Research Article

On the Existence of Positive Periodic Solutions for Second-Order Functional Differential Equations with Multiple Delays

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The existence results of positive ω-periodic solutions are obtained for the second-order functional differential equation with multiple delays $u''(t) + a(t)u(t) = f(t, u(t), u(t - \tau_1(t)), \ldots, u(t - \tau_n(t)))$, where $a(t) \in C(\mathbb{R})$ is a positive ω-periodic function, $f : \mathbb{R} \times [0, +\infty)^{n+1} \to [0, +\infty)$ is a continuous function which is ω-periodic in t, and $\tau_1(t), \ldots, \tau_n(t) \in C(\mathbb{R}, [0, +\infty))$ are ω-periodic functions. The existence conditions concern the first eigenvalue of the associated linear periodic boundary problem. Our discussion is based on the fixed-point index theory in cones.

1. Introduction

In this paper, we deal with the existence of positive periodic solution of the second-order functional differential equation with multiple delays

$$u''(t) + a(t)u(t) = f(t, u(t), u(t - \tau_1(t)), \ldots, u(t - \tau_n(t))), \quad t \in \mathbb{R}, \tag{1.1}$$

where $a(t) \in C(\mathbb{R})$ is a positive ω-periodic function, $f : \mathbb{R} \times [0, +\infty)^{n+1} \to [0, +\infty)$ is a continuous function which is ω-periodic in t, and $\tau_1(t), \ldots, \tau_n(t) \in C(\mathbb{R}, [0, +\infty))$ are ω-periodic functions $\omega > 0$ is a constant.

In recent years, the existence of periodic solutions for second-order functional differential equations has been researched by many authors see [1–8] and references therein. In some practice models, only positive periodic solutions are significant. In [4–8], the authors obtained the existence of positive periodic solutions for some second-order functional

differential equations by using fixed-point theorems of cone mapping. Especially in [5], Wu considered the second-order functional differential equation

$$u^{''}(t) + a(t)u(t) = \lambda f(t, u(t - \tau_1(t)), \ldots, u(t - \tau_n(t))), \quad t \in \mathbb{R}, \tag{1.2}$$

and obtained the existence results of positive periodic solutions by using the Krasnoselskii fixed-point theorem of cone mapping when the coefficient $a(t)$ satisfies the condition that $0 < a(t) < \pi^2/\omega^2$ for every $t \in \mathbb{R}$. And in [8], Li obtained the existence results of positive ω-periodic solutions for the second-order differential equation with constant delays

$$-u^{''}(t) + a(t)u(t) = f(t, u(t - \tau_1), \ldots, u(t - \tau_n)), \quad t \in \mathbb{R}, \tag{1.3}$$

by employing the fixed-point index theory in cones. For the second-order differential equations without delay, the existence of positive periodic solutions has been discussed by more authors, see [9–14].

Motivated by the paper mentioned above, we research the existence of positive periodic solutions of (1.1). We aim to obtain the essential conditions on the existence of positive periodic solution of (1.1) by constructing a special cone and applying the fixed-point index theory in cones.

In this paper, we assume the following conditions:

(H1) $a \in C(\mathbb{R}, (0, +\infty))$ is ω-periodic function and there exists a constant $1 \leq p \leq +\infty$ such that

$$\|a\|_p \leq K(2p^*), \tag{1.4}$$

where $\|a\|_p$ is the p-norm of a in $L^p[0, \omega]$, p^* is the conjugate exponent of p defined by $(1/p) + (1/p^*) = 1$, and the function $K(q)$ is defined by

$$K(q) = \begin{cases} \dfrac{2\pi}{q\omega^{1+2/q}} \left(\dfrac{2}{2+q} \right)^{1-2/q} \left(\dfrac{\Gamma(1/q)}{\Gamma(1/2 + 1/q)} \right)^2, & \text{if } 1 \leq q < +\infty, \\ \dfrac{4}{\omega}, & \text{if } q = +\infty \end{cases} \tag{1.5}$$

in which Γ is the Gamma function.

(H2) $f \in C(\mathbb{R} \times [0, +\infty)^{n+1}, [0, +\infty))$ and $f(t, x_0, x_1, \ldots, x_n)$ is ω-periodic in t.

(H3) $\tau_1(t), \ldots, \tau_n(t) \in C(\mathbb{R}, [0, \infty))$ are ω-periodic functions.

