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ON THE EXISTENCE OF POSITIVE INVARIANT FUNCTIONS FOR SEMIGROUPS OF OPERATORS

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1. Introduction. Let S be a semigroup. $B(S)$ will denote the space of all bounded real-valued functions on S. A linear functional φ on $B(S)$ is called a *left invariant mean* on S if for any $f \in B(S)$ and any $a \in S$,

$$\inf \{f(s); s \in S\} \leqq \varphi(f) \leqq \sup \{f(s); s \in S\}$$

and

$$\varphi({}_a f) = \varphi(f) ,$$

where ${}_a f$ is defined by ${}_a f(s) = f(as)$ for $s \in S$. The semigroup S is said to be *left amenable* if it has a left invariant mean. In what follows we shall always assume that S is left amenable. LIM will denote the set of all left invariant means on S. If $f \in B(S)$, we define

$$M(f) = \sup \{\varphi(f); \varphi \in LIM\} .$$

Let $(X, \mathscr{M}, m)$ be a probability space and $L_p(X) = L_p(X, \mathscr{M}, m)$, $1 \leqq p \leqq \infty$, the usual Banach spaces. Let $\mathscr{S} = \{T_s; s \in S\}$ be a representation of S as a semigroup of positive linear operators on $L_p(X)$ for some fixed p with $1 \leqq p \leqq \infty$. Thus $T_{s_1} T_{s_2} = T_{s_1 s_2}$ for $s_1, s_2 \in S$. Here if $p = \infty$, we shall assume, throughout this paper, that each T_s is *countably additive*, i.e., $T(\lim_n f_n) = \lim_n Tf_n$ provided (f_n) is an increasing sequence of non-negative functions in $L_\infty(X)$ such that $\lim_n f_n \in L_\infty(X)$; hence T_s is the adjoint of an operator on $L_1(X)$ and T_s^* restricted to $L_1(X)$ is an L_1-operator. A function f in $L_p(X)$ is called *$\mathscr{S}$-invariant* if $T_s f = f$ for all $s \in S$. In the case of $p = 1$, the problem of finding necessary and sufficient conditions for the existence of a strictly positive $\mathscr{S}$-invariant function has been studied by many authors (see, for example, [1], [2], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13]). It is known that if $\| T_s \|_1 \leqq 1$ for all $s \in S$, then the following conditions are equivalent:

(0) There exists a function $f_0 \in L_1(X)$ with $f_0 > 0$ a.e. and $T_s f_0 = f_0$ for all $s \in S$.

(i) $A \in \mathscr{M}$ and $m(A) > 0$ imply $\inf \left\{ \int_A T_s 1 \, dm; s \in S \right\} > 0$.

(ii) $A \in \mathscr{M}$ and $m(A) > 0$ imply $M\left(\int_A T_s 1\, dm\right) > 0$.

The purpose of the present paper is to prove similar results for L_p-operator semigroups $\mathscr{S}$, without the restriction of norm condition.

In the last section we assume that $\sup\{\|T_s\|_p; s \in S\} < \infty$ and also that there exists a strictly positive function e in $L_q(X)$, where $p^{-1} + q^{-1} = 1$, such that $T_s^* e \leqq e$ a.e. for each $s \in S$. Under these assumptions we obtain a generalization of Neveu's decomposition theorem [9] for the particular semigroup generated by a single positive linear contraction on $L_1(X)$.

2. Existence of positive invariant functions.

THEOREM 1. *Let $\mathscr{S} = \{T_s; s \in S\}$ be a representation of S as a semigroup of positive linear operators on $L_1(X)$. Then the following conditions are equivalent.*

(0) *There exists a function $f_0 \in L_1(X)$ with $f_0 > 0$ a.e. and $T_s f_0 = f_0$ for all $s \in S$.*

(i) *There exists a non-negative function h in $L_1(X)$ such that the set $\{T_s h; s \in S\}$ is weakly sequentially compact in $L_1(X)$ and for any $0 \leqq u \in L_\infty(X)$ with $\|u\|_\infty > 0$,*

$$\inf\left\{\int (T_s h) u\, dm; s \in S\right\} > 0 .$$

PROOF. Since the implication (0) $\Rightarrow$ (i) is obvious, we prove here only the converse implication (i) $\Rightarrow$ (0).

