, one has to make sure that Ω_Φ is not too restrictive to allow for usable necessary conditions. Necessary conditions for the control path and a discussion of problems I and II are given in [4].

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