In Assumption (H1), if $p = +\infty$, since $K(2) = \pi^2/\omega^2$, then (1.4) implies that a satisfies the condition

$$0 < a(t) \leq \frac{\pi^2}{\omega^2}, \quad t \in [0, \omega]. \tag{1.6}$$

This condition includes the case discussed in [5].

The techniques used in this paper are completely different from those in [5]. Our results are more general than those in [5] in two aspects. Firstly, we relax the conditions of the coefficient $a(t)$ appeared in an equation in [5] and expand the range of its values. Secondly, by constructing a special cone and applying the fixed-point index theory in cones, we obtain the essential conditions on the existence of positive periodic solutions of (1.1). The conditions concern the first eigenvalue of the associated linear periodic boundary problem, which improve and optimize the results in [5]. To our knowledge, there are very few works on the existence of positive periodic solutions for the above functional differential equations under the conditions concerning the first eigenvalue of the corresponding linear equation.

Our main results are presented and proved in Section 3. Some preliminaries to discuss (1.1) are presented in Section 2.

2. Preliminaries

In order to discuss (1.1), we consider the existence of ω-periodic solution of the corresponding linear differential equation

$$u'' + a(t)u = h(t), \quad t \in \mathbb{R}, \tag{2.1}$$

where $h \in C(\mathbb{R})$ is a ω-periodic function. It is obvious that finding an ω-periodic solution of (2.1) is equivalent to finding a solution of the linear periodic boundary value problem

$$u'' + a(t)u = h(t), \quad t \in [0, \omega],$$
$$u(0) = u(\omega), \quad u'(0) = u'(\omega). \tag{2.2}$$

In [14], Torres show the following existence resulted.

Lemma 2.1. *Assume that (H1) holds, then for every* $h \in C[0, \omega]$*, the linear periodic boundary problem* (2.2) *has a unique solution expressed by*

$$u(t) = \int_0^\omega G(t, s)h(s)ds, \quad t \in [0, \omega], \tag{2.3}$$

where $G(t, s) \in C([0, \omega] \times [0, \omega])$ *is the Green function of the linear periodic boundary problem* (2.2)*, which satisfies the positivity:* $G(t, s) > 0$ *for every* $(t, s) \in [0, \omega] \times [0, \omega]$*.*

For the details, see [14, Theorem 2.1 and Corollary 2.3].

For $m \in \mathbb{N}$, we use $C_\omega^m(\mathbb{R})$ to denote the mth-order continuous differentiable ω-periodic functions space. Let $X = C_\omega(\mathbb{R})$ be the Banach space of all continuous ω-periodic functions equipped the norm $\|u\| = \max_{0 \le t \le \omega}|u(t)|$.

Let

$$K_0 = \{u \in X \mid u(t) \ge 0, \ t \in \mathbb{R}\} \tag{2.4}$$

be the cone of all nonnegative functions in X. Then X is an ordered Banach space by the cone K_0. K_0 has a nonempty interior

$$\text{int}(K_0) = \{u \in X \mid u(t) > 0, \ t \in \mathbb{R}\}. \tag{2.5}$$

Let

$$\underline{G} = \min_{0 \le t, s \le \omega} G(t, s), \quad \overline{G} = \max_{0 \le t, s \le \omega} G(t, s); \quad \sigma = \underline{G}/\overline{G}. \tag{2.6}$$

Lemma 2.2. *Assume that (H1) holds, then for every $h \in X$, (2.1) has a unique ω-periodic solution u. Let $T : h \mapsto u$, then $T : X \to X$ is a completely continuous linear operator, and when $h \in K_0$, Th has the positivity estimate*

$$Th(t) \ge \sigma \|Th\|, \quad \forall t \in \mathbb{R}. \tag{2.7}$$

Proof. Let $h \in X$. By Lemma 2.1, the linear periodic boundary problem (2.2) has a unique solution $u \in C^2[0, \omega]$ given by (2.3). We extend u to a ω-periodic function, which is still denoted by u, then $u := Th \in C^2_\omega(\mathbb{R})$ is a unique ω-periodic solution of (2.1). By (2.3),

$$Th(t) = \int_0^\omega G(t, s) h(s) ds, \quad t \in [0, \omega]. \tag{2.8}$$

From this we see that T maps every bounded set in X to a bounded equicontinuous set of X. Hence, by the Ascoli-Arzelà theorem, $T : X \to X$ is completely continuous.