Suppose (i) holds. It follows that $\sup\{\|T_s h\|_1; s \in S\} < \infty$. Hence if $\varphi \in LIM$, we can define, for $A \in \mathscr{M}$,

$$\mu(A) = \varphi\left(\int_A T_s h\, dm\right) .$$

The condition (i) implies that μ is a finite measure on $(X, \mathscr{M})$ equivalent with m. Let $f_0 = d\mu/dm$. Then, clearly, $f_0 > 0$ a.e., and $T_s f_0 = f_0$ for all $s \in S$, since φ is a left invariant mean. This completes the proof.

COROLLARY 1. *Let $\mathscr{S} = \{T_s; s \in S\}$ be a representation of S as a semigroup of positive linear operators on $L_1(X)$. Suppose* $\sup\{\|T_s\|_1; s \in S\} < \infty$. *Then the following conditions are equivalent.*

(0) *There exists a function $f_0 \in L_1(X)$ with $f_0 > 0$ a.e. and $T_s f_0 = f_0$ for all $s \in S$.*

(i) *$A \in \mathscr{M}$ and $m(A) > 0$ imply* $\inf\left\{\int_A T_s 1\, dm; s \in S\right\} > 0$.

PROOF. The implication (0) $\Rightarrow$ (i) is easy (cf. [2] or [9]), so we prove only the implication (i) $\Rightarrow$ (0).

Suppose (i) holds. By Theorem 1 it suffices to prove that the set $\{T_s 1; s \in S\}$ is weakly sequentially compact in $L_1(X)$. If this is not the case, then there exists an $\varepsilon > 0$, a sequence (A_n) in $\mathscr{M}$, and a sequence (s_n) in S such that $A_1 \supset A_2 \supset \cdots$, $\bigcap_{n=1}^{\infty} A_n = \emptyset$, and $\int_{A_n} T_{s_n} 1 \, dm \geqq \varepsilon$ for all $n \geqq 1$. But a slight modification of the proof of Lemma 9 of Hajian and Ito [5] demonstrates that this is impossible, and hence $\{T_s 1; s \in S\}$ must be weakly sequentially compact in $L_1(X)$. The proof is complete.

THEOREM 2. *Let $1 < p \leqq \infty$, and let $\mathscr{S} = \{T_s; s \in S\}$ be a representation of S as a semigroup of positive linear operators on $L_p(X)$. Then the following conditions are equivalent.*

(0) *There exists a function $f_0 \in L_p(X)$ with $f_0 > 0$ a.e. and $T_s f_0 = f_0$ for all $s \in S$.*

(i) *There exists a non-negative function h in $L_p(X)$ such that for any $0 \leqq u \in L_q(X)$ with $\|u\|_q > 0$,*

$$0 < \inf\left\{\int (T_s h) u \, dm; s \in S\right\} \leqq \sup\left\{\int (T_s h) u \, dm; s \in S\right\} < \infty ,$$

where $p^{-1} + q^{-1} = 1$.

(ii) *There exists a non-negative function h in $L_p(X)$ such that for any $0 \leqq u \in L_q(X)$ with $\|u\|_q > 0$,*

$$\sup\left\{\int (T_s h) u \, dm; s \in S\right\} < \infty \quad \text{and} \quad M\left(\int (T_s h) u \, dm\right) > 0 .$$

If $A \in \mathscr{M}$ then 1_A is the indicator function of A and $L_p(A)$ denotes the Banach space of all $L_p(X)$-functions that vanish a.e. on $X - A$. For the proof of Theorem 2 we need the following