Let $h \in K_0$. For every $t \in [0, \omega]$, from (2.8) it follows that

$$Th(t) = \int_0^\omega G(t, s) h(s) ds \le \overline{G} \int_0^\omega h(s) ds, \tag{2.9}$$

and therefore,

$$\|Th\| \le \overline{G} \int_0^\omega h(s) ds. \tag{2.10}$$

Using (2.8) and this inequality, we have that

$$\begin{aligned} Th(t) = \int_0^\omega G(t, s) h(s) ds &\ge \underline{G} \int_0^\omega h(s) ds \\ &= \left(\underline{G}/\overline{G}\right) \cdot \overline{G} \int_0^\omega h(s) ds \\ &\ge \sigma \|Th\|. \end{aligned} \tag{2.11}$$

Hence, by the periodicity of u, (2.7) holds for every $t \in \mathbb{R}$. □

From (2.7) we easily see that $T(K_0) \subset \text{int}(K_0)$; namely, $T : X \to X$ is a strongly positive linear operator. By the well-known Krein-Rutman theorem, the spectral radius $r(T) > 0$ is a simple eigenvalue of T, and T has a corresponding positive eigenfunction $\phi \in K_0$; that is,

$$T\phi = r(T)\phi. \tag{2.12}$$

Since ϕ can be replaced by $c\phi$, where $c > 0$ is a constant, we can choose $\phi \in K_0$ such that

$$\int_0^\omega \phi(t)dt = 1. \tag{2.13}$$

Set $\lambda_1 = 1/r(T)$, then $\phi = T(\lambda_1 \phi)$. By Lemma 2.2 and the definition of T, $\phi \in C_\omega^2(\mathbb{R})$ satisfies the differential equation

$$\phi''(t) + a(t)\phi(t) = \lambda_1\phi(t), \quad t \in \mathbb{R}. \tag{2.14}$$

Thus, λ_1 is the minimum positive real eigenvalue of the linear equation (2.1) under the ω-periodic condition. Summarizing these facts, we have the following lemma.

Lemma 2.3. *Assume that (H1) holds, then there exist $\phi \in K_0 \cap C_\omega^2(\mathbb{R})$ such that (2.13) and (2.14) hold.*

Let $f : \mathbb{R} \times [0, \infty)^{n+1} \to [0, \infty)$ satisfy the assumption (H2). For every $u \in X$, set

$$F(u)(t) := f(t, u(t), u(t - \tau_1), \ldots, u(t - \tau_n)), \quad t \in \mathbb{R}. \tag{2.15}$$

Then $F : K_0 \to K_0$ is continuous. Define a mapping $A : K_0 \to X$ by

$$A = T \circ F. \tag{2.16}$$

By the definition of operator T, the ω-periodic solution of (1.1) is equivalent to the fixed point of A. Choose a subcone of K_0 by

$$K = \{u \in K_0 \mid u(t) \geq \sigma \|u\|_C, \ t \in \mathbb{R}\}. \tag{2.17}$$

By the strong positivity (2.7) of T and the definition of A, we easily obtain the following.

Lemma 2.4. *Assume that (H1) holds, then $A(K_0) \subset K$ and $A : K \to K$ is completely continuous.*

Hence, the positive ω-periodic solution of (1.1) is equivalent to the nontrivial fixed point of A. We will find the nonzero fixed point of A by using the fixed-point index theory in cones.

We recall some concepts and conclusions on the fixed-point index in [15, 16]. Let X be a Banach space and $K \subset X$ a closed convex cone in X. Assume that Ω is a bounded open subset of X with boundary $\partial\Omega$, and $K \cap \Omega \neq \emptyset$. Let $A : K \cap \overline{\Omega} \to K$ be a completely continuous

mapping. If $Au \neq u$ for any $u \in K \cap \partial\Omega$, then the fixed-point index $i(A, K \cap \Omega, K)$ has definition. One important fact is that if $i(A, K \cap \Omega, K) \neq 0$, then A has a fixed point in $K \cap \Omega$. The following two lemmas in [16] are needed in our argument.

Lemma 2.5. *Let Ω be a bounded open subset of X with $\theta \in \Omega$, and $A : K \cap \overline{\Omega} \to K$ a completely continuous mapping. If $\mu Au \neq u$ for every $u \in K \cap \partial\Omega$ and $0 < \mu \leq 1$, then $i(A, K \cap \Omega, K) = 1$.*

Lemma 2.6. *Let Ω be a bounded open subset of X and $A : K \cap \overline{\Omega} \to K$ a completely continuous mapping. If there exists an $e \in K \setminus \{\theta\}$ such that $u - Au \neq \mu e$ for every $u \in K \cap \partial\Omega$ and $\mu \geq 0$, then $i(A, K \cap \Omega, K) = 0$.*

3. Main Results

We consider the existence of positive ω-periodic solutions of (1.1). Assume that $f : \mathbb{R} \times [0, \infty)^{n+1} \to [0, \infty)$ satisfy (H2). To be convenient, we introduce the notations

$$
\begin{aligned}
f_0 &= \liminf_{\overline{x} \to 0^+} \min_{t \in [0,\omega]} \frac{f(t, x_0, x_1, \ldots, x_n)}{\overline{x}}, \\
f^0 &= \limsup_{\overline{x} \to 0^+} \max_{t \in [0,\omega]} \frac{f(t, x_0, x_1, \ldots, x_n)}{\underline{x}}, \\
f_\infty &= \liminf_{\underline{x} \to +\infty} \min_{t \in [0,\omega]} \frac{f(t, x_0, x_1, \ldots, x_n)}{\overline{x}}, \\
f^\infty &= \limsup_{\underline{x} \to +\infty} \max_{t \in [0,\omega]} \frac{f(t, x_0, x_1, \ldots, x_n)}{\underline{x}},
\end{aligned}
\tag{3.1}
$$

where $\overline{x} = \max\{x_0, x_1, \ldots, x_n\}$ and $\underline{x} = \min\{x_0, x_1, \ldots, x_n\}$. Our main results are as follows.

Theorem 3.1. *Suppose that (H1)–(H3) hold. If f satisfies the condition*

(H4) $f^0 < \lambda_1 < f_\infty$,

then (1.1) has at least one positive ω-periodic solution.

Theorem 3.2. *Suppose that (H1)–(H3) hold. If f satisfies the condition*

(H5) $f^\infty < \lambda_1 < f_0$,

then (1.1) has at least one positive ω-periodic solution.

In Theorem 3.1, the condition (H4) allows $f(t, x_0, x_1, \ldots, x_n)$ to be superlinear growth on $x_0, x_1, \ldots, x_n$. For example,

$$
f(t, x_0, x_1, \ldots, x_n) = b_0(t)x_0^2 + b_1(t)x_1^2 + \cdots + b_n(t)x_n^2
\tag{3.2}
$$

satisfies (H4) with $f^0 = 0$ and $f_\infty = +\infty$, where $b_0, b_1, \ldots, b_n \in C_\omega(\mathbb{R})$ are positive ω-periodic functions.

In Theorem 3.2, the condition (H5) allows $f(t, x_0, x_1, \ldots, x_n)$ to be sublinear growth on $x_0, x_1, \ldots, x_n$. For example,

$$f(t, x_0, x_1, \ldots, x_n) = c_0(t)\sqrt{|x_0|} + c_1(t)\sqrt{|x_1|} + \cdots + c_n(t)\sqrt{|x_n|} \tag{3.3}$$

satisfies (H5) with $f_0 = +\infty$ and $f^\infty = 0$, where $c_0, c_1, \ldots, c_n \in C_\omega(\mathbb{R})$ are positive ω-periodic solution.

Applying Theorems 3.1 and 3.2 to (1.2), we have the following.

Corollary 3.3. *Suppose that (H1)–(H3) hold. If the parameter λ satisfies one of the following conditions*

(1) $\lambda_1/f_\infty < \lambda < \lambda_1/f^0$,

(2) $\lambda_1/f_0 < \lambda < \lambda_1/f^\infty$,

then (1.2) has at least one positive ω-periodic solution.

This result improves and extends [5, Theorem 1.3].

Proof of Theorem 3.1. Let $K \subset X$ be the cone defined by (2.17) and $A : K \to K$ the operator defined by (2.16). Then the positive ω-periodic solution of (1.1) is equivalent to the nontrivial fixed point of A. Let $0 < r < R < +\infty$ and set

$$\Omega_1 = \{u \in X \mid \|u\| < r\}, \qquad \Omega_2 = \{u \in X \mid \|u\| < R\}. \tag{3.4}$$

We show that the operator A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ when r is small enough and R large enough.