LEMMA. *Let $1 < p \leqq \infty$, and let $\mathscr{S} = \{T_s; s \in S\}$ be a representation of S as a semigroup of positive linear operators on $L_p(X)$. Then the space X is uniquely decomposed into two sets Y and Z in $\mathscr{M}$ such that*

(a) *there exists a function $g \in L_p(Y)$ with $g > 0$ a.e. on Y and $T_s g = g$ for all $s \in S$,*

(b) *if $0 \leqq h \in L_p(X)$ satisfies $\sup\left\{\int (T_s h) u \, dm; s \in S\right\} < \infty$ for any $0 \leqq u \in L_q(X)$, then*

$$M\left(\int (T_s h) v \, dm\right) = 0$$

for any $0 \leqq v \in L_q(Z)$.

PROOF. Since the T_s are positive, there exists a non-negative $\mathscr{S}$-invariant function g in $L_p(X)$ such that for any non-negative $\mathscr{S}$-invariant function f in $L_p(X)$, $\operatorname{supp} f \subset \operatorname{supp} g$. Let us denote $Y = \operatorname{supp} g$ and $Z = X - Y$. To prove (b), let $0 \leqq h \in L_p(X)$ and $\sup\left\{\int (T_s h)u\,dm; s \in S\right\} < \infty$ for any $0 \leqq u \in L_q(X)$. If $\varphi \in LIM$ and $u \in L_q(X)$, define

$$\Phi(u) = \varphi\left(\int (T_s h)u\,dm\right).$$

Then Φ is a positive linear functional on $L_q(X)$ and, since the dual space of $L_q(X)$ is the space of $L_p(X)$, there exists a non-negative function f in $L_p(X)$ such that $\Phi(u) = \int fu\,dm$ for any $u \in L_q(X)$. Since $\Phi(T_s^* u) = \Phi(u)$ for any $s \in S$ and any $u \in L_q(X)$, it follows that $T_s f = f$ for all $s \in S$, and hence $\operatorname{supp} f \subset \operatorname{supp} g = Y$. Consequently we have $\Phi(v) = \int fv\,dm = 0$ for any $v \in L_q(Z)$. This proves (b), and the uniqueness of such a decomposition is easily checked. The proof is complete.

PROOF OF THEOREM 2. The implications (0) $\Rightarrow$ (i) $\Rightarrow$ (ii) are obvious, hence we prove only the implication (ii) $\Rightarrow$ (0).

Suppose (ii) holds. By Lemma it is sufficient to prove that $m(Z) = 0$. To see this, let $v = 1_Z$. Then, since $M\left(\int (T_s h)v\,dm\right) = 0$, the condition (ii) implies that $\| v \|_q = 0$ and hence $m(Z) = 0$. The proof is complete.

COROLLARY 2. *Let $1 < p \leqq \infty$, and let $\mathscr{S} = \{T_s; s \in S\}$ be a representation of S as a positive linear operators on $L_p(X)$. Suppose* $\sup\{\| T_s \|_p; s \in S\} < \infty$. *Then the following conditions are equivalent.*

(0) *There exists a function $f_0 \in L_p(X)$ with $f_0 > 0$ a.e. and $T_s f_0 = f_0$ for all $s \in S$.*

(i) *$A \in \mathscr{M}$ and $m(A) > 0$ imply* $\inf\left\{\int_A T_s 1\,dm; s \in S\right\} > 0$.

(ii) *$A \in \mathscr{M}$ and $m(A) > 0$ imply* $M\left(\int_A T_s 1\,dm\right) > 0$.

PROOF. Immediate from Theorem 2.

3. Decomposition theorem. Let $1 \leqq p \leqq \infty$, and let $\mathscr{S} = \{T_s; s \in S\}$ be a representation of S as a semigroup of positive linear operators on $L_p(X)$. Throughout this section we shall assume that

$$\sup\{\| T_s \|_p; s \in S\} < \infty, \tag{1}$$

and that there exists a strictly positive function e in $L_q(X)$ such that

$$T_s^* e \leqq e \text{ a.e. for each } s \in S. \tag{2}$$

PROPOSITION 1. *The following conditions are equivalent.*

(0) *There exists a function $f_0 \in L_p(X)$ with $f_0 > 0$ a.e. and $T_s f_0 = f_0$ for all $s \in S$.*

(i) *$A \in \mathscr{M}$ and $m(A) > 0$ imply* $\inf\left\{\int_A T_s 1\, dm;\, s \in S\right\} > 0$.