Since $f^0 < \lambda_1$, by the definition of f^0, there exist $\varepsilon \in (0, \lambda_1)$ and $\delta > 0$, such that

$$f(t, x_0, x_1, \ldots, x_n) \leq (\lambda_1 - \varepsilon)\underline{x}, \quad t \in [0, \omega], \ \overline{x} \in (0, \delta), \tag{3.5}$$

where $\overline{x} = \max\{x_0, x_1, \ldots, x_n\}$ and $\underline{x} = \min\{x_0, x_1, \ldots, x_n\}$. Choosing $r \in (0, \delta)$, we prove that A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_1$; namely, $\mu Au \neq u$ for every $u \in K \cap \partial\Omega_1$ and $0 < \mu \leq 1$. In fact, if there exist $u_0 \in K \cap \partial\Omega_1$ and $0 < \mu_0 \leq 1$ such that $\mu_0 Au_0 = u_0$ and since $u_0 = T(\mu_0 F(u_0))$, by definition of T and Lemma 2.2, $u_0 \in C_\omega^2(\mathbb{R})$ satisfies the differential equation

$$u_0''(t) + a(t)u_0(t) = \mu_0 F(u_0)(t), \quad t \in \mathbb{R}, \tag{3.6}$$

where $F(u)$ is defined by (2.15). Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, we have

$$0 < \sigma \|u_0\| \leq u_0(\tau) \leq \|u_0\| = r < \delta, \quad \forall \tau \in \mathbb{R}. \tag{3.7}$$

This implies that

$$0 < \max\{u_0(t), u_0(t - \tau_1(t)), \ldots, u_0(t - \tau_n(t))\} < \delta, \quad t \in \mathbb{R}. \tag{3.8}$$

From this and (3.5), it follows that

$$
\begin{aligned}
F(u_0)(t) &= f(t, u_0(t), u_0(t - \tau_1(t)), \dots, u_0(t - \tau_n(t))) \\
&\leq (\lambda_1 - \varepsilon) \cdot \min\{u_0(t), u_0(t - \tau_1(t)), \dots, u_0(t - \tau_n(t))\} \\
&\leq (\lambda_1 - \varepsilon) u_0(t), \quad t \in \mathbb{R}.
\end{aligned}
\tag{3.9}
$$

By this inequality and (3.6), we have

$$
u_0''(t) + a(t) u_0(t) = \mu_0 F(u_0)(t) \leq (\lambda_1 - \varepsilon) u_0(t), \quad t \in \mathbb{R}.
\tag{3.10}
$$

Let $\phi \in K \cap C_\omega^2(\mathbb{R})$ be the function given in Lemma 2.4. Multiplying the inequality (3.10) by $\phi(t)$ and integrating on $[0, \omega]$, we have

$$
\int_0^\omega \left[u_0''(t) + a(t) u_0(t) \right] \phi(t) dt \leq (\lambda_1 - \varepsilon) \int_0^\omega u_0(t) \phi(t) dt.
\tag{3.11}
$$

For the left side of the above inequality using integration by parts, then using the periodicity of u_0 and ϕ and (2.14), we have

$$
\begin{aligned}
\int_0^\omega \left[u_0''(t) + a(t) u_0(t) \right] \phi(t) dt &= \int_0^\omega u_0(t) \left[\phi''(t) + a(t) \phi(t) \right] dt \\
&= \lambda_1 \int_0^\omega u_0(x) \phi(t) dt.
\end{aligned}
\tag{3.12}
$$

Consequently, we obtain that

$$
\lambda_1 \int_0^\omega u_0(x) \phi(t) dt \leq (\lambda_1 - \varepsilon) \int_0^\omega u_0(x) \phi(t) dt.
\tag{3.13}
$$

Since $u_0 \in K \cap \partial \Omega_1$, by the definition of K and (2.13),

$$
\int_0^\omega u_0(x) \phi(t) dt \geq \sigma \|u_0\| \int_0^\omega \phi(t) dt = \sigma \|u_0\| > 0.
\tag{3.14}
$$

From this and (3.13), we conclude that $\lambda_1 \leq \lambda_1 - \varepsilon$, which is a contradiction. Hence, A satisfies the condition of Lemma 2.5 in $K \cap \partial \Omega_1$. By Lemma 2.5, we have

$$
i(A, K \cap \Omega_1, K) = 1.
\tag{3.15}
$$

On the other hand, since $f_\infty > \lambda_1$, by the definition of f_∞, there exist $\varepsilon_1 > 0$ and $H > 0$ such that

$$
f(t, x_0, x_1, \dots, x_n) \geq (\lambda_1 + \varepsilon_1) \, \overline{x}, \quad t \in [0, \omega], \; \underline{x} > H,
\tag{3.16}
$$

where $\bar{x} = \max\{x_0, x_1, \ldots, x_n\}$ and $\underline{x} = \min\{x_0, x_1, \ldots, x_n\}$. Choose $R > \max\{H/\sigma, \delta\}$ and $e(t) \equiv 1$. Clearly, $e \in K \setminus \{\theta\}$. We show that A satisfies the condition of Lemma 2.6 in $K \cap \partial\Omega_2$; namely, $u - Au \neq \mu\phi$ for every $u \in K \cap \partial\Omega_2$ and $\mu \geq 0$. In fact, if there exist $u_1 \in K \cap \partial\Omega_2$ and $\mu_1 \geq 0$ such that $u_1 - Au_1 = \mu_1 e$, since $u_1 - \mu_1 e = Au_1 = T(F(u_1))$, by the definition of T and Lemma 2.2, $u_1 \in C_\omega^2(\mathbb{R})$ satisfies the differential equation

$$u_1''(t) + a(t)\big(u_1(t) - \mu_1\big) = F(u_1)(t), \quad t \in \mathbb{R}. \tag{3.17}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definitions of K and Ω_2, we have

$$u_1(\tau) \geq \sigma\|u_1\| = \sigma R > H, \quad \forall \tau \in \mathbb{R}. \tag{3.18}$$

This means that

$$\min\{u_1(t), u_1(t - \tau_1(t)), \ldots, u_1(t - \tau_n(t))\} > H, \quad t \in \mathbb{R}. \tag{3.19}$$

Combining this with (3.16), we have that

$$\begin{aligned}
F(u_1)(t) &= f(t, u_1(t), u_1(t - \tau_1(t)), \ldots, u_1(t - \tau_n(t))) \\
&\geq (\lambda_1 + \varepsilon_1) \cdot \max\{u_1(t), u_1(t - \tau_1(t)), \ldots, u_1(t - \tau_n(t))\} \\
&\geq (\lambda_1 + \varepsilon_1)u_1(t), \quad t \in \mathbb{R}.
\end{aligned} \tag{3.20}$$

From this inequality and (3.17), it follows that

$$u_1''(t) + a(t)u_1(t) = \mu_1 a(t) + F(u_1)(t) \geq (\lambda_1 + \varepsilon_1)u_1(t), \quad t \in \mathbb{R}. \tag{3.21}$$

Multiplying this inequality by $\phi(t)$ and integrating on $[0, \omega]$, we have

$$\int_0^\omega \big[u_1''(t) + a(t)u_1(t)\big]\phi(t)dt \geq (\lambda_1 + \varepsilon_1)\int_0^\omega u_1(t)\phi(t)dt. \tag{3.22}$$

For the left side of the above inequality using integration by parts and (2.14), we have

$$\begin{aligned}
\int_0^\omega \big[u_1''(t) + a(t)u_1(t)\big]\phi(t)dt &= \int_0^\omega u_1(t)\big[\phi''(t) + a(t)\phi(t)\big]dt \\
&= \lambda_1 \int_0^\omega u_1(x)\phi(t)dt.
\end{aligned} \tag{3.23}$$

From this and (3.22), it follows that

$$\lambda_1 \int_0^\omega u_1(x)\phi(t)dt \geq (\lambda_1 + \varepsilon_1)\int_0^\omega u_1(x)\phi(t)dt. \tag{3.24}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of K and (2.13), we have

$$\int_0^\omega u_1(x)\phi(t)dt \geq \sigma\|u_1\| \int_0^\omega \phi(t)dt = \sigma\|u_1\| > 0. \tag{3.25}$$

Hence, from (3.24) it follows that $\lambda_1 \geq \lambda_1 + \varepsilon_1$, which is a contradiction. Therefore, A satisfies the condition of Lemma 2.6 in $K \cap \partial\Omega_2$. By Lemma 2.6, we have

$$i(A, K \cap \Omega_2, K) = 0. \tag{3.26}$$

Now by the additivity of the fixed-point index (3.15), and (3.26), we have

$$i\left(A, K \cap \left(\Omega_2 \setminus \overline{\Omega}_1\right), K\right) = i(A, K \cap \Omega_2, K) - i(A, K \cap \Omega_1, K) = -1. \tag{3.27}$$

Hence A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive ω-periodic solution of (1.1). □

Proof of Theorem 3.2. Let $\Omega_1, \Omega_2 \subset X$ be defined by (3.4). We prove that the operator A defined by (2.16) has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ if r is small enough and R is large enough.