(ii) *$A \in \mathscr{M}$ and $m(A) > 0$ imply* $M\left(\int_A T_s 1\, dm\right) > 0$.

(iii) *$f \in L_p(X)$ and $f > 0$ a.e. imply $\sum_{n=1}^{\infty} T_{s_n} f = \infty$ a.e. for any sequence (s_n) in S.*

(iv) *$0 \leqq u \in L_q(X)$ and $\sum_{n=1}^{\infty} T_{s_n}^* u < \infty$ a.e. for some sequence (s_n) in S imply $u = 0$.*

(v) *$0 \leqq u \in L_q(X)$ and $\sum_{n=1}^{\infty} T_{s_n}^* u \in L_q(X)$ for some sequence (s_n) in S imply $u = 0$.*

PROOF. By Corollaries 1 and 2, it is sufficient to prove that (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i) and (i) $\Rightarrow$ (ii) $\Rightarrow$ (v).

(i) $\Rightarrow$ (iii): If $s \in S$ and $f \in L_p(X)$, define

$$V_s(ef) = e(T_s f) .$$

Since $\{ef;\, f \in L_p(X)\}$ is dense in $L_1(X)$ in the L_1-norm topology and $\| V_s(ef) \|_1 \leqq \| (T_s^* e) f \|_1 \leqq \| ef \|_1$, V_s may be considered to be a positive linear operator on $L_1(X)$ such that $\| V_s \|_1 \leqq 1$. It is clear that $V_{s_1} V_{s_2} = V_{s_1 s_2}$ for $s_1, s_2 \in S$. Thus $\{ V_s;\, s \in S\}$ is a representation of S as a semigroup of positive linear contractions on $L_1(X)$. By using an argument analogous to that of Fong [3, p. 79], it may be readily seen that (i) implies that

(i)′ $A \in \mathscr{M}$ and $m(A) > 0$ imply $\inf\left\{\int_A V_s 1\, dm;\, s \in S\right\} > 0$.

Let $f \in L_p(X)$, $f > 0$ a.e., and let $\xi \in L_\infty(X)$, $\xi > 0$ a.e.. Then define, as in Neveu [9],

$$h = \xi \Big/ \left(1 + \sum_{n=1}^{\infty} V_{s_n}(ef)\right)$$

where (s_n) is an arbitrary sequence in S. It follows that $0 \leqq h \in L_\infty(X)$ and

$$\sum_{n=1}^{\infty} \int (V_{s_n}(ef)) h\, dm = \int \left(\sum_{n=1}^{\infty} V_{s_n}(ef)\right) h\, dm < \infty .$$

Hence $\inf\left\{\int (V_s(ef)) h\, dm;\, s \in S\right\} = 0$. But since $ef > 0$ a.e. and $\| V_s \|_1 \leqq 1$ for all $s \in S$, it follows that

$$\inf\left\{\int (V_s 1) h\, dm;\, s \in S\right\} = 0 ,$$

and hence $h = 0$ a.e. by (i)′. This demonstrates that

$$\sum_{n=1}^{\infty} T_{s_n} f = \frac{1}{e} \sum_{n=1}^{\infty} V_{s_n}(ef) = \infty \quad \text{a.e.} .$$

(iii) $\Rightarrow$ (iv): If $0 \leqq u \in L_q(X)$ and $\sum_{n=1}^{\infty} T_{s_n}^* u < \infty$ a.e. for some sequence

(s_n) in S, define

$$f = \xi \Big/ \left(1 + \sum_{n=1}^{\infty} T_{s_n}^* u\right).$$

It follows that $f \in L_p(X)$, $f > 0$ a.e., and $\sum_{n=1}^{\infty} \int f(T_{s_n}^* u) dm < \infty$. Since $\sum_{n=1}^{\infty} T_{s_n} f = \infty$ a.e. by (iii), we observe that $u = 0$ a.e..