By $f_0 > \lambda_1$ and the definition of f_0, there exist $\varepsilon > 0$ and $\delta > 0$, such that

$$f(t, x_0, x_1, \ldots, x_n) \geq (\lambda_1 + \varepsilon)\overline{x}, \quad t \in [0, \omega], \ \overline{x} \in (0, \delta), \tag{3.28}$$

where $\overline{x} = \max\{x_0, x_1, \ldots, x_n\}$. Let $r \in (0, \delta)$ and $e(t) \equiv 1$. We prove that A satisfies the condition of Lemma 2.6 in $K \cap \partial\Omega_1$; namely, $u - Au \neq \mu e$ for every $u \in K \cap \partial\Omega_1$ and $\mu \geq 0$. In fact, if there exist $u_0 \in K \cap \partial\Omega_1$ and $\mu_0 \geq 0$ such that $u_0 - Au_0 = \mu_0 e$ and since $u_0 - \mu_0 e = Au_0 = T(F(u_0))$, by the definition of T and Lemma 2.2, $u_0 \in C^2_\omega(\mathbb{R})$ satisfies the differential equation

$$u_0''(t) + a(t)(u_0(t) - \mu_0) = F(u_0)(t), \quad t \in \mathbb{R}. \tag{3.29}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, u_0 satisfies (3.7), and hence (3.8) holds. From (3.8) and (3.28), it follows that

$$\begin{aligned}
F(u_0)(t) &= f(t, u_0(t), u_0(0 - \tau_1(t)), \ldots, u_0(t - \tau_n(t))) \\
&\geq (\lambda_1 + \varepsilon) \cdot \max\{u_0(t), u_0(t - \tau_1(t)), \ldots, u_0(t - \tau_n(t))\} \\
&\geq (\lambda_1 + \varepsilon)u_0(t), \quad t \in \mathbb{R}.
\end{aligned} \tag{3.30}$$

By this and (3.29), we obtain that

$$u_0''(t) + a(t)u_0(t) = \mu_0 a(t) + F(u_0)(t) \geq (\lambda_1 + \varepsilon)u_0(t), \quad t \in \mathbb{R}. \tag{3.31}$$

Multiplying this inequality by $\phi(t)$ and integrating on $[0, \omega]$, we have

$$\int_0^\omega \left[u_0''(t) + a(t)u_0(t)\right]\phi(t)dt \geq (\lambda_1 + \varepsilon) \int_0^\omega u_0(t)\phi(t)dt. \tag{3.32}$$

For the left side of this inequality using integration by parts and (2.14), we have

$$
\int_0^\omega \left[u_0^{''}(t) + a(t)u_0(t) \right] \phi(t) dt = \int_0^\omega u_0(t) \left[\phi'(t) + a(t)\phi(t) \right] dt
$$
$$
= \lambda_1 \int_0^\omega u_0(x)\phi(t) dt. \tag{3.33}
$$

From this and (3.32), it follows that

$$
\lambda_1 \int_0^\omega u_1(x)\phi(t) dt \geq (\lambda_1 + \varepsilon_1) \int_0^\omega u_1(x)\phi(t) dt. \tag{3.34}
$$

Since $u_0 \in K \cap \partial\Omega_1$, from the definition of K and (2.13) it follows that (3.14) holds. By (3.14) and (3.34), we see that $\lambda_1 \geq \lambda_1 + \varepsilon$, which is a contradiction. Hence, A satisfies the condition of Lemma 2.6 in $K \cap \partial\Omega_1$. By Lemma 2.6, we have

$$
i(A, K \cap \Omega_1, K) = 0. \tag{3.35}
$$

Since $f^\infty < \lambda_1$, by the definition of f^∞, there exist $\varepsilon_1 \in (0, \lambda_1)$ and $H > 0$ such that