(iv) $\Rightarrow$ (v): Obvious.

(v) $\Rightarrow$ (i): Let $0 \leqq h \in L_\infty(X)$ and $\sum_{n=1}^{\infty} V_{s_n}^* h \in L_\infty(X)$ for some sequence (s_n) in S. Since $V_{s_n}^* h = (1/e) T_{s_n}^*(eh)$ for each $n \geqq 1$, it follows that

$$\sum_{n=1}^{\infty} T_{s_n}^*(eh) \in L_q(X) .$$

Since $e > 0$ a.e., (v) implies that $h = 0$ a.e.. This and Theorem 3.3 of Sachdeva [10] imply that (i)′ holds. Hence an argument analogous to that of Fong [3, p. 79] implies that (i) holds too.

(i) $\Rightarrow$ (ii): Obvious.

(ii) $\Rightarrow$ (v): If $\varphi \in LIM$ and $0 \leqq u \in L_q(X)$, define

$$\Phi(u) = \varphi\left(\int (T_s 1) u \, dm\right) .$$

Here if $\sum_{n=1}^{\infty} T_{s_n}^* u \in L_q(X)$ for some sequence (s_n) in S, then for each $k \geqq 1$ we have

$$k\Phi(u) = \Phi\left(\sum_{n=1}^{k} T_{s_n}^* u\right) \leqq \Phi\left(\sum_{n=1}^{\infty} T_{s_n}^* u\right) < \infty ,$$

since φ is a left invariant mean. Thus $\Phi(u) = 0$, and so $M\left(\int (T_s 1) u \, dm\right) = 0$. Consequently (ii) implies that $u = 0$ a.e.. This completes the proof of Proposition 1.

The following proposition is a counterpart to Proposition 1.

PROPOSITION 2. *The following conditions are equivalent.*

(0) *The only* $g \in L_p(X)$ *such that* $T_s g = g$ *for all* $s \in S$ *is* 0.

(i) *There exists a strictly positive function* u *in* $L_q(X)$ *such that*

$$\inf\left\{\int (T_s 1) u \, dm;\ s \in S\right\} = 0 .$$

(ii) *For each strictly positive function* f *in* $L_p(X)$ *there exists a sequence* (s_n) *in* S, *dependent on* f, *such that* $\sum_{n=1}^{\infty} T_{s_n} f < \infty$ *a.e.*.

(iii) *There exists a strictly positive function* u *in* $L_q(X)$ *and a sequence* (s_n) *in* S *such that* $\sum_{n=1}^{\infty} T_{s_n}^* u < \infty$ *a.e.*.

(iv) *There exists a strictly positive function* u *in* $L_q(X)$ *and a sequence* (s_n) *in* S *such that* $\sum_{n=1}^{\infty} T_{s_n}^* u \in L_q(X)$.

PROOF. (0) $\Rightarrow$ (i): Let $\{V_s; s \in S\}$ be the same as in the proof of Proposition 1. It follows from (0) and Proposition 1 that the only g in $L_1(X)$ such that $V_s g = g$ for all $s \in S$ is 0. Let $\varphi \in LIM$ and define, for $h \in L_\infty(X)$, $\Phi(h) = \varphi\left(\int (V_s e) h\, dm\right)$. Since $\Phi(V_s h) = \Phi(h)$ for any $s \in S$ and any $h \in L_\infty(X)$, and since $\| V_s \|_1 \leqq 1$ for any $s \in S$, it follows from Lemma 1 of Neveu [9] that for some strictly positive function h in $L_\infty(X)$,

$$\inf\left\{\int (V_s e) h\, dm;\, s \in S\right\} = 0 .$$

Here if we let $u = eh$, then $\inf\left\{\int (T_s 1) u\, dm;\, s \in S\right\} = 0$.