$$
f(t, x_0, x_1, \ldots, x_n) \leq (\lambda_1 - \varepsilon_1)\underline{x}, \quad t \in [0, \omega], \ \underline{x} > H, \tag{3.36}
$$

where $\underline{x} = \min\{x_0, x_1, \ldots, x_n\}$. Choosing $R > \max\{H/\sigma, \delta\}$, we show that A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_2$; namely, $\mu Au \neq u$ for every $u \in K \cap \partial\Omega_2$ and $0 < \mu \leq 1$. In fact, if there exist $u_1 \in K \cap \partial\Omega_2$ and $0 < \mu_1 \leq 1$ such that $\mu_1 Au_1 = u_1$, since $u_1 = T(\mu_1 F(u_1))$, by the definition of T and Lemma 2.2, $u_1 \in C_\omega^2(\Omega)$ satisfies the differential equation

$$
u_1^{''}(t) + a(t)u_1(t) = \mu_1 F(u_1)(t), \quad t \in \mathbb{R}. \tag{3.37}
$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definitions of K and Ω_2, u_1 satisfies (3.18), and hence (3.19) holds. By (3.19) and (3.36), we have

$$
F(u_1)(t) = f(t, u_1(t), u_1(t - \tau_1(t)), \ldots, u_1(t - \tau_n(t)))
$$
$$
\leq (\lambda_1 - \varepsilon_1) \cdot \min\{u_1(t), u_1(t - \tau_1(t)), \ldots, u_1(t - \tau_n(t))\} \tag{3.38}
$$
$$
\leq (\lambda_1 - \varepsilon_1)u_1(t), \quad t \in \mathbb{R}.
$$

From this inequality and (3.37), it follows that

$$
u_1^{''}(t) + a(t)u_1(t) = \mu_1 F(u_1)(t) \leq (\lambda_1 - \varepsilon_1)u_1(t), \quad t \in \mathbb{R}. \tag{3.39}
$$

Multiplying this inequality by $\phi(t)$ and integrating on $[0, \omega]$, we have

$$
\int_0^\omega \left[u_1^{''}(t) + a(t)u_1(t) \right] \phi(t) dt \leq (\lambda_1 - \varepsilon_1) \int_0^\omega u_1(t)\phi(t) dt. \tag{3.40}
$$

For the left side of this inequality using integration by parts and (2.14), we have

$$
\int_0^\omega \left[u_1''(t) + a(t)u_1(t) \right] \phi(t)dt = \int_0^\omega u_1(t) \left[\phi''(t) + a(t)\phi(t) \right] dt
$$
$$
= \lambda_1 \int_0^\omega u_1(x)\phi(t)dt. \tag{3.41}
$$

Consequently, we obtain that

$$
\lambda_1 \int_0^\omega u_1(x)\phi(t)dt \ \leq (\lambda_1 - \varepsilon_1) \int_0^\omega u_1(x)\phi(t)dt. \tag{3.42}
$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of K and (2.13) we see that (3.25) holds. From (3.25) and (3.42), we see that $\lambda_1 \leq \lambda_1 - \varepsilon_1$, which is a contradiction. Hence, A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_2$. By Lemma 2.5 we have

$$
i(A, K \cap \Omega_2, K) = 1. \tag{3.43}
$$

Now, from (3.35) and (3.43) it follows that

$$
i\left(A, K \cap \left(\Omega_2 \setminus \overline{\Omega}_1 \right), K \right) = i(A, K \cap \Omega_2, K) - i(A, K \cap \Omega_1, K) = 1. \tag{3.44}
$$

Hence, A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive ω-periodic solution of (1.1). $\quad\square$
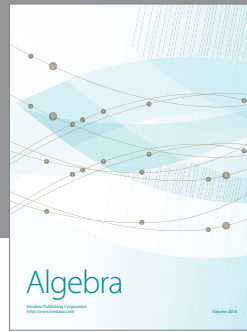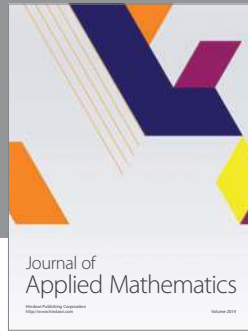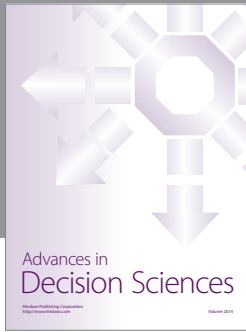
Acknowledgment

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