(i) $\Rightarrow$ (0): By (2), if $T_s g = g$ for all $s \in S$, then $T_s | g | = | g |$ for all $s \in S$. Thus (i) and (1) imply that

$$\int | g |\, u\, dm = \inf\left\{\int (T_s | g |) u\, dm;\, s \in S\right\} = 0 ,$$

and hence $g = 0$ a.e..

(i) $\Rightarrow$ (ii): Let $f \in L_p(X)$ and $f > 0$ a.e.. Since the $\| T_s \|_p$ are bounded, (i) implies that $\inf\left\{\int (T_s f) u\, dm;\, s \in S\right\} = 0$, and so there exists a sequence (s_n) in S such that $\sum_{n=1}^\infty \int (T_{s_n} f) u\, dm < \infty$. Since $u > 0$ a.e., it follows that $\sum_{n=1}^\infty T_{s_n} f < \infty$ a.e..

(ii) $\Rightarrow$ (i): Let (s_n) be a sequence in S such that $\sum_{n=1}^\infty T_{s_n} 1 < \infty$ a.e.. Let $\xi \in L_\infty(X)$ and $\xi > 0$ a.e.. Define $u = \xi/(1 + \sum_{n=1}^\infty T_{s_n} 1)$. Then $u \in L_q(X)$, $u > 0$ a.e., and $\inf\left\{\int (T_s 1) u\, dm;\, s \in S\right\} = 0$.

(i) $\Rightarrow$ (iv): Since, by (i), the only g in $L_1(X)$ such that $V_s g = g$ for all $s \in S$ is 0, there exists a strictly positive function h in $L_\infty(X)$ with $h \leqq 1$ such that $\inf\left\{\int (V_s 1) h\, dm;\, s \in S\right\} = 0$. Then, as in Sachdeva [10, p. 203] (see also Takahashi [12, Lemma 4]), we can choose $s_n \in S$, $n = 1, 2, \cdots$, such that

$$\int \left(\sum_{i=1}^n V_{s_n} \cdots V_{s_i} 1\right) h\, dm < \frac{1}{2^n} .$$

For $j \geqq 0$, define

$$h_j = \left[h - \sum_{n=j+1}^\infty \left(\sum_{i=1}^n (V_{s_n} \cdots V_{s_i})^* h\right)\right]^+ .$$

It is clear that $0 \leqq h_j \leqq h$, and

$$\int (h - h_j) dm \leqq \sum_{n=j+1}^\infty \int \sum_{i=1}^n (V_{s_n} \cdots V_{s_i})^* h\, dm < \frac{1}{2^j} .$$

It follows that $m(\bigcap_{j=0}^\infty \{x \in X;\, h_j(x) = 0\}) = 0$. Next we prove that for each $j \geqq 0$,

$$\sum_{n=1}^{\infty} (V_{s_n} \cdots V_{s_1})^* h_j \in L_\infty(X) . \tag{3}$$

To see this, define the operators S_{ji}, where $j \geqq i \geqq 0$, as follows:

$$S_{ji} = \begin{cases} V_{s_j} \cdots V_{s_{i+1}} & \text{if } j > i \geqq 0 , \\ I & \text{if } j = i \geqq 0 . \end{cases}$$

It follows, as in [10, p. 204], that

$$\sum_{m=j+1}^{\infty} (S_{mj})^* h_j \leqq 1 \quad \text{a.e.} .$$

Thus

$$\sum_{m=j+1}^{\infty} V_{s_1}^* \cdots V_{s_m}^* h_j = (V_{s_1}^* \cdots V_{s_j}^*)\Big(\sum_{m=j+1}^{\infty} (S_{mj})^* h_j\Big) \in L_\infty(X) ,$$

from which (3) follows easily. Since $T_s^*(eh_j) = e(V_s^* h_j)$ for any $s \in S$, we have

$$\sum_{n=1}^{\infty} (T_{s_n} \cdots T_{s_1})^* (eh_j) \in L_q(X) . \tag{4}$$

Let $a_j = \| eh_j \|_q + \| \sum_{n=1}^{\infty} (T_{s_n} \cdots T_{s_1})^* (eh_j) \|_q + 1$, and put

$$v = \sum_{j=0}^{\infty} (eh_j / 2^j a_j) .$$

Then $v \in L_q(X)$, $v > 0$ a.e., and $\sum_{n=1}^{\infty} (T_{s_n} \cdots T_{s_1})^* v \in L_q(X)$.

(iv) $\Rightarrow$ (iii): Obvious.

(iii) $\Rightarrow$ (i): Let u be a strictly positive function in $L_q(X)$ and (s_n) a sequence in S such that $\sum_{n=1}^{\infty} T_{s_n}^* u < \infty$ a.e.. Let $\xi \in L_\infty(X)$ and $\xi > 0$ a.e.. Define

$$f = \xi \Big/ \Big(1 + \sum_{n=1}^{\infty} T_{s_n}^* u\Big) .$$

Then $f \in L_p(X)$ and $f > 0$ a.e.. Since $\int (\sum_{n=1}^{\infty} T_{s_n} f) u \, dm = \int f (\sum_{n=1}^{\infty} T_{s_n}^* u) dm < \infty$, $\sum_{n=1}^{\infty} T_{s_n} f < \infty$ a.e.. Thus if we let

$$h = \xi \Big/ \Big(1 + \sum_{n=1}^{\infty} V_{s_n}(ef)\Big) ,$$

then $h \in L_\infty(X)$ and $h > 0$ a.e.. Moreover, since $\sum_{n=1}^{\infty} \int V_{s_n}(ef) h \, dm < \infty$,

$$\inf \Big\{ \int V_s(ef) h \, dm ; s \in S \Big\} = 0 .$$

Therefore, it follows that

$$\inf\left\{\int (T_s1)eh\,dm;\, s\in S\right\} = \inf\left\{\int (V_se)h\,dm;\, s\in S\right\} = 0\ .$$

This completes the proof of Proposition 2.

Combining Propositions 1 and 2, we have the following decomposition of the space X.

THEOREM 3. *The space X is the disjoint union of two uniquely determined sets P and N in $\mathscr{M}$ such that*

(a) *there exists a function g in $L_p(P)$ with $g > 0$ a.e. on P and $T_sg = g$ for all $s \in S$,*

(b) *if $T_sf = f$ for all $s \in S$, then $f \in L_p(P)$,*

(c) *if f is a strictly positive function in $L_p(X)$, then for any sequence (s_n) in S,*

$$\sum_{n=1}^{\infty} T_{s_n}f = \infty \quad a.e. \quad on\ P\ ,$$

and for some sequence (s'_n) in S,

$$\sum_{n=1}^{\infty} T_{s'_n}f < \infty \quad a.e. \quad on\ N = X - P\ .$$

A positive operator T on $L_p(X)$ is called *conservative* if $\sum_{n=0}^{\infty} T^nf = 0$ or ∞ a.e. for any $0 \leqq f \in L_p(X)$. The following proposition is an extension of results due to Sachdeva [10] and Fong [3].

PROPOSITION 3. *If there exists a strictly positive function f_0 in $L_p(X)$ such that $T_sf_0 = f_0$ for all $s \in S$, then the T_s are conservative and for each $A \in \mathscr{M}$, the left invariant means of $\int_A T_s1\,dm$ coincide. Conversely, if S is countably generated, if the T_s are conservative, and if for each $A \in \mathscr{M}$, the left invariant means of $\int_A T_s1\,dm$ coincide, then there exists a strictly positive function f_0 in $L_p(X)$ such that $T_sf_0 = f_0$ for all $s \in S$.*

PROOF. Using techniques given in Sachdeva [10] and Fong [3], it is now easy to prove the proposition, and hence we omit the details.